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FIBREWISE COMPACTNESS AND UNIFORMITIES

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1. INTRODUCTION

In this report, firstly, we consider the relationship between fibrewise compactness and fibrewise uniformities. For this, I. M. James obtained the following result in [4].

Proposition 17.1([4]). Let X be a fibrewise compact and fibrewise regular space over B, with B regular. Then there exists a unique fibrewise uniform structure Ω on X, compatible with the fibrewise topology, in which the members of Ω are the nbds of the diagonal.

This proposition is false in a strict sense of the definition of "fibrewise uniform structure". We relieve this proposition by using the notion "fibrewise entourage uniformity" in [6].

Secondary, in section 3, we introduce a new notion of fibrewise quasi-uniform spaces which is a common extended one of both fibrewise uniform spaces ([6]) and quasi-uniform spaces ([2]). Further we investigate the fibrewise quasi-uniformaziblity of fibrewise spaces.

Throughout this report, we use the following notation and terminology.

For a set X, a function $p : X \to B$, $W \subset B$ and $b \in B$, $p^{-1}(W) = X_W$, $p^{-1}(b) = X_b$, $X_W \times X_W = X_W^2$ and $X \times X = X^2$. For $D, E \subset X^2$, $D \circ E = \{(x,z) | \exists y \in X$ such that $(x,y) \in D, (y,z) \in E\}$, $D^{-1} = \{(y,x) | (x,y) \in D\}$ and $D[x] = \{y | (x,y) \in D\}$. For a quasi-uniformity $\mathcal{U}$ on X, let $\mathcal{U}^{-1} = \{U^{-1} | U \in \mathcal{U}\}$, and $\mathcal{U}^*$ be the fibrewise quasi-uniformity generated by $\{U \cap U^{-1} | U \in \mathcal{U}\}$. For a (fibrewise) quasi-uniform space $(X, \mathcal{U})$, $\tau(\mathcal{U})$, $\tau(\mathcal{U}^{-1})$ and $\tau(\mathcal{U}^*)$ are (fibrewise) topologies induced by $\mathcal{U}$, $\mathcal{U}^{-1}$ and $\mathcal{U}^*$, respectively.

Let B be a fixed topological space (as the base space) with a topology τ. We will use the abbreviation *nbd(s)* for *neighborhood(s)*. For $b \in B$, $N(b)$ is the family of all open nbds of b.

For other terminology and definitions in the topological category TOP and the fibrewise category TOP_B, one can consult [1] and [4], respectively, and for quasi-uniform spaces, see [2].

This report is a part of our paper [3].

2. Fibrewise compactness and uniformities

In this section, we discuss the difference of fibrewise uniformities of [4] and [6], and show that the assertion of Proposition 17.1 in [4] is false in the strict sense of definition of [4], and relieve it from difficulty by using the notion of "fibrewise entourage uniformity" in [6].

First, we begin with the definition of fibrewise uniform structure.

Definition 2.1. Let X be a fibrewise set over B. By a *fibrewise uniform structure* on X we mean a filter Ω on X^2 satisfying three conditions, as follows.

(FU1) Each $D \in \Omega$ contains the diagonal Δ of X.

(FU2) For any $D \in \Omega$ and $b \in B$, there exist $W \in N(b)$ and $E \in \Omega$ such that $X^2_W \cap E \subset D^{-1}$.

(FU3) For any $D \in \Omega$ and $b \in B$, there exist $W \in N(b)$ and $E \in \Omega$ such that $(X^2_W \cap E) \circ (X^2_W \cap E) \subset D$.

In the fibrewise compact spaces, I.M.James (in [4]) obtained Proposition 17.1 (see Section 1). But this proposition is false in a strict sense of the definition of "fibrewise uniform structure". In fact, we can construct the following examples.

Example 2.2. Let $X = B$ be the set of all positive real numbers with the usual topology and $p : X \to B$ be the identity map. Then X is a fibrewise compact and fibrewise regular space over B. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two families of X^2 constructed as follows:

$$\mathcal{B}_1 = \{U_\epsilon | U_\epsilon = \{(x,y) | x - \epsilon < y < x + \epsilon\},\ \epsilon > 0\},$$
$$\mathcal{B}_2 = \{U_{\epsilon,a} | U_{\epsilon,a} = \{(x,y) | x - \epsilon < y < \sqrt{x^2 + a}\},\ \epsilon > 0, a > 0\}.$$

Let Ω_1 and Ω_2 be the filters on X^2 generated by $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and let Ω be the filter on X^2 which contains all nbds of the diagonal. Then it is easy to see that Ω_1, Ω_2 and Ω are different each other. □

On the other hand, we introduced a notion of slightly stronger fibrewise uniformity (called by *fibrewise entourage uniformity*) in [6] in order to discuss the relationship between the fibrewise uniformities by using entourages and coverings. This notion of fibrewise entourage uniformity seems to relieve the difficulty in the above.

