<table>
<thead>
<tr>
<th>Title</th>
<th>METRIC CONVEXITY OF #-SYMMETRIC CONES (Application of Geometry to Operator Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>LIM, YONGDO</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2009), 1632: 8-13</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2009-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/140429">http://hdl.handle.net/2433/140429</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
METRIC CONVEXITY OF \(\#\)-SYMMETRIC CONES

YONGDO LIM

ABSTRACT. In this paper we introduce a general notion of a symmetric cone, valid for the finite and infinite dimensional case, and prove that one can deduce the seminegative curvature of the Thompson part metric in this general setting: the distance function between points evolving in time on two geodesics is a convex function.

1. Symmetric sets with midpoints

A \(\#\)-symmetric set consists of a binary system \((X, \bullet)\), with left translation \(S_x y := x \bullet y\) representing the point symmetry through \(x\), satisfying for all \(a, b, c \in X\):

(S1) \(a \bullet a = a (S_a a = a)\);
(S2) \(a \bullet (a \bullet b) = b (S_a S_a = \text{id}_X)\);
(S3) \(a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c) (S_a S_b S_a)\);
(S4) the equation \(x \bullet a = b (S_x a = b)\) has a unique solution \(x \in X\), called the midpoint or mean of \(a\) and \(b\), and denoted by \(a \# b\).

The axioms bear close resemblance to the Loos axioms for a symmetric space. A binary system \((X, \bullet)\) satisfying (S1), (S2), and (S3) also satisfies (S4) if and only if it is a quasigroup. Thus the preceding structures are also referred to as symmetric quasigroups. Systems satisfying only Axioms (1)-(3) are called symmetric sets (or involutive quandles in knot theory circles)

A pointed \#-symmetric set is a triple \((X, \bullet, \epsilon)\), where \((X, \bullet)\) is a \#-symmetric set and \(\epsilon \in X\) is some distinguished point, called the base point. In this setting we define

\(x^0 = \epsilon,\ x^{-1} := S_\epsilon x,\ x^2 := S_x \epsilon,\ x^{1/2} := \epsilon \# x\)

and inductively from these definitions all dyadic powers are defined so that the following rules are satisfied:

\((x^r)^s = x^{rs},\ x^r \# x^s = x^{(r+s)/2}\).
If we consider the dyadic rationals $\mathbb{D}$ endowed with the $\#$-symmetric structure $a \bullet b = 2a - b$ (the reflection of $b$ through $a$), then $a \# b = (a + b)/2$, the usual midpoint, and the map $t \mapsto x^t : \mathbb{D} \to X$ is both a $\bullet$-homomorphism and $\#$-homomorphism. From this fact the preceding rules (and others) easily follow.

The displacement group $G(X)$ (also called the transvection group) of a $\#$-symmetric set $X$ is the group generated under the composition by all transformations of the form $S_x S_y$, $x, y \in X$. If $X$ is pointed with base point $\varepsilon$, then $G(X)$ is generated by all $S_x S_\varepsilon$ and $X$ embeds into $G(X)$ as a twisted subgroup (closed under $g \bullet h = gh^{-1}g$) via the quadratic representation $Q : X \to G(X)$ defined by $Q(x) = S_x S_\varepsilon$. The image $Q(X)$ is a pointed $\#$-symmetric set under the preceding $\bullet$-operation and the quadratic representation is an isomorphism between $X$ and $Q(X)$. In particular, $Q(X)$ is uniquely 2-divisible and $Q(x \# y) = Q(x) \# Q(y), Q(x^{1/2}) = Q(x)^{1/2} ([1, 2])$. For $x, y \in X$, we write interchangeably as convenient

$$x \cdot y = Q(x)y = Q(x)(y).$$

**Example 1.1.** Let $\mathbb{R}$ be equipped with the standard $\#$-symmetric operation $x \bullet y := 2x - y$ and the usual metric. Then $x \# y = (x + y)/2$, the usual midpoint operation, and the metric is convex. Thus $(\mathbb{R}, \bullet, 0)$ is a pointed symmetric space with convex metric.

**Definition 1.2.** A pointed symmetric space with convex metric is a pointed $\#$-symmetric set $P$ equipped with a complete metric $d(\cdot, \cdot)$ satisfying for all $x, y \in P$ and $g \in G(P)$

(i) $d(gx, gy) = d(x, y)$,
(ii) $d(x^{-1}, y^{-1}) = d(x, y)$,
(iii) $d(x^{1/2}, y^{1/2}) \leq \frac{1}{2}d(x, y)$,
(iv) $x \mapsto x^2 : P \to P$ is continuous.

