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A uniform way of formulating various results concerning l-equivalence

Jan Spěvák

If V is a topological vector space over $\mathbb{R}$, by the symbol $V^\star$ we denote its weak dual, i.e. the topological vector space of all real-valued continuous linear functionals endowed with the topology of point-wise convergence. Having a Tychonoff space X we denote by $C_p(X)$ the set of all real-valued functions on X endowed with the topology of point-wise convergence. It is well known that $C_p(X)$ forms (together with the point-wise addition and scalar multiplication) a topological vector space. By $L_p(X)$ we denote the topological vector space $C_p(X)^\star$. We say that $B \subset L_p(X)$ is a *base* of $L_p(X)$ provided that for every $a \in L_p(X)$ there exists unique $n \in \mathbb{N} \cup \{0\}$, $\{b_1, \ldots, b_n\} \subset B$ and $\{r_1, \ldots, r_n\} \in \mathbb{R}$ such that $a = \sum_{i=1}^{n} r_i b_i$. If moreover B has the property that every $f \in C_p(B)$ can be extended (uniquely) to a continuous linear functional on $L_p(X)$, we call B an *l-base* of $L_p(X)$.

It is well known (see [1]) that X is always embedded in $L_p(X)$ as its l-base by the canonical homeomorphism ψ, where $\psi(x)(f) = f(x)$ for $x \in X$ and $f \in C_p(X)$. It follows easily that $C_p(X) = L_p(X)^\star$.

Recall that Tychonoff spaces X, Y are l-equivalent ($X \overset{l}{\sim} Y$) provided that $C_p(X)$ and $C_p(Y)$ are isomorphic as topological vector spaces. From what was written so far it follows that $X \overset{l}{\sim} Y$ if and only if $L_p(X)$ and $L_p(Y)$ are isomorphic as topological vector spaces. In other words, if and only if Y is an l-base of $L_p(X)$ (and vice versa).

We say that a topological property $\mathcal{T}$ is preserved by l-equivalence provided that if $X \overset{l}{\sim} Y$ and X has the property $\mathcal{T}$ then so does Y.

In 1982 it was proved by Arkhangelskii that pseudocompactness and compactness are preserved by l-equivalence ([2]). This result was reproved by Uspenskii in [5]. The same year Pestov proved that the Čech-Lebesgue (covering) dimension is preserved by l-equivalence. In 1998 Velichko showed that the Lindelöf property is also preserved by l-equivalence ([6]) and this result was generalized in 2000 by Bouziad for arbitrary Lindelöf degree ([3]).

The proofs of the above mentioned results are nontrivial and they use different tools. In this short remark we present kind of a uniform reformulation of the above results by introducing the notion of B-cover. Before doing so, recall that for $f \in L_p(X)^\star$ its kernel $\ker f$ is defined by $\ker f = \{x \in L_p(X) : f(x) = 0\}$.

Definition 1. Given a Tychonoff space X and $B \subset L_p(X)$ we say that $\mathcal{F} \subset L_p(X)^\star$ is a *B-cover* provided that $\bigcap_{f\in\mathcal{F}} \ker f \cap B = \emptyset$. Having B-covers $\mathcal{F}, \mathcal{G}$ we say that $\mathcal{G}$ is a *B-subcover of $\mathcal{F}$* provided that $\mathcal{G} \subset \mathcal{F}$, and we say that $\mathcal{G}$ is *B-refinement of $\mathcal{F}$* if for every $g \in \mathcal{G}$ there exists $f \in \mathcal{F}$ such that $(\ker f \cap B) \subset (\ker g \cap B)$. Finally, we say that a B-cover $\mathcal{F}$ *has order less or equal to n* if for every $b \in B$ it holds that $|\{f \in \mathcal{F} : b \notin \ker f\}| \leq n$.