Definition 2.3. ([6]) Let X be a fibrewise set over B. By a *fibrewise entourage uniformity* on X we mean a filter Ω on X^2 satisfying four conditions: (FU1), (FU2) and (FU3) in Definition 2.1, and

(FU4) If $D \subset X^2$ satisfies that for each $b \in B$, there exist $W \in N(b)$ and $E \in \Omega$ such that $X^2_W \cap E \subset D$, then $D \in \Omega$.

We call X with Ω a *fibrewise entourage uniform space*, and denoted by (X, Ω).

It is easily verified that, in Example 2.2, Ω_1 and Ω_2 are fibrewise uniform structures but not fibrewise entourage uniformities on X, and Ω is a fibrewise entourage uniformity on X.

To relieve Proposition 17.1 ([4]), we shall introduce some notions.

For a fibrewise entourage uniformity Ω on X, a subfamily $\mathcal{B}$ of Ω is said to be a *fibrewise uniform base* (briefly say, *fibrewise u-base*) if $\mathcal{B}$ is a filter-base and satisfies the conditions (FU1), (FU2), (FU3) and the following:

For each $D \in \Omega$ and $b \in B$, there exist $W \in N(b)$ and $E \in \mathcal{B}$ such that $X^2_W \cap E \subset D$.

A subfamily $\mathcal{S}$ of Ω is said to be a *fibrewise uniform subbase* (briefly say, *fibrewise u-subbase*) if $\mathcal{S}$ is a filter-base and the family of all finite intersections of members of $\mathcal{S}$ is a fibrewise u-base of Ω.

A family $\mathcal{G}$ of subsets of X^2 is said to be a *fibrewise uniform germ* (briefly say, *fibrewise u-germ*) if $\mathcal{G}$ is a filter-base and satisfies the conditions (FU1), (FU2) and (FU3). A family $\mathcal{S}$ of subsets of X^2 is said to be a *fibrewise uniform subgerm* (briefly say, *fibrewise u-subgerm*) if $\mathcal{S}$ is a filter-base and the family of all finite intersections of members of $\mathcal{S}$ is a fibrewise u-germ.

It is clear that, for a fibrewise u-germ $\mathcal{G}$, the family

$$\Omega = \{D | \forall b \in B, \exists W \in N(b), \exists E \in \mathcal{G} \text{ such that } X^2_W \cap E \subset D\}$$

is a fibrewise entourage uniformity on X. Then it is clear that $\mathcal{G}$ is a fibrewise u-base of Ω. (Ω is said to be the fibrewise entourage uniformity *generated* by $\mathcal{G}$).

In Example 2.2, Ω_1 and Ω_2 are fibrewise u-germs and the fibrewise entourage uniformities generated by Ω_1 and Ω_2 are equal to the fibrewise entourage uniformity Ω.

We can relieve Proposition 17.1 and Corollary 17.2 ([4]) as the following forms. The fibrewise uniform topology is the fibrewise topology induced by the (entourage) uniformity (cf. [4] Section 13 and [6] Section 3). Proofs of theorems are almost all same as those in [4].

Theorem 2.4. Let X be a fibrewise compact and fibrewise regular space over B, with B regular. Then there exists a unique fibrewise entourage uniformity Ω on X, compatible with the fibrewise topology, in which the members of Ω are the nbds of the diagonal.

Theorem 2.5. Let $f : X \to Y$ be a fibrewise function, where X and Y are fibrewise entourage uniform space over B, with B regular. Suppose that X is fibrewise compact over B in the fibrewise uniform topology. If f is continuous, in the fibrewise uniform topology, then f is fibrewise uniformly continuous.

3. Fibrewise quasi-uniformities

In this section, we define a new notion of fibrewise quasi-uniform spaces, and study fibrewise quasi-uniformizability of fibrewise spaces.

Definition 3.1. Let X be a fibrewise set over B. By a *fibrewise quasi-uniformity* on X, we mean a filter $\mathcal{U}$ on X^2 satisfying the conditions (FU1), (FU3) and (FU4) in Definitions 2.1 and 2.3.

By a *fibrewise quasi-uniform space* $(X,\mathcal{U})$ we mean a fibrewise set X with a fibrewise quasi-uniformity $\mathcal{U}$.

Fibrewise quasi-uniform spaces over a point can be regarded as a quasi-uniform spaces in the ordinary sense. If $\mathcal{U}$ is a fibrewise quasi-uniformity, then $\mathcal{U}^{-1}$ is also a fibrewise quasi-uniformity and is called the *conjugate* of $\mathcal{U}$.

Further, note that our definition of fibrewise quasi-uniformity is an extended version of a fibrewise entourage uniformity (Definition 2.3), and is not an extended one of fibrewise uniform structure (Definition 2.1).

It is easily verified that for a fibrewise quasi-uniformity $\mathcal{U}$ on X the filter $\mathcal{U}^*$ is a fibrewise entourage uniformity on X.

For a fibrewise quasi-uniformity $\mathcal{U}$ on X, a subfamily $\mathcal{B}$ of $\mathcal{U}$ is said to be a *fibrewise quasi-uniform base* (briefly say, *fibrewise qu-base*) if $\mathcal{B}$ is a filter-base and satisfies the conditions (FU1), (FU3) and the following:

For each $U \in \mathcal{U}$ and $b \in B$, there exist $W \in N(b)$ and $V \in \mathcal{B}$ such that $X_W^2 \cap V \subset U$.