A symmetric space with convex metric is a $\#$-symmetric set equipped with a complete metric that is a pointed symmetric space with convex metric with respect to some pointing.

We recall some basic results about symmetric spaces with convex metrics from [3].
Theorem 1.3 ([3]). Let $P$ be a symmetric space with convex metric. Then for distinct $x, y \in P$, there exists a unique continuous homomorphism $\alpha_{x,y}$ (called an $s$-geodesic) of $\#$-symmetric sets from $\mathbb{R}$ into $P$ satisfying $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$. Furthermore, the maps

$$(x, y) \mapsto x \bullet y : P \times P \to P, \quad (t, x, y) \mapsto \alpha_{x,y}(t) := x \# t y : \mathbb{R} \times P \times P \to P$$

are continuous.

The element $x \# t y$ is called the $t$-weighted mean of $x$ and $y$. Note that $x \# y = x \# 1/2 y$.

Theorem 1.4 ([3]). Let $P$ be a symmetric space with convex metric. For every pair $(\beta, \gamma)$ of $s$-geodesics, the real function $t \mapsto d(\beta(t), \gamma(t))$ is a convex function.

Remark 1.5. We note that the unique $s$-geodesic line satisfying $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$ is

$$\alpha_{x,y}(t) = x^{1/2}.(x^{-1/2}.y)^{t}$$

and $\alpha_{y,x}(1-t) = \alpha_{x,y}(t), t \in \mathbb{R}$ ([3]). In particular,

$$\left(Q(y)x\right)^{t} = Q(y)Q(x^{1/2})(Q(x^{1/2})y^{2})^{t-1}. \quad (1.1)$$

2. $\#$-SYMMETRIC CONES

Let $V$ be a real Banach space and let $\Omega$ henceforth denote a non-empty open convex cone of $V$: $t\Omega \subset \Omega$ for all $t > 0$, $\Omega + \Omega \subset \Omega$, and $\overline{\Omega} \cap -\overline{\Omega} = \{0\}$, where $\overline{\Omega}$ denotes the closure of $\Omega$. We consider the partial order on $V$ defined by

$$x \leq y \text{ if and only if } y - x \in \overline{\Omega}.$$ 

We further assume that $\Omega$ is a normal cone: there exists a constant $K$ with $||x|| \leq K||y||$ for all $x, y \in \Omega$ with $x \leq y$. Any member $a$ of $\Omega$ is an order unit for the ordered space $(V, \leq)$, and hence $|x|_{a} := \inf\{\lambda > 0 : -\lambda a \leq x \leq \lambda a\}$ defines a norm. By Proposition 1.1 in [6], for a normal cone $\overline{\Omega}$, the order unit norm $| \cdot |_{a}$ is equivalent to $|| \cdot ||$. 
A. C. Thompson [7] (cf. [5], [6]) has proved that $\Omega$ is a complete metric space with respect to the Thompson part metric defined by

$$d(x, y) = \max\{\log M(x/y), \log M(y/x)\}$$

where $M(x/y) := \inf\{\lambda > 0 : x \leq \lambda y\} = |x|_y$. Furthermore, the topology induced by the Thompson metric agrees with the relative Banach space topology.

**Theorem 2.1 ([4]).** Let $\Omega$ be an open convex normal cone in a Banach space $V$. Suppose that there is a pointed $\#$-symmetric structure on $\Omega$ satisfying

(i) $2x \leq \varepsilon + x^2$

(ii) the squaring map $x \mapsto x^2 = Q(x)\varepsilon$ is continuous,

(iii) every basic displacement $Q(x)$ is continuous and linear on $\Omega$.

Then $\Omega$ is a pointed symmetric space with convex metric, the Thompson metric whose metric topology agrees with the relative topology.

