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Determinacy of Wadge classes in the Baire space and simple iteration of inductive definition

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Abstract

In [4], we introduced determinacy schemata motivated by Wadge classes in descriptive set theory. In this paper, we prove that a simple iteration of Σ_1^1 inductive definition implies $\mathsf{Sep}(\Delta_2^0, \Sigma_2^0)$ determinacy in the Baire space over RCA_0.

1 Introduction

We consider the following type of game: Two players, say player I and player II, alternately choose an element of X to form an infinite sequence f of elements of X. Both players can refer the history of their plays. Player I wins iff a given formula $\varphi(f)$ of f holds. Player II wins iff I does not win.

In [4], we introduced various determinacy schemata motivated by Wadge classes in descriptive set theory and we investigated the strength of them in the Cantor space, i.e., the case of $X = \{0,1\}$ in the framework of *reverse*

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mathematics (cf. [6]). Actually, it is proved that there is a hierarchy of determinacy between Σ_2^0 determinacy and $\Sigma_2^0 \wedge \Pi_2^0$ determinacy, the determinacy of games defined by conjunctions of Σ_2^0 formulae and Π_2^0 formulae.

It is natural to ask whether we have a proper hierarchy between Σ_2^0 and $\Sigma_2^0 \wedge \Pi_2^0$ determinacies also in the Baire space, i. e., the case of $X = \mathbb{N}$.

In this paper, we will have a partial answer to it. We will give a rough sketch of the proof that a simple iteration of inductive definition implies $\mathsf{Sep}(\Delta_2^0, \Sigma_2^0)$ determinacy in the Baire space.

At the end of this paper, we will conjecture that the simple iteration of inductive definition have actually the same strength as a single inductive definition.

2 Preliminaries

2.1 The base theory RCA_0

The language L_2 of second order arithmetic consists of $+, \cdot, 0, 1, =, <$, number variables $x, y, ...$, propositional connectives and number quantifiers, set variables $X, Y, ...$, set quantifiers and $\in$. Terms and formulae are defined in the usual way. A formula is Π_0^0, Σ_0^0 or Δ_0^0 if it is built up from atomic formulae by propositional connectives and bounded number quantifiers $\forall x < t$ and $\exists x < t$. A Σ_n^0 (resp. Π_n^0) formula is one consisting of n number quantifiers beginning with an existential (resp. universal) one followed by a Π_0^0 formula. A formula is Σ_0^1, Π_0^1 or *arithmetical* if it does not contain set quantifiers. A Σ_n^1 (resp. Π_n^1) formula is one consisting of n set quantifier beginning with an existential (resp. universal) one followed by a Π_0^1 formula.

Note that formulae in these classes may have set parameters. In this paper, we consider only boldface classes, i. e., they allow formulae to have set parameters.

We use the following base theory.

Definition 2.1 (RCA_0). RCA_0 is the formal system in the language L_2 which consists of the axioms of discrete ordered semi-ring for $(\mathbb{N}, +, \cdot, 0, 1, <)$ plus the schemata of Σ^0_1 induction and of Δ^0_1 comprehension:

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists Y \forall n(\varphi(n) \leftrightarrow n \in Y),$$

where $\varphi(x)$ is a Σ^0_1 formula without free occurrences of Y, and where $\psi(x)$ is a Π^0_1 formula.

In [4], we use a weaker base theory RCA^*_0, which consists of the axioms of discrete ordered semi-ring with exponentiation, Δ^0_1 comprehension and Σ^0_0 induction. Roughly speaking, RCA^*_0 can be regarded as the theory RCA_0 minus Σ^0_1 induction. In [4], it is shown that, over RCA_0, Σ^0_1 comprehension is equivalent to $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}^*$, which asserts the determinacy of games defined by conjunctions of Σ^0_1 formulae Π^0_1 formulae. Because RCA^*_0 proves that Σ^0_1 comprehension implies Σ^0_1 induction, when we consider equivalences between determinacy schemata stronger than $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}^*$ and set existence axiom stronger than Σ^0_1 comprehension, we need not to care about the difference between base theories RCA_0 and RCA^*_0.

Notation 2.2 (sequences). The set of all infinite sequences from X (i.e., functions from $\mathbb{N}$ to X) is denoted by $X^{\mathbb{N}}$. We call $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$ the *Cantor space* and $\mathbb{N}^{\mathbb{N}}$ the *Baire space.* The set of all finite sequences from X is denoted by $X^{<\mathbb{N}}$. Note that the *empty sequence*, denoted by $\langle\rangle$, belongs to $X^{<\mathbb{N}}$.