Since the proofs of the following four theorems proceed in the same way, we will present only the proof of the first theorem. Other follow easily from what was written so far and from the observation that if $\mathcal{F} \subset L_p(X)^\star (= C_p(X))$ is a B-cover, then the set $\mathcal{U}_\mathcal{F} = \{B \setminus \ker f : f \in \mathcal{F}\}$ is a functionally open cover of B. In the case of pseudocompactness one has to use its characterization saying that X is pseudocompact if and only if every countable functionally open cover of X has a finite subcover.

Theorem 1. *For a Tychonoff space X the following statements are equivalent:*

1. $\dim X \leq n$
2. *There exists an l-base $B \subset L_p(X)$ such that every finite B-cover has a finite B-refinement of order less or equal to n.*
3. *For every l-base $B \subset L_p(X)$ it holds that every finite B-cover has a finite B-refinement of order less or equal to n.*
4. *For every l-base $B \subset L_p(X)$ it holds that* $\dim B \leq n$.

Proof. (1) $\Leftrightarrow$ (2) Since X is an l-base, we can put $B = X$. Observe that if $\mathcal{F} \subset L_p(X)^\star (= C_p(X))$ is an X-cover, then the set $\mathcal{U}_\mathcal{F} = \{X \setminus \ker f : f \in \mathcal{F}\}$ is a functionally open cover of X. Conversely if $\mathcal{U}$ is a functionally open cover of X then there is some X-cover $\mathcal{F} \subset L_p(X)^\star$ with $\mathcal{U}_\mathcal{F} = \mathcal{U}$. Moreover, $\mathcal{G}$ is an X-refinement of $\mathcal{F}$ if and only if $\mathcal{U}_\mathcal{G}$ is a refinement of $\mathcal{U}_\mathcal{F}$. Now the equivalence follows from the definition of covering dimension and from the fact that the order of $\mathcal{F}$ is less or equal to n if and only if the order of $\mathcal{U}_\mathcal{F}$ is less or equal to n.

(2) $\Rightarrow$ (3) This is the only nontrivial implication which follows from the same argumentation as in the proof of equivalence (1) $\Leftrightarrow$ (2), from the fact that B is an l-base of $L_p(X)$ iff $B \overset{l}{\sim} X$ and from the fact that covering dimension is preserved by l-equivalence.

The proof of (3) $\Leftrightarrow$ (4) is similar to (1) $\Leftrightarrow$ (2) and the implication (4) $\Rightarrow$ (1) is obvious. $\square$

Recall that the Lindelöf degree $l(X)$ of a space X is defined by $l(X) = \min\{\kappa : \text{every open cover of } X \text{ has a subcover of cardinality} \leq \kappa\} + \omega$

Theorem 2. *For a Tychonoff space X the following statements are equivalent:*

1. $l(X) \leq \kappa$
2. *There exists an l-base $B \subset L_p(X)$ such that every B-cover has a B-subcover of cardinality $\leq \kappa$.*
3. *For every l-base $B \subset L_p(X)$ it holds that every B-cover has a B-subcover of cardinality $\leq \kappa$.*
4. *Every l-base $B \subset L_p(X)$ satisfies $l(B) \leq \kappa$.*

Theorem 3. *For a Tychonoff space X the following statements are equivalent:*

1. *X is pseudocompact.*
2. *There exists an l-base $B \subset L_p(X)$ such that every countable B-cover has a finite B-subcover.*
3. *For every l-base $B \subset L_p(X)$ it holds that every countable B-cover has a finite B-subcover.*
4. *Every l-base of $L_p(X)$ is pseudocompact.*

Theorem 4. *For a Tychonoff space X the following statements are equivalent:*

1. *X is compact.*
2. *There exists an l-base $B \subset L_p(X)$ such that every B-cover has a finite B-subcover.*
3. *For every l-base $B \subset L_p(X)$ it holds that every B-cover has a finite B-subcover.*
4. *Every l-base of $L_p(X)$ is compact.*

In the above theorems we have expressed some topological properties of a Tychonoff space X only by means of $L_p(X)$ (and its dual $C_p(X)$). However, to do so, we needed essentially some deep results about preservation of those properties by l-equivalence. Our way of formulation gives us the hope that there may exists a uniform approach to prove the theorems directly.

References

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