<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>Title</td>
<td>On Telgarski's formula (Research trends on general and geometric topology and their problems)</td>
</tr>
<tr>
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<td>Oka, Shinpei</td>
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On Telgárski’s formula

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The following formula due to R. Telgársky ([3]) is indispensable in dealing with products of scattered spaces.

**Theorem 1.** Let $X$, $Y$ be scattered spaces with $\text{leng}(X) = \alpha$ and $\text{leng}(Y) = \beta$. Then

1. $(X \times Y)^{(\sigma)} = \bigcup_{\tau \oplus v = \sigma} X^{(\tau)} \times Y^{(v)}$ for every $\sigma$.
2. $(X \times Y)^{(\sigma)} = \bigcup_{\tau \oplus v = \sigma} X^{(\tau)} \times Y^{(v)}$ for every $\sigma$.
3. $\text{leng}(X \times Y) = \sup \{\tau \oplus v + 1 | \tau < \alpha \text{ and } v < \beta\}$.

The symbol $\oplus$ means Hessenberg’s sum defined as follows:

**Definition 1.** Let $\alpha > 0$, $\beta > 0$ be ordinals. Using Cantor’s normal form, represent $\alpha$, $\beta$ uniquely as

$$\alpha = \omega^{\gamma_1}n_1 + \omega^{\gamma_2}n_2 + \cdots + \omega^{\gamma_k}n_k, \quad \beta = \omega^{\gamma_1}m_1 + \omega^{\gamma_2}m_2 + \cdots + \omega^{\gamma_k}m_k,$$

$\gamma_1 > \gamma_2 > \cdots > \gamma_k$, $0 \leq n_i < \omega$, $0 \leq m_i < \omega$, so that $n_i = 0 = m_i$ does not occur. Define

$$\alpha \oplus \beta = \Sigma_{i=1}^{k} \omega^{\gamma_i}(n_i + m_i) .$$

Also define $\alpha \oplus 0 = 0 \oplus \alpha = 0$ for every $\alpha$.

Hessenberg’s sum is certainly convenient for describing the derivatives $(X \times Y)^{(\sigma)}$ but not for describing $\text{leng}(X \times Y)$. With an emphasis on the length of product spaces, we define a binary operation $\pi$ as follows:

**Definition 2.** Let $\alpha > 0$, $\beta > 0$ be ordinals. Represent $\alpha$, $\beta$ as in Definition 1. Put

$$l = \min\{\max\{i | n_i \neq 0\}, \max\{j | m_j \neq 0\}\}$$

and define

$$\pi(\alpha, \beta) = \begin{cases} \Sigma_{i=1}^{l} \omega^{\gamma_i}(n_i + m_i) & \text{if } l < k \\ (\Sigma_{i=1}^{k-1} \omega^{\gamma_i}(n_i + m_i)) + \omega^{\gamma_k}(n_k + m_k - 1) & \text{if } l = k \end{cases} ,$$

where $l = k$ is, of course, equivalent to $n_k \neq 0 \neq m_k$.

For convenience, define $\pi(\alpha, 0) = \pi(0, \alpha) = 0$ for every ordinal $\alpha$.

It is to be noted that, unlike Hessenberg’s sum, the operation $\pi$ is a continuous operation with respect to the order topology.

Now we can restate Telgársky’s formula as follows:
Theorem 2. Let \( X, Y \) be scattered spaces with \( \text{leng}(X) = \alpha \), \( \text{leng}(Y) = \beta \). Then

1. \( \text{leng}(X \times Y) = \pi(\alpha, \beta) \).
2. \( (X \times Y)^{[\sigma]} = \cup \{X_{(\tau)} \times Y_{(\nu)} \mid \pi(\tau + 1, \nu + 1) = \sigma + 1\} \) for every ordinal \( \sigma \).
3. \( (X \times Y)^{[\sigma]} = \cup \{X_{(\tau)} \times Y_{(\nu)} \mid \pi(\tau + 1, \nu + 1) = \sigma + 1\} \) for every ordinal \( \sigma \).

We write simply \( \pi(\alpha, \beta) = \alpha \ast \beta \).

Proposition 1. \( \alpha \ast \beta = \beta \ast \alpha \). \( (\alpha \ast \beta) \ast \gamma = \alpha \ast (\beta \ast \gamma) \).

Definition 3. A factorization \( \alpha = \beta \ast \gamma \) of an ordinal \( \alpha \) is called trivial if one of \( \beta \), \( \gamma \) is 1 (and the other is \( \alpha \)). An ordinal \( \alpha \) is called a prime ordinal if \( \alpha > 1 \) and it does not admit a non-trivial factorization.

As far as I know, the following notion was first defined and used by K. Borsuk ([1] also see [2]).

Definition 4. A factorization \( X \approx Y \times Z \) of a space \( X \) is called trivial if one of \( Y \), \( Z \) is a one point space (and the other is homeomorphic to \( X \)). A space \( X \) is called a prime space (or simply a prime) if \( |X| > 1 \) and it does not admit a non-trivial factorization.

