数理解析研究所講究録
第 1634 巻 2009 年 41-44

Infinite-dimensions and rings of continuous functions

Chieko Komoda

School of Health & Sports Science, Juntendo University, Inba, Chiba, 270-1695, Japan

This is a joint study with Takashi Kimura.

We assume that all spaces are normal and all rings are commutative unless otherwise stated. We refer the readers to [3] for dimension theory.

Let aR denote the principal right ideal generated in a ring R by an element a.

Canfell defined the dimension of a ring as follows.

Definition 1([2]). A set of principal ideals a_iR , $i=1,\ldots,n$, is *uniquely generated* if whenever $a_iR=b_iR$, $i=1,\ldots,n$, there exist elements u_i of R such that $a_i=b_iu_i$ $i=1,\ldots,n$, and $\sum_{i=1}^{n}u_iR=R$. The *dimension* of R — denoted by $\dim R$ — is the least integar n such that every set of $n+1$ principal ideals is uniquely generated.

Let $C(X)$ be the ring of real-valued continuous functions defined on a space X. For $f\in C(X)$, the zero set $Z(f)$ of f, is defined by $Z(f)=\{x\in X: f(x)=0\}$. Clearly $\sum_{i=1}^{n}f_iC(X)=C(X)$ if and only if $\bigcap_{i=1}^{n}Z(f_i)=\emptyset$.

Canfell [2] proved the following theorem.

Theorem 1 (Canfell [2]). *For every space X we have* $\dim X = \dim C(X)$.

We consider transfinite extensions of the dimension of a ring R.

Borst defined the transfinite dimension of a space X as follows.

Definition 2(cf. [3]). For every set L we denote by Fin$L$ the collection of all non-empty finite subsets of L and for every $\sigma \in \mathrm{Fin}L$ and $M \subset \mathrm{Fin}L$ let

$$M^\sigma = \{\tau \in \mathrm{Fin}L : \sigma \cup \tau \in M \text{ and } \sigma \cap \tau = \emptyset\}.$$

If $\sigma = \{a\}$, we write M^a instead of $M^{\{a\}}$.

For every subcollection M of Fin$L$ the *large order* Ord$M$ of M, which is an ordinal number or the "infinite number" ∞, is defined by the following conditions:

(O1) $\mathrm{Ord}M = 0$ if and only if $M = \emptyset$;

(O2) $\mathrm{Ord}M \leq \alpha$, where α is an ordinal number > 0, if $\mathrm{Ord}M^a < \alpha$ for every $a \in L$;

(O3) $\mathrm{Ord}M = \alpha$ if $\mathrm{Ord}M \leq \alpha$ and the inequality $\mathrm{Ord}M \leq \beta$ holds for no $\beta < \alpha$;

(O4) $\mathrm{Ord}M = \infty$ if $\mathrm{Ord}M \leq \alpha$ holds for no ordinal number α.

Let Γ be an index set. A collection $\tau = \{(A_i, B_i) : i \in \Gamma\}$ of pairs of disjoint closed subsets of X is called *essential* if for every $\{L_i : i \in \Gamma\}$, where L_i is a partition in X between A_i and B_i for every $i \in \Gamma$, we have $\bigcap_{i\in\Gamma} L_i \neq \emptyset$; if τ is not essential then it is called *inessential.*

Definition 3(cf. [3]). For a space X we denote by L the set of all pairs (A, B) of disjoint closed subsets of X. Let us set $M = \{\sigma \in \mathrm{Fin}L : \sigma \text{ is essential}\}$.

The number Ord$M$ is denoted by trdim$X$ and called the *transfinite covering dimension* of a space X.

Arenas defined transfinite extensions of the dimension of a ring R.

Definition 4([1]). For a ring R we denote by L the set of principal ideal aR. Let us set $M = \{\sigma \in \mathrm{Fin}L : \sigma \text{ is not uniquely generated}\}$.

The number Ord$M$ is denoted by trdim$R$ and called the *transfinite dimension* of a ring R.

Arenas [1] proved the following theorem, which is a partial transfinite generalization of Canfell's theorm.

Theorem 2 (Arenas [1]). *For every space X we have* trdim$X \leq$ trdim$C(X)$.

Arenas asked whether the equality in theorem 2 is true.

We gave a negative answer.

Theorem 3. *For every metric space X with* trdim$X \geq \omega$ *we have* trdim$C(X) = \infty$.

We redefine Arenas' definition as follows and prove the transfinite dimensions of the space X to be equal to the transfinite dimensions of the ring $C(X)$.

Definition 5. Let Γ be an index set. A collection $\sigma = \{(a_i, b_i) : i \in \Gamma\}$ of pairs of principal ideals of R is called *essential* if for every $\{u_i : i \in \Gamma\}$, where $u_i \in R$ and $a_i = b_i u_i$ for every $i \in \Gamma$, we have $\sum_{i \in \Gamma} u_i R \neq R$; if σ is not essential then it is called *inessential*.

For a ring R we denote by L the set of all pairs (a, b) of principal ideals of R with $aR = bR$. Let us set $M = \{\sigma \in \text{Fin}L : \sigma \text{ is essential}\}$.

The number Ord$M$ is denoted by trdim$R$ and called the *transfinite dimension* of a ring R.

Theorem 4 . *For every space X we have* trdim$X =$ trdim$C(X)$.

References

[1] F. G. Arenas, *Transfinite dimension and rings of continuous functions*, Questions and Answers in General Topology 13(1995), 157-167.

[2] M. J. Canfell, *Uniqueness of generators of principal ideals in rings of continuous functions*, Proc. Amer. Math. Soc. 26(1970), 571-573.

[3] R. Engelking, Theory of Dimensions, Finite and Infinite, Heldermann Verlag, 1995.

Chieko Komoda

Department of Health Science, School of Health & Sports Science, Juntendo University

Inba, Chiba, 270-1695, Japan

E-mail address: chieko_komoda@sakura.juntendo.ac.jp