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On Telgárski's formula

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The following formula due to R. Telgársky ([3]) is indispensable in dealing with products of scattered spaces.

Theorem 1. *Let* X, Y *be scattered spaces with* $\mathrm{leng}(X)=\alpha$ *and* $\mathrm{leng}(Y)=\beta$. *Then*
(1) $(X\times Y)^{(\sigma)}=\cup_{\tau\oplus v=\sigma}X^{(\tau)}\times Y^{(v)}$ *for every* σ .
(2) $(X\times Y)_{(\sigma)}=\cup_{\tau\oplus v=\sigma}X_{(\tau)}\times Y_{(v)}$ *for every* σ .
(3) $\mathrm{leng}(X\times Y)=\sup\{\tau\oplus v+1\,|\,\tau<\alpha \text{ and } v<\beta\}$.

The symbol $\oplus$ means Hessenberg's sum defined as follows :

Definition 1. Let $\alpha>0,\ \beta>0$ be ordinals. Using Cantor's normal form, represent $\alpha,\ \beta$ uniquely as

$$\alpha=\omega^{\gamma_1}n_1+\omega^{\gamma_2}n_2+\cdots+\omega^{\gamma_k}n_k\ ,\quad \beta=\omega^{\gamma_1}m_1+\omega^{\gamma_2}m_2+\cdots+\omega^{\gamma_k}m_k\ ,$$

$\gamma_1>\gamma_2>\cdots>\gamma_k$, $0\le n_i<\omega$, $0\le m_i<\omega$, so that $n_i=0=m_i$ does not occur. Define

$$\alpha\oplus\beta=\Sigma_{i=1}^{k}\,\omega^{\gamma_i}(n_i+m_i)\quad .$$

Also define $\alpha\oplus 0=0\oplus\alpha=0$ for every α .

Hessenberg's sum is certainly convinient for describing the derivatives $(X\times Y)^{(\sigma)}$ but not for describing $\mathrm{leng}(X\times Y)$. With an emphasis on the length of product spaces, we define a binary operation π as follows :

Definition 2. Let $\alpha>0,\ \beta>0$ be ordinals. Represent $\alpha,\ \beta$ as in Definition 1. Put

$$l=\min\{\max\{i\,|\,n_i\neq 0\}\ ,\ \max\{j\,|\,m_j\neq 0\}\}$$

and define

$$\pi(\alpha,\beta)=\begin{cases}\Sigma_{i=1}^{l}\,\omega^{\gamma_i}(n_i+m_i) & \text{if } l<k\\ (\Sigma_{i=1}^{k-1}\,\omega^{\gamma_i}(n_i+m_i))+\omega^{\gamma_k}(n_k+m_k-1) & \text{if } l=k\end{cases}\ ,$$

where $l=k$ is, of course, equivalent to $n_k\neq 0\neq m_k$.

For convenience, define $\pi(\alpha,0)=\pi(0,\alpha)=0$ for every ordinal α .

It is to be noted that , unlike Hessenberg's sum, the operation π is a countinuous operation with respect to the order topology.

Now we can restate Telgársky's formula as follows :

Theorem 2. *Let* X, Y *be scattered spaces with* $\mathrm{leng}(X)=\alpha$, $\mathrm{leng}(Y)=\beta$. *Then*
(1) $\mathrm{leng}(X\times Y)=\pi(\alpha,\beta)$.
(2) $(X\times Y)_{(\sigma)}=\cup\{X_{(\tau)}\times Y_{(v)}\mid \pi(\tau+1,v+1)=\sigma+1\}$ for every ordinal σ .
(3) $(X\times Y)^{(\sigma)}=\cup\{X^{(\tau)}\times Y^{(v)}\mid \pi(\tau+1,v+1)=\sigma+1\}$ for every ordinal σ .

We write simply $\pi(\alpha,\beta)=\alpha*\beta$.

Proposition 1. $\alpha*\beta=\beta*\alpha$. $(\alpha*\beta)*\gamma=\alpha*(\beta*\gamma)$.

Definition 3. A factorization $\alpha=\beta*\gamma$ of an ordinal α is called *trivial* if one of β, γ is 1 (and the other is α) . An ordinal α is called a *prime* ordinal if $\alpha>1$ and it does not admit a non-trivial factorization.

As far as I know, the following notion was first defined and used by K. Borsuk ([1] also see [2]).

Definition 4. A factorization $X\approx Y\times Z$ of a space X is called *trivial* if one of Y, Z is a one point space (and the other is homeomorphic to X) . A space X is called a *prime* space (or simply a *prime*) if $|X|>1$ and it does not admit a non-trivial factorization.