A subfamily $\mathcal{S}$ of $\mathcal{U}$ is said to be a *fibrewise quasi-uniform subbase* (briefly say, *fibrewise qu-subbase*) if $\mathcal{S}$ is a filter-base and the family of all finite intersections of members of $\mathcal{S}$ is a fibrewise qu-base of $\mathcal{U}$.

A family $\mathcal{G}$ of subsets of X^2 is said to be a *fibrewise quasi-uniform germ* (briefly say, *fibrewise qu-germ*) if $\mathcal{G}$ is a filter-base and satisfies the conditions (FU1) and (FU3). A family $\mathcal{S}$ of subsets of X^2 is said to be a *fibrewise quasi-uniform subgerm* (briefly say, *fibrewise qu-subgerm*) if $\mathcal{S}$ is a filter-base and the family of all finite intersections of members of $\mathcal{S}$ is a fibrewise qu-germ.

It is clear that, for a fibrewise qu-germ $\mathcal{G}$, the family

$$\mathcal{U} = \{U \mid \forall b \in B, \exists V \in \mathcal{G} \text{ such that } V \cap X_W^2 \subset U\}$$

is a fibrewise quasi-uniformity on X. Then it is clear that $\mathcal{G}$ is a fibrewise qu-base of $\mathcal{U}$. ($\mathcal{U}$ is said to be the fibrewise quasi-uniformity *generated* by $\mathcal{G}$).

Now, we prove that every fibrewise space is fibrewise quasi-uniformizable; that is, there exists a fibrewise quasi-uniformity $\mathcal{U}$ on X such that $\tau(\mathcal{U}) = \tau_X$. This idea is an analogous one of Pervin quasi-uniformity [2]. Further, we refer to the definition of "quasi-uniform space over B" in Park and Lee [7].

Let X be a set. For every subset A of X, let

$$S(A) := A \times A \cup (X - A) \times X.$$

Theorem 3.2. Let (X, τ_X) be a fibrewise space over B. Then $\mathcal{S} = \{S(A) \mid A \in \tau_X\}$ is a fibrewise qu-subgerm for a fibrewise quasi-uniformity on X compatible with τ_X.

Proof. For each $A \in \tau_X$, it is clear that $\Delta \subset S(A)$, and we can easily show that $S(A) \circ S(A) = S(A)$. Thus $\mathcal{S}$ is a fibrewise qu-subgerm for a fibrewise quasi-uniformity on X.

Let $\tau(\mathcal{U})$ be the topology defined by the fibrewise quasi-uniformity $\mathcal{U}$ which is generated by the qu-subgerm $\mathcal{S}$. Now we shall show that $\tau(\mathcal{U}) = \tau_X$.

Let $O \in \tau_X$ and $x \in O$. Then $x \in S(O)[x] = O$. Thus $O \in \tau(\mathcal{U})$.

Conversely, let $O \in \tau(\mathcal{U})$ and $x \in O$. Then there exist $W \in N(p(x))$ and $O_1, \cdots, O_n \in \tau$ such that $x \in \bigcap_{i=1}^n S(O_i)[x] \cap X_W \subset O$. In fact, If $x \notin \bigcup_{i=1}^n O_i$, then $X = \bigcap_{i=1}^n S(O_i)[x] \subset U[x]$. Therefore $U[x] = X \in \tau_X$. If $x \in \bigcup_{i=1}^n O_i$, then $\bigcap_{i=1}^n S(O_i)[x] = \bigcap_{i=1}^n \{O_i | x \in O_i\}$ is a τ-open set and X_W is also τ-open. Thus $\bigcap_{i=1}^n S(O_i)[x] \cap X_W$ is a τ-open set. Hence $O \in \tau$. □

We call the fibrewise quasi-uniformity constructed in this theorem *fibrewise Pervin quasi-uniformity.*

Last, we shall note the definition of "quasi-uniform space over B" in Park and Lee [7]. Their definition is as follow: By a *quasi-uniform space X over B* they mean a function $p : X \to B$ in which both of X and B are quasi-uniform spaces and p is a quasi-uniformly continuous map. This definifion is a generalization of I.M.James' in [5]. In [5], he studied $p : X \to B$ in the situation that both of X and B are uniform spaces and p is a uniformly continuous map. On the other hand, our definition of fibrewise quasi-uniformity in this section is a generalization along Konami and Miwa in [6] (and James' in [4] as an idea).

In connection with the Pervin quasi-uniformity [2], the following proposition was obtained in [2].

Proposition 2.17 ([2]). For every continuous map $f : (X, \tau_X) \to (B, \tau_B)$, let $\mathcal{U}$ and $\mathcal{V}$ be the Pervin quasi-uniformities on X and B respectively, then $f : (X, \mathcal{U}) \to (B, \mathcal{V})$ is quasi-uniformly continuous.

If we consider this proposition, we can say that every fibrewise space X over B can be considered as "quasi-uniform space X over B" (in [7]) if we introduce the Pervin quasi-uniformities to X and B.

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