Fix any sequences s and t in $X^{\mathbb{N}}$. $|s|$ denotes the *length* of s and $s(i)$ denotes the $(i+1)$-th element of s for $i < |s|$. The *concatenation* of s and t, denoted by $s*t$, is $\langle s(0), s(1), ..., s(|s|-1), t(0), t(1), ..., t(|t|-1)\rangle$. If $f \in X^{\mathbb{N}}$, $s * f$ denotes $\langle s(0), s(1), ..., s(|s|-1), f(0), f(1), ..., f(n), ...\rangle$. For $s \in X^{<\mathbb{N}}$ and $n \leq |s|$, $s[n]$ is the *n-th initial segment* of s, i.e., $\langle s(0), ..., s(n-1)\rangle$. If f is an infinite sequence, $f[n]$ denotes $\langle f(0), ..., f(n-1)\rangle$. If $s = t[k]$ for some $k \leq |t|$, s is called an *initial segment* of t, $s \subseteq t$ for notation. If $n \leq |s|$, $s \ominus n$ is the sequence with the first n elements removed from s, i.e., $\langle s(n), s(n+1), ..., s(|s|-1)\rangle$. If f is in $X^{\mathbb{N}}$, $f \ominus n$ is g defined by $g(k) = f(n+k)$ for $k \in \mathbb{N}$. For $s \in X^{<\mathbb{N}}$, $(s)_X$ denotes the set $\{t \in X^{<\mathbb{N}} : s \subseteq t\}$.

2.2 Determinacy in second order arithmetic

In this paper, a game is a formula $\varphi(f)$ with a distinguished function variable f. Since L_2 does not formally have function variables, $\varphi(f)$ is an abbreviation of $\varphi(X) \land \forall n \exists m((n,m) \in X \land (\forall l(n,l) \in X \to m = l))$. Therefore, the complexity of the formula $\varphi(f)$ is not always the same as $\varphi(X)$. However, we do not need to care about this point when we work in a system stronger than ACA_0, the system RCA_0 plus *arithmetical comprehension*

$$\exists X \forall n(n \in X \leftrightarrow \psi(n)),$$

where $\psi(n)$ is an arithmetical formula in which X does not occur freely.

For a given formula $\varphi(f)$ with a distinguished function variable $f \in X^{\mathbb{N}}$, a *game* $\varphi(f)$ *in* $X^{\mathbb{N}}$ is defined as follows: Two players, say player I and player II, alternately choose an element x in X to form $f \in X^{\mathbb{N}}$ which is called the resulting play. Player I wins if and only if $\varphi(f)$ holds. Player II wins if and only if player I does not win. In this paper, we assume that player I is male and that player II is female.

We regard a class Γ of formulae with a distinguished function variable as a class of games.

Notation 2.3 (strategy). For a game in $X^{\mathbb{N}}$, a *strategy* σ for player I (or II) is a function assigning an element of X to each even-length (resp. odd-length) $t \in X^{<\mathbb{N}}$. $\mathcal{S}_{\mathrm{I}}^X$ (resp. $\mathcal{S}_{\mathrm{II}}^X$) is the set of all the strategies for player I (resp. II) in a game in $X^{\mathbb{N}}$. Note that $\mathcal{S}_{\mathrm{I}}^X$ and $\mathcal{S}_{\mathrm{II}}^X$ can be regarded as $\mathbb{N}^{\mathbb{N}}$ if $X = \mathbb{N}$ and $2^{\mathbb{N}}$ if $X = \{0,1\}$ in RCA_0, by a suitable coding of finite sequences. If players I and II follow strategies σ and τ respectively, the resulting play is uniquely determined and denoted by $\sigma \otimes \tau$.

For any strategy τ for player II, k^τ is the finite play of length $2k$ in which player I plays 0 at all his turns and in which player II plays following τ. For example, 2^τ is the sequence $\langle 0, \tau(\langle 0\rangle), 0, \tau(\langle 0, \tau(\langle 0\rangle), 0\rangle)\rangle$.

A strategy σ for a player is a *winning strategy* if the player wins $\varphi(f)$ as long as he or she plays following it. The assertion that σ is a winning strategy for player I (resp. II) in game $\varphi(f)$ in $X^{\mathbb{N}}$ can be written $\forall\tau \in \mathcal{S}_{\mathrm{II}}^{X}\varphi(\sigma \otimes \tau)$ (resp. $\forall\tau \in \mathcal{S}_{\mathrm{I}}^{X}\neg\varphi(\tau \otimes \sigma)$). A game $\varphi(f)$ is *determinate* if one of the players has a winning strategy. For a game $\varphi(f)$ in $X^{\mathbb{N}}$, we use the following abbreviation:

$$\mathsf{Det}^{X}[\varphi] \equiv \exists\sigma \in \mathcal{S}_{\mathrm{I}}^{X}\forall\tau \in \mathcal{S}_{\mathrm{II}}^{X}\varphi(\sigma \otimes \tau) \vee \exists\tau \in \mathcal{S}_{\mathrm{II}}^{X}\forall\sigma \in \mathcal{S}_{\mathrm{I}}^{X}\neg\varphi(\sigma \otimes \tau),$$

which asserts that $\varphi(f)$ is determinate. The following schema of Γ *determinacy* asserts that all the Γ games are determinate.

Γ determinacy in $X^{\mathbb{N}}$: $\mathsf{Det}^{X}[\varphi]$ for any Γ game $\varphi(f)$ in $X^{\mathbb{N}}$.

Det^{*} and Det abbreviate determinacy in the Cantor space and that in the Baire space, respectively.

An *s-strategy for player I* (or II) is a function σ which assigns an element of X to each even-length (resp. odd-length) $t \in (s)_X$.

