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On Telgárski’s formula

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The following formula due to R. Telgársky ([3]) is indispensable in dealing with products of scattered spaces.

**Theorem 1.** Let $X$, $Y$ be scattered spaces with $\text{leng}(X) = \alpha$ and $\text{leng}(Y) = \beta$. Then

1. $(X \times Y)^{(\sigma)} = \bigcup_{\tau \oplus v = \sigma} X^{(\tau)} \times Y^{(v)}$ for every $\sigma$.
2. $(X \times Y)_{(\sigma)} = \bigcup_{\tau \oplus v = \sigma} X_{(\tau)} \times Y_{(v)}$ for every $\eta \sigma$.
3. $\text{leng}(X \times Y) = \sup \{\tau \oplus v + 1 | \tau < \alpha \text{ and } v < \beta\}$.

The symbol $\oplus$ means Hessenberg’s sum defined as follows:

**Definition 1.** Let $\alpha > 0$, $\beta > 0$ be ordinals. Using Cantor’s normal form, represent $\alpha$, $\beta$ uniquely as

\[
\alpha = \omega^{n_1} + \omega^{n_2} + \cdots + \omega^{n_k}, \quad \beta = \omega^{m_1} + \omega^{m_2} + \cdots + \omega^{m_k},
\]

$\gamma_1 > \gamma_2 > \cdots > \gamma_k$, $0 \leq n_i < \omega$, $0 \leq m_i < \omega$, so that $n_i = 0 = m_i$ does not occur. Define

\[
\alpha \oplus \beta = \Sigma_{i=1}^{k} \omega^{n_i + m_i}.
\]

Also define $\alpha \oplus 0 = 0 \oplus \alpha = 0$ for every $\alpha$.

Hessenberg’s sum is certainly convenient for describing the derivatives $(X \times Y)^{(\sigma)}$ but not for describing $\text{leng}(X \times Y)$. With an emphasis on the length of product spaces, we define a binary operation $\pi$ as follows:

**Definition 2.** Let $\alpha > 0$, $\beta > 0$ be ordinals. Represent $\alpha$, $\beta$ as in Definition 1. Put

\[
l = \min\{\max\{i | n_i \neq 0\}, \max\{j | m_j \neq 0\}\}
\]

and define

\[
\pi(\alpha, \beta) = \begin{cases} 
\Sigma_{i=1}^{l} \omega^{n_i + m_i} & \text{if } l < k \\
(\Sigma_{i=1}^{k-1} \omega^{n_i + m_i}) + \omega^{n_k + m_k - 1} & \text{if } l = k
\end{cases}
\]

where $l = k$ is, of course, equivalent to $n_k \neq 0 \neq m_k$.

For convenience, define $\pi(\alpha, 0) = \pi(0, \alpha) = 0$ for every ordinal $\alpha$.

It is to be noted that, unlike Hessenberg’s sum, the operation $\pi$ is a countinuous operation with respect to the order topology.

Now we can restate Telgársky’s formula as follows:
Theorem 2. Let $X$, $Y$ be scattered spaces with $\text{leng}(X) = \alpha$, $\text{leng}(Y) = \beta$. Then
(1) $\text{leng}(X \times Y) = \pi(\alpha, \beta)$ .
(2) $(X \times Y)^{\sigma} = \cup\{X_{(\tau)} \times Y_{(\upsilon)} \mid \pi(\tau + 1, \upsilon + 1) = \sigma + 1\}$ for every ordinal $\sigma$ .
(3) $(X \times Y)^{\sigma} = \cup\{X^{(\tau)} \times Y^{(\upsilon)} \mid \pi(\tau + 1, \upsilon + 1) = \sigma + 1\}$ for every ordinal $\sigma$ .

We write simply $\pi(\alpha, \beta) = \alpha \ast \beta$ .

Proposition 1. $\alpha \ast \beta = \beta \ast \alpha$ .
$(\alpha \ast \beta) \ast \gamma = \alpha \ast (\beta \ast \gamma)$ .