By the definition of the operation π we have

Proposition 2. *An ordinal* α *is a prime ordinal if and only if* $\alpha=\omega^\gamma+1$ *with* γ *an ordinal.*

The compact countable metric space X satisfying $\mathrm{leng}(X)=\alpha$ and $|X^{(\alpha-1)}|=|X_{(\alpha-1)}|=n$ is denoted by $MS(\alpha,n)$. By the uniqueness of $MS(\alpha,n)$ we have $MS(\alpha,n)\approx MS(\alpha,1)\times MS(1,n)$ and, by Theorem 2, $MS(\alpha,1)\times MS(\beta,1)\approx MS(\alpha*\beta,1)$. These facts are combined with Proposition 2 to give

Proposition 3. *A compact countable metric space* X *is a prime if and only if* $X\approx MS(1,p)$ *with* p *a prime number or* $X\approx MS(\omega^\gamma+1,1)$ *with* γ *an ordinal.*

Theorem 3. *Every non-limit ordinal* $\alpha>1$ *is factorized uniquely as*

$$\alpha=\alpha_1*\alpha_2*\cdots*\alpha_j$$

into prime ordinals $\alpha_1\geq\alpha_2\geq\cdots\geq\alpha_j$. *In full detail, if the normal form of* α *is*

$$\alpha=\omega^{\zeta_1}n_1+\omega^{\zeta_2}n_2+\cdots+\omega^{\zeta_k}n_k$$

with $\zeta_k = 0$ *then*

$$\begin{aligned}\alpha = \ & \overbrace{(\omega^{\zeta_1}+1) * (\omega^{\zeta_1}+1) * \cdots * (\omega^{\zeta_1}+1)}^{n_1} * \\ & \overbrace{(\omega^{\zeta_2}+1) * (\omega^{\zeta_2}+1) * \cdots * (\omega^{\zeta_2}+1)}^{n_2} * \\ & \qquad\qquad \vdots \\ & \overbrace{(\omega^{\zeta_{k-1}}+1) * (\omega^{\zeta_{k-1}}+1) * \cdots * (\omega^{\zeta_{k-1}}+1)}^{n_{k-1}} * \\ & \overbrace{2 * 2 * \cdots * 2}^{n_k - 1} \ . \qquad \textit{(Do not confuse } n_{k-1} \textit{ with } n_k - 1 \textit{ .)}\end{aligned}$$

A translation of the theorem into product spaces is as follows :

Corollary 1. *Every compact countable metric space* X *with* $|X| > 1$ *is factorized uniquely as*

$$X \approx X_1 \times X_2 \times \cdots \times X_j$$

into primes $X_1, X_2, \ldots, X_j$. *In further detail, if* $X = MS(\alpha, n)$, $\alpha = \omega^{\zeta_1}n_1 + \omega^{\zeta_2}n_2 + \cdots + \omega^{\zeta_k}n_k$ *with* $\zeta_k = 0$, *then*

$$\begin{aligned}X \approx \ & \overbrace{MS(\omega^{\zeta_1}+1,\,1) \times MS(\omega^{\zeta_1}+1,\,1) \times \cdots \times MS(\omega^{\zeta_1}+1,\,1)}^{n_1} \times \\ & \overbrace{MS(\omega^{\zeta_2}+1,\,1) \times MS(\omega^{\zeta_2}+1,\,1) \times \cdots \times MS(\omega^{\zeta_2}+1,\,1)}^{n_2} \times \\ & \qquad\qquad \vdots \\ & \overbrace{MS(\omega^{\zeta_{k-1}}+1,\,1) \times MS(\omega^{\zeta_{k-1}}+1,\,1) \times \cdots \times MS(\omega^{\zeta_{k-1}}+1,\,1)}^{n_{k-1}} \times \\ & \overbrace{MS(2,\,1) \times MS(2,\,1) \times \cdots \times MS(2,\,1)}^{n_k - 1} \times \\ & MS(1,\,p_1) \times MS(1,\,p_2) \times \cdots \times MS(1,\,p_r) \quad ,\end{aligned}$$

where $n = p_1p_2\cdots p_r$ *is the usual factorization of the natural number* n *into primes.*

Examples. Put, for example, $\alpha = \omega^2 + \omega 2 + 3$. Then

$$\omega^2 + \omega 2 + 3 = (\omega^2+1) * (\omega+1) * (\omega+1) * 2 * 2 \ .$$

Thus

$$\begin{aligned}MS(\omega^2 + \omega 2 + 3,\,1) = \ & MS(\omega^2+1,\,1) \times MS(\omega+1,\,1) \times MS(\omega+1,\,1) \times \\ & MS(2,\,1) \times MS(2,\,1) \ .\end{aligned}$$

$$\begin{aligned} MS(\omega^2+\omega 2+3,\, 6) = \;& MS(\omega^2+1,\, 1) \times MS(\omega+1,\, 1) \times MS(\omega+1,\, 1)\times \\ & MS(2,\, 1) \times MS(2,\, 1) \times MS(1,\, 3) \times MS(1,\, 2)\ . \end{aligned}$$

References

[1] K. Borsuk, *Sur la décomposition des polyèdres en produits cartésiens*, Fund. Math. **31** (1938), 137-148.

[2] K. Borsuk, *On the decomposition of a locally connected compactum into Cartesian product of a curve and a manifold*, Fund. Math. **40** (1953), 140-159.

[3] R. Telgársky, *Derivatives of Cartesian product and dispersed spaces*, Colloq. Math. **19** (1968), 59-66.