FIBREWISE COMPACTNESS AND UNIFORMITIES

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1. INTRODUCTION

In this report, firstly, we consider the relationship between fibrewise compactness and fibrewise uniformities. For this, I. M. James obtained the following result in [4].

Proposition 17.1([4]). Let $X$ be a fibrewise compact and fibrewise regular space over $B$, with $B$ regular. Then there exists a unique fibrewise uniform structure $\Omega$ on $X$, compatible with the fibrewise topology, in which the members of $\Omega$ are the nbds of the diagonal. This proposition is false in a strict sense of the definition of “fibrewise uniform structure”. We relieve this proposition by using the notion “fibrewise entourage uniformity” in [6].

Secondary, in section 3, we introduce a new notion of fibrewise quasi-uniform spaces which is a common extended one of both fibrewise uniform spaces ([6]) and quasi-uniform spaces ([2]). Further we investigate the fibrewise quasi-uniformability of fibrewise spaces.

Throughout this report, we use the following notation and terminology.

For a set $X$, a function $p : X \to B$, $W \subset B$ and $b \in B$, $p^{-1}(W) = X_w$, $p^{-1}(b) = X_b$, $X_w \times X_w = X_w^2$ and $X \times X = X^2$. For $D, E \subset X^2$, $D \circ E = \{(x, z) | \exists y \in X \text{ such that } (x, y) \in D, (y, z) \in E\}$, $D^{-1} = \{(y, x) | (x, y) \in D\}$ and $D[x] = \{y | (x, y) \in D\}$. For a quasi-uniformity $\mathcal{U}$ on $X$, let $\mathcal{U}^{-1} = \{U^{-1} | U \in \mathcal{U}\}$, and $\mathcal{U}^*$ be the fibrewise quasi-uniformity generated by $\{U \cap U^{-1} | U \in \mathcal{U}\}$. For a (fibrewise) quasi-uniform space $(X, \mathcal{U}, \tau(\mathcal{U}), \tau(\mathcal{U}^{-1}))$ and $\tau(\mathcal{U}^*)$ are (fibrewise) topologies induced by $\mathcal{U}, \mathcal{U}^{-1}$ and $\mathcal{U}^*$, respectively.

Let $B$ be a fixed topological space (as the base space) with a topology $\tau$. We will use the abbreviation nbd(s) for neighborhood(s). For $b \in B$, $N(b)$ is the family of all open nbds of $b$.

For other terminology and definitions in the topological category $TOP$ and the fibrewise category $TOP_B$, one can consult [1] and [4], respectively, and for quasi-uniform spaces, see [2].

This report is a part of our paper [3].
2. Fibrewise compactness and uniformities

In this section, we discuss the difference of fibrewise uniformities of [4] and [6], and show that the assertion of Proposition 17.1 in [4] is false in the strict sense of definition of [4], and relieve it from difficulty by using the notion of "fibrewise entourage uniformity" in [6].

First, we begin with the definition of fibrewise uniform structure.

Definition 2.1. Let $X$ be a fibrewise set over $B$. By a fibrewise uniform structure on $X$ we mean a filter $\Omega$ on $X^2$ satisfying three conditions, as follows.

(FU1) Each $D \in \Omega$ contains the diagonal $\Delta$ of $X$.

(FU2) For any $D \in \Omega$ and $b \in B$, there exist $W \in N(b)$ and $E \in \Omega$ such that $X_b^2 \cap E \subset D^{-1}$.

(FU3) For any $D \in \Omega$ and $b \in B$, there exist $W \in N(b)$ and $E \in \Omega$ such that $(X_b^2 \cap E) \circ (X_b^2 \cap E) \subset D$.

In the fibrewise compact spaces, I.M.James (in [4]) obtained Proposition 17.1 (see Section 1). But this proposition is false in a strict sense of the definition of "fibrewise uniform structure". In fact, we can construct the following examples.