By the definition of the operation \( \pi \) we have

Proposition 2. An ordinal \( \alpha \) is a prime ordinal if and only if \( \alpha = \omega^\gamma + 1 \) with \( \gamma \) an ordinal.

The compact countable metric space \( X \) satisfying \( \text{leng}(X) = \alpha \) and \( |X^{(\alpha-1)}| = |X_{(\alpha-1)}| = n \) is denoted by \( MS(\alpha, n) \). By the uniqueness of \( MS(\alpha, n) \) we have \( MS(\alpha, n) \approx MS(\alpha, 1) \times MS(1, n) \) and, by Theorem 2, \( MS(\alpha, 1) \times MS(\beta, 1) \approx MS(\alpha \ast \beta, 1) \).

These facts are combined with Proposition 2 to give

Proposition 3. A compact countable metric space \( X \) is a prime if and only if \( X \approx MS(1, p) \) with \( p \) a prime number or \( X \approx MS(\omega^\gamma + 1, 1) \) with \( \gamma \) an ordinal.

Theorem 3. Every non-limit ordinal \( \alpha > 1 \) is factorized uniquely as

\[ \alpha = \alpha_1 \ast \alpha_2 \ast \cdots \ast \alpha_j \]

into prime ordinals \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_j \). In full detail, if the normal form of \( \alpha \) is

\[ \alpha = \omega^{\xi_1}n_1 + \omega^{\xi_2}n_2 + \cdots + \omega^{\xi_k}n_k \]
with $\zeta_k = 0$ then

$$
\alpha = \frac{n_1}{(\omega^{\zeta_1} + 1) \times (\omega^{\zeta_1} + 1) \times \cdots \times (\omega^{\zeta_1} + 1)} \times \\
\frac{n_2}{(\omega^{\zeta_2} + 1) \times (\omega^{\zeta_2} + 1) \times \cdots \times (\omega^{\zeta_2} + 1)} \times \\
\vdots \\
\frac{n_{k-1}}{(\omega^{\zeta_{k-1}} + 1) \times (\omega^{\zeta_{k-1}} + 1) \times \cdots \times (\omega^{\zeta_{k-1}} + 1)} \times \\
\frac{2 \times 2 \times \cdots \times 2}{(Do \ not \ confuse \ n_{k-1} \ with \ n_{k-1} \ .)}
$$

A translation of the theorem into product spaces is as follows:

**Corollary 1.** Every compact countable metric space $X$ with $|X| > 1$ is factorized uniquely as

$$
X \approx X_1 \times X_2 \times \cdots \times X_j
$$

into primes $X_1, X_2, \ldots, X_j$. In further detail, if $X = MS(\alpha, n)$, $\alpha = \omega^{\zeta_1}n_1 + \omega^{\zeta_2}n_2 + \cdots + \omega^{\zeta_k}n_k$ with $\zeta_k = 0$, then

$$
X \approx \frac{n_1}{MS(\omega^{\zeta_1} + 1, 1) \times MS(\omega^{\zeta_1} + 1, 1) \times \cdots \times MS(\omega^{\zeta_1} + 1, 1)} \times \\
\frac{n_2}{MS(\omega^{\zeta_2} + 1, 1) \times MS(\omega^{\zeta_2} + 1, 1) \times \cdots \times MS(\omega^{\zeta_2} + 1, 1)} \times \\
\vdots \\
\frac{n_{k-1}}{MS(\omega^{\zeta_{k-1}} + 1, 1) \times MS(\omega^{\zeta_{k-1}} + 1, 1) \times \cdots \times MS(\omega^{\zeta_{k-1}} + 1, 1)} \times \\
\frac{2 \times 2 \times \cdots \times 2}{MS(2, 1) \times MS(2, 1) \times \cdots \times MS(2, 1)} \\
\times \frac{MS(1, p_1) \times MS(1, p_2) \times \cdots \times MS(1, p_r)}{
$$

where $n = p_1p_2 \cdots p_r$ is the usual factorization of the natural number $n$ into primes.

**Examples.** Put, for example, $\alpha = \omega^2 + \omega 2 + 3$. Then

$$
\omega^2 + \omega 2 + 3 = (\omega^2 + 1) \times (\omega + 1) \times (\omega + 1) \times 2 \times 2
$$

Thus

$$
MS(\omega^2 + \omega 2 + 3, 1) = MS(\omega^2 + 1, 1) \times MS(\omega + 1, 1) \times MS(\omega + 1, 1) \times \\
MS(2, 1) \times MS(2, 1)
$$
\[ MS(\omega^2 + \omega + 2 + 3, 6) = MS(\omega^2 + 1, 1) \times MS(\omega + 1, 1) \times MS(\omega + 1, 1) \times MS(2, 1) \times MS(2, 1) \times MS(1, 3) \times MS(1, 2). \]

References