A JB-algebra $V$ is a Jordan algebra with unit $e$ endowed with a complete norm $|| \cdot ||$ such that

$$||zw|| \leq ||z|| \cdot ||w||,$$

$$||z^2|| = ||z||^2,$$

$$||z||^2 \leq ||z^2 + w^2||.$$

**Example 2.2.** (1) The positive cone of hermitian elements of a $C^*$-algebra.

(2) Spin factors and Lorentz cones: Let $(H, \langle \cdot | \cdot \rangle)$ be a real Hilbert space with $\dim H \geq 2$ and let $V = \mathbb{R} \times H$ equipped with the Banach space norm $||(t, x)|| = |t| + \sqrt{\langle x|x \rangle}$. We define the Jordan product on $V$ by

$$(s, y) \circ (t, x) = (st + \langle y|x \rangle, sx + ty).$$

The element $e = (1, 0) \in V$ acts as a unit element. The corresponding symmetric cone is given by (the Lorentz cone, forward light cone)

$$\Omega = \{(t, x) \in V : t > ||x|| = \sqrt{\langle x|x \rangle}\}.$$

For $x \in V$ we write $L(x)(y) = xy$, the multiplication operator. We consider the set

$$\Omega := \{x \in V : \text{Spec}(L(x)) \subset (0, \infty)\}.$$
Then $\Omega$ is an open convex cone of $V$ and is realized as $\Omega = \exp(V) := \{\exp(x) : x \in V\}$.

The Banach algebra norm agrees with the order unit norm $|x|_\epsilon := \inf\{t > 0 : t\epsilon \pm x \geq 0\}$, or equivalently $\Omega$ is a normal cone. The quadratic representation of the Jordan algebra is defined by $P(z) = 2L(z)^2 - L(z^2)$. It is well-known that for each $z \in \Omega$, $P(z) \in G(\Omega)$ the linear automorphism group of $\Omega$. In fact, there is a polar decomposition $G(\Omega) = P(\Omega) \text{Aut}(V)$ where $\text{Aut}(V)$ denotes the Jordan automorphism group of $V$. We further note that $\text{Aut}(V) = \{g \in G(\Omega) : g(e) = e\}$. The basic properties

\[ P(z)z^{-1} = z, \quad P(z)^{-1} = P(z^{-1}), \quad P(P(z)w) = P(z)P(w)P(z) \]

yield a pointed symmetric set structure $x \bullet y = P(x)y^{-1}$ with $e = \epsilon$ as base point on the set of invertible elements, in particular on the cone $\Omega$. In symmetric set notation, $P(a) = Q(a)$ and the symmetric set inverse $a^{-1} := e \bullet a$ agrees with the Jordan inverse of $a$.

Next, we show that the pointed symmetric space $(\Omega, \epsilon = e)$ is #-symmetric. Let $x, y \in \Omega$ such that $x^2 = y^2$. Then by the commutativity of Jordan products, $0 = x^2 + y^2 = L(x + y)(x - y)$. Since $L(z)$ is invertible for all $z \in \Omega$, $x - y = 0$. This implies that each element of $\Omega$ has a unique square root. Note that if $a = \exp(x), x \in V$ then $a^{1/2} = \exp(\frac{1}{2}x)$. Moreover, if $a, b \in \Omega$ then the quadratic equation $P(x)a^{-1} = b$ has a unique solution in $\Omega$. Note that $x = P(1/2)(P(a^{-1/2})b)^{1/2} \in \Omega$ solves the equation. Suppose that $x$ and $y$ are solutions in $\Omega$. Then

\[
(P(a^{-1/2})x)^2 = P(P(a^{-1/2})x)\epsilon = P(a^{-1/2})P(x)P(a^{-1/2})\epsilon \\
= P(a^{-1/2})(P(x)a^{-1}) \\
= P(a^{-1/2})b = P(a^{-1/2})(P(y)a^{-1}) \\
= P(a^{-1/2})P(y)P(a^{-1/2})\epsilon \\
= (P(a^{-1/2})y)^2
\]

and hence $P(a^{-1/2})x = P(a^{-1/2})y$, so $x = y$. We conclude that the open convex cone $\Omega$ is a #-symmetric set under the operation $x \bullet y = P(x)y^{-1}$. In this case the dyadic
power $a^t$ of $a = \exp(x)$ agrees with $\exp(tx)$ and the geometric mean $a\# b$ of $a$ and $b$ is

$$a\# b = P(a^{1/2})(P(a^{-1/2})b)^{1/2}.$$ 

**Corollary 2.3.** Let $V$ be a JB-algebra and let $\Omega$ be the associated symmetric cone. Then $\Omega$ is a symmetric space with convex metric with respect to the Thompson metric. In particular, the distance function between points evolving in time on two geodesics is a convex function.

**REFERENCES**


**E-mail address:** ylim@knu.ac.kr