Example 2.2. Let $X = B$ be the set of all positive real numbers with the usual topology and $p : X \to B$ be the identity map. Then $X$ is a fibrewise compact and fibrewise regular space over $B$. Let $B_1$ and $B_2$ be two families of $X^2$ constructed as follows:

$$B_1 = \{U_\epsilon|U_\epsilon = \{(x, y)|x - \epsilon < y < x + \epsilon\}, \epsilon > 0\},$$

$$B_2 = \{U_{\epsilon,a}|U_{\epsilon,a} = \{(x, y)|x - \epsilon < y < \sqrt{x^2 + a}\}, \epsilon > 0, a > 0\}.$$

Let $\Omega_1$ and $\Omega_2$ be the filters on $X^2$ generated by $B_1$ and $B_2$, respectively, and let $\Omega$ be the filter on $X^2$ which contains all nbds of the diagonal. Then it is easy to see that $\Omega_1$, $\Omega_2$ and $\Omega$ are different each other.

On the other hand, we introduced a notion of slightly stronger fibrewise uniformity (called by fibrewise entourage uniformity) in [6] in order to discuss the relationship between the fibrewise uniformities by using entourages and coverings. This notion of fibrewise entourage uniformity seems to relieve the difficulty in the above.

Definition 2.3. ([6]) Let $X$ be a fibrewise set over $B$. By a fibrewise entourage uniformity on $X$ we mean a filter $\Omega$ on $X^2$ satisfying four conditions: (FU1), (FU2) and (FU3) in Definition 2.1, and

(FU4) If $D \subset X^2$ satisfies that for each $b \in B$, there exist $W \in N(b)$ and $E \in \Omega$ such that $X_b^2 \cap E \subset D$, then $D \in \Omega$.

We call $X$ with $\Omega$ a fibrewise entourage uniform space, and denoted by $(X, \Omega)$.

It is easily verified that, in Example 2.2, $\Omega_1$ and $\Omega_2$ are fibrewise uniform structures but not fibrewise entourage uniformities on $X$, and $\Omega$ is a fibrewise entourage uniformity on $X$. 
To relieve Proposition 17.1 ([4]), we shall introduce some notions.

For a fibrewise entourage uniformity $\Omega$ on $X$, a subfamily $\mathcal{B}$ of $\Omega$ is said to be a fibrewise uniform base (briefly say, fibrewise u-base) if $\mathcal{B}$ is a filter-base and satisfies the conditions (FU1), (FU2), (FU3) and the following:

For each $D \in \Omega$ and $b \in B$, there exist $W \in N(b)$ and $E \in \mathcal{B}$ such that $X^2_W \cap E \subset D$.

A subfamily $\mathcal{S}$ of $\Omega$ is said to be a fibrewise uniform subbase (briefly say, fibrewise u-subbase) if $\mathcal{S}$ is a filter-base and the family of all finite intersections of members of $\mathcal{S}$ is a fibrewise u-base of $\Omega$.

A family $\mathcal{G}$ of subsets of $X^2$ is said to be a fibrewise uniform germ (briefly say, fibrewise u-germ) if $\mathcal{G}$ is a filter-base and satisfies the conditions (FU1), (FU2) and (FU3). A family $\mathcal{S}$ of subsets of $X^2$ is said to be a fibrewise uniform subgerm (briefly say, fibrewise u-subgerm) if $\mathcal{S}$ is a filter-base and the family of all finite intersections of members of $\mathcal{S}$ is a fibrewise u-germ.

It is clear that, for a fibrewise u-germ $\mathcal{G}$, the family

$$\Omega = \{D | \forall b \in B, \exists W \in N(b), \exists E \in \mathcal{G} \text{ such that } X^2_W \cap E \subset D\}$$

is a fibrewise entourage uniformity on $X$. Then it is clear that $\mathcal{G}$ is a fibrewise u-base of $\Omega$. ($\Omega$ is said to be the fibrewise entourage uniformity generated by $\mathcal{G}$).

In Example 2.2, $\Omega_1$ and $\Omega_2$ are fibrewise u-germs and the fibrewise entourage uniformities generated by $\Omega_1$ and $\Omega_2$ are equal to the fibrewise entourage uniformity $\Omega$.