Definition 3. A factorization $\alpha = \beta \ast \gamma$ of an ordinal $\alpha$ is called trivial if one of $\beta$, $\gamma$ is 1 (and the other is $\alpha$) . An ordinal $\alpha$ is called a prime ordinal if $\alpha > 1$ and it does not admit a non-trivial factorization.

As far as I know, the following notion was first defined and used by K. Borsuk ([1] also see [2]).

Definition 4. A factorization $X \approx Y \times Z$ of a space $X$ is called trivial if one of $Y$, $Z$ is a one point space (and the other is homeomorphic to $X$) . A space $X$ is called a prime space (or simply a prime) if $|X| > 1$ and it does not admit a non-trivial factorization.

By the definition of the operation $\pi$ we have

Proposition 2. An ordinal $\alpha$ is a prime ordinal if and only if $\alpha = \omega^\gamma + 1$ with $\gamma$ an ordinal.

The compact countable metric space $X$ satisfying $\text{leng}(X) = \alpha$ and $|X^{(\alpha - 1)}| = |X_{(\alpha - 1)}| = n$ is denoted by $MS(\alpha, n)$ . By the uniqueness of $MS(\alpha, n)$ we have $MS(\alpha, n) \approx MS(\alpha, 1) \times MS(1, n)$ and, by Theorem 2, $MS(\alpha, 1) \times MS(\beta, 1) \approx MS(\alpha \ast \beta, 1)$ . These facts are combined with Proposition 2 to give

Proposition 3. A compact countable metric space $X$ is a prime if and only if $X \approx MS(1, p)$ with $p$ a prime number or $X \approx MS(\omega^\gamma + 1, 1)$ with $\gamma$ an ordinal.

Theorem 3. Every non-limit ordinal $\alpha > 1$ is factorized uniquely as

$$\alpha = \alpha_1 \ast \alpha_2 \ast \cdots \ast \alpha_j$$

into prime ordinals $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_j$ . In full detail, if the normal form of $\alpha$ is

$$\alpha = \omega^{\zeta_1}n_1 + \omega^{\zeta_2}n_2 + \cdots + \omega^{\zeta_k}n_k$$
with $\zeta_k = 0$ then

$$\alpha = \frac{\omega^{n_1} + 1}{\omega^{n_2} + 1} \cdots \frac{\omega^{n_k} + 1}{\omega^{n_k-1} + 1} \cdots \frac{\omega^{n_k} + 1}{\omega^{n_k-1} + 1} \cdots$$

(Do not confuse $n_{k-1}$ with $n_k - 1$.)

A translation of the theorem into product spaces is as follows:

**Corollary 1.** Every compact countable metric space $X$ with $|X| > 1$ is factorized uniquely as

$$X \approx X_1 \times X_2 \times \cdots \times X_j$$

into primes $X_1$, $X_2$, \ldots, $X_j$. In further detail, if $X = MS(\alpha, n)$, $\alpha = \omega^{n_1} + \omega^{n_2} + \cdots + \omega^{n_k}$ with $\zeta_k = 0$, then

$$X \approx MS(\omega^{n_1} + 1, 1) \times MS(\omega^{n_2} + 1, 1) \times \cdots \times MS(\omega^{n_k} + 1, 1) \times \cdots$$

where $n = p_1 p_2 \cdots p_r$ is the usual factorization of the natural number $n$ into primes.

**Examples.** Put, for example, $\alpha = \omega^2 + \omega 2 + 3$. Then

$$\omega^2 + \omega 2 + 3 = (\omega^2 + 1) * (\omega + 1) * (\omega + 1) * 2 * 2 .$$

Thus

$$MS(\omega^2 + \omega 2 + 3, 1) = MS(\omega^2 + 1, 1) \times MS(\omega + 1, 1) \times MS(\omega + 1, 1) \times MS(2, 1) \times MS(2, 1) .$$
\[ MS(\omega^2 + \omega + 3, 6) = MS(\omega^2 + 1, 1) \times MS(\omega + 1, 1) \times MS(\omega + 1, 1) \times MS(2, 1) \times MS(2, 1) \times MS(1, 3) \times MS(1, 2). \]

References