We can relieve Proposition 17.1 and Corollary 17.2 ([4]) as the following forms. The fibrewise uniform topology is the fibrewise topology induced by the (entourage) uniformity (cf. [4] Section 13 and [6] Section 3). Proofs of theorems are almost all same as those in [4].

**Theorem 2.4.** Let $X$ be a fibrewise compact and fibrewise regular space over $B$, with $B$ regular. Then there exists a unique fibrewise entourage uniformity $\Omega$ on $X$, compatible with the fibrewise topology, in which the members of $\Omega$ are the nbds of the diagonal.

**Theorem 2.5.** Let $f : X \rightarrow Y$ be a fibrewise function, where $X$ and $Y$ are fibrewise entourage uniform space over $B$, with $B$ regular. Suppose that $X$ is fibrewise compact over $B$ in the fibrewise uniform topology. If $f$ is continuous, in the fibrewise uniform topology, then $f$ is fibrewise uniformly continuous.

### 3. Fibrewise quasi-uniformities

In this section, we define a new notion of fibrewise quasi-uniform spaces, and study fibrewise quasi-uniformizability of fibrewise spaces.

**Definition 3.1.** Let $X$ be a fibrewise set over $B$. By a fibrewise quasi-uniformity on $X$, we mean a filter $\mathcal{U}$ on $X^2$ satisfying the conditions (FU1), (FU3) and (FU4) in Definitions 2.1 and 2.3.
By a fibrewise quasi-uniform space $(X, \mathcal{U})$ we mean a fibrewise set $X$ with a fibrewise quasi-uniformity $\mathcal{U}$.

Fibrewise quasi-uniform spaces over a point can be regarded as a quasi-uniform spaces in the ordinary sense. If $\mathcal{U}$ is a fibrewise quasi-uniformity, then $\mathcal{U}^{-1}$ is also a fibrewise quasi-uniformity and is called the conjugate of $\mathcal{U}$.

Further, note that our definition of fibrewise quasi-uniformity is an extended version of a fibrewise entourage uniformity (Definition 2.3), and is not an extended one of fibrewise uniform structure (Definition 2.1).

It is easily verified that for a fibrewise quasi-uniformity $\mathcal{U}$ on $X$ the filter $\mathcal{U}^*$ is a fibrewise entourage uniformity on $X$.

For a fibrewise quasi-uniformity $\mathcal{U}$ on $X$, a subfamily $B$ of $\mathcal{U}$ is said to be a fibrewise quasi-uniform base (briefly say, fibrewise qu-base) if $B$ is a filter-base and satisfies the conditions (FU1), (FU3) and the following:

For each $U \in \mathcal{U}$ and $b \in B$, there exist $W \in N(b)$ and $V \in B$ such that $X^2_W \cap V \subset U$.

A subfamily $\mathcal{S}$ of $\mathcal{U}$ is said to be a fibrewise quasi-uniform subbase (briefly say, fibrewise qu-subbase) if $\mathcal{S}$ is a filter-base and the family of all finite intersections of members of $\mathcal{S}$ is a fibrewise qu-base of $\mathcal{U}$.

A family $\mathcal{G}$ of subsets of $X^2$ is said to be a fibrewise quasi-uniform germ (briefly say, fibrewise qu-germ) if $\mathcal{G}$ is a filter-base and satisfies the conditions (FU1) and (FU3). A family $\mathcal{S}$ of subsets of $X^2$ is said to be a fibrewise quasi-uniform subgerm (briefly say, fibrewise qu-subgerm) if $\mathcal{S}$ is a filter-base and the family of all finite intersections of members of $\mathcal{S}$ is a fibrewise qu-germ.

It is clear that, for a fibrewise qu-germ $\mathcal{G}$, the family

$$\mathcal{U} = \{U|\forall b \in B, \exists V \in \mathcal{G} \text{ such that } V \cap X^2_W \subset U\}$$

is a fibrewise quasi-uniformity on $X$. Then it is clear that $\mathcal{G}$ is a fibrewise qu-base of $\mathcal{U}$. ($\mathcal{U}$ is said to be the fibrewise quasi-uniformity generated by $\mathcal{G}$).

Now, we prove that every fibrewise space is fibrewise quasi-uniformizable; that is, there exists a fibrewise quasi-uniformity $\mathcal{U}$ on $X$ such that $\tau(\mathcal{U}) = \tau_X$. This idea is an analogous one of Pervin quasi-uniformity [2]. Further, we refer to the definition of "quasi-uniform space over $B$" in Park and Lee [7].

Let $X$ be a set. For every subset $A$ of $X$, let

$$S(A) := A \times A \cup (X - A) \times X.$$

**Theorem 3.2.** Let $(X, \tau_X)$ be a fibrewise space over $B$. Then $S = \{S(A)|A \in \tau_X\}$ is a fibrewise qu-subgerm for a fibrewise quasi-uniformity on $X$ compatible with $\tau_X$. 
Proof. For each $A \in \tau_X$, it is clear that $\Delta \subset S(A)$, and we can easily show that $S(A) \circ S(A) = S(A)$. Thus $S$ is a fibrewise quasi-subgerm for a fibrewise quasi-uniformity on $X$.

Let $\tau(\mathcal{U})$ be the topology defined by the fibrewise quasi-uniformity $\mathcal{U}$ which is generated by the quasi-subgerm $S$. Now we shall show that $\tau(\mathcal{U}) = \tau_X$.

Let $O \in \tau_X$ and $x \in O$. Then $x \in S(O)[x] = O$. Thus $O \in \tau(\mathcal{U})$.

Conversely, let $O \in \tau(\mathcal{U})$ and $x \in O$. Then there exist $W \in N(p(x))$ and $O_1, \cdots , O_n \in \tau$ such that $x \in \bigcap_{i=1}^{n} S(O_i)[x] \cap X_W \subset O$. In fact, if $x \not\in \bigcup_{i=1}^{n} O_i$, then $X = \bigcap_{i=1}^{n} S(O_i)[x] \subset U[x]$. Therefore $U[x] = X \in \tau_X$. If $x \in \bigcup_{i=1}^{n} O_i$, then $\bigcap_{i=1}^{n} S(O_i)[x] = \bigcap_{i=1}^{n} \{ O_i[ x \in O_i] \}$ is a $\tau$-open set and $X_W$ is also $\tau$-open. Thus $\bigcap_{i=1}^{n} S(O_i)[x] \cap X_W$ is a $\tau$-open set. Hence $O \in \tau_X$.

We call the fibrewise quasi-uniformity constructed in this theorem fibrewise Pervin quasi-uniformity.

Last, we shall note the definition of “quasi-uniform space over $B$” in Park and Lee [7]. Their definition is as follow: By a quasi-uniform space $X$ over $B$ they mean a function $p : X \rightarrow B$ in which both of $X$ and $B$ are quasi-uniform spaces and $p$ is a quasi-uniformly continuous map. This definition is a generalization of I.M.James’ in [5]. In [5], he studied $p : X \rightarrow B$ in the situation that both of $X$ and $B$ are uniform spaces and $p$ is a uniformly continuous map. On the other hand, our definition of fibrewise quasi-uniformity in this section is a generalization along Konami and Miwa in [6] (and James’ in [4] as an idea).

In connection with the Pervin quasi-uniformity [2], the following proposition was obtained in [2].

Proposition 2.17 ([2]). For every continuous map $f : (X, \tau_X) \rightarrow (B, \tau_B)$, let $\mathcal{U}$ and $\mathcal{V}$ be the Pervin quasi-uniformities on $X$ and $B$ respectively, then $f : (X, \mathcal{U}) \rightarrow (B, \mathcal{V})$ is quasi-uniformly continuous.

If we consider this proposition, we can say that every fibrewise space $X$ over $B$ can be considered as “quasi-uniform space $X$ over $B$” (in [7]) if we introduce the Pervin quasi-uniformities to $X$ and $B$.

REFERENCES