For s-strategies σ for player I and τ for player II, $\sigma \otimes \tau$ denotes the sequence f such that $f(i) = s(i)$ for all $i < |s|$, $f(2i) = \sigma(f[2i])$ for all $2i \geq |s|$, $f(2i+1) = \tau(f[2i+1])$ for all $2i+1 \geq |s|$, in other words, the play, starting from s, in which player I follows σ and player II follows τ. Note that if $s = \langle\rangle$, the definition of $\sigma \otimes \tau$ coincides with the previous definition. For a game $\varphi(f)$ in $X^{\mathbb{N}}$, a s-strategy σ for player I (or II) is winning if, for every s-strategy τ for player II (resp. I), $\varphi(\sigma \otimes \tau)$ (resp. $\neg\varphi(\tau \otimes \sigma)$). *Player I (or II) wins at s in $\varphi(f)$* if (1) there is a winning s-strategy for player I (resp. II), or equivalently, (2) either (i) $|s|$ is even and player I (resp. II) has a winning strategy in $\varphi(s * f)$ or (ii) $|s|$ is odd and player II (resp. I) has a winning strategy in $\neg\varphi(s * f)$. Note that in (ii) of (2), the role of two players are exchanged.

3 Inductive definition and determinacy

First, we see the notion of *inductive definition* in set theory. An operator $\Gamma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is in a class $\mathcal{C}$ of formulae if $\Gamma(X)$ is equivalent to a formula in $\mathcal{C}$. For a operator $\Gamma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ and a set V, consider the following sequence $\langle V_0 : \alpha \in \mathrm{On}\rangle$ of sets:

- $V_0 = V$
- $V_\alpha = \Gamma(\bigcup_{\beta<\alpha} V_\beta) \cup \bigcup_{\beta<\alpha} V_\beta$

Since $\langle V_\alpha : \alpha \in \mathrm{On}\rangle$ is an increasing sequence, there exists, in ZFC, $\alpha < \omega_1$ such that $V_\alpha = V_\beta$ for all $\beta > \alpha$ and such V_α is called the *fixed point of* Γ *starting from* V.

However, in a weak set theory, such as KP (cf. [1]), the existence of the fixed point is not always guaranteed. $\mathcal{C}$ *inductive definition* asserts that, for any operator Γ in $\mathcal{C}$, there exists a fixed point of Γ.

The following formalization of inductive definition in second order arithmetic was introduced by [7].

We need the notion of pre-wellordering.

Definition 3.1 (pre-wellordering). A binary relation $(W, <_W)$ is *pre-ordering* on its *field* $\mathrm{field}(W) = \{x : \exists y(x <_W y \vee y <_W x)\}$ if it satisfies the following properties:

Reflexivity $\forall a \in \mathrm{field}(W)(a <_W a)$

Connectivity $\forall a \in \mathrm{field}(W)\forall b \in \mathrm{field}(W)((a <_W b) \vee (b <_W a) \in W)$

Transitivity
$\forall a \in \mathrm{field}(W)\forall b \in \mathrm{field}(W)\forall c \in \mathrm{field}(W)((a <_W b) \wedge (b <_W c) \to (a <_W c))$

For a binary relation $(W, <_W)$, $W_x = \{y : y <_W x\}$ and $W_{<_W x} = \{y : y <_W x \wedge x \not<_W y\}$. A pre-ordering $(W, <_W)$ is a *pre-wellordering* if it is well-founded, i.e., there is no $f : \mathbb{N} \to \mathrm{field}(W)$ such that, for all n, $f(n+1) <_W f(n)$ and $f(n) \not<_W f(n+1)$.

In second order arithmetic, an *operator* is a formula $\eta(x, X)$ with a distinguished number variable x and a distinguished set variable X and sequence

$\langle V_\beta : \beta \leq \alpha \rangle$ toward the least fixed point V_α of $\eta(x, X)$ is given as a pre-wellordering $(W, <_W)$ intuitively defined as follows: $x_0 <_W x_1$ if and only if $\alpha_0 < \alpha_1$, where α_i is the least ordinal β with $x_i \in V_\beta$.

Definition 3.2 ($\mathcal{C}$ inductive definition)**.** Let $\mathcal{C}$ be a class of L_2 formulae with a . $\mathcal{C}$ *inductive definition* ($\mathcal{C}$-ID) asserts that for any $\eta(x, X)$ in $\mathcal{C}$, called a $\mathcal{C}$ *operator*, there exists a set $(W, <_W)$ such that

1. $(W, <_W)$ is a pre-wellordering on field(W),
2. $\forall x(x \in \text{field}(W) \leftrightarrow \eta(x, W_{<_W x}))$,
3. $\forall x(\eta(x, \text{field}(W)) \rightarrow x \in \text{field}(W))$.

For an operator η, field(W) of the set W with the above three properties is called the *fixed point of* η.

An operator η is *monotone* if, for all X and Y and for all x, $X \subseteq Y$ implies $\forall x(\eta(x, X) \rightarrow \eta(x, Y))$. $\mathcal{C}$ *monotone inductive definition* ($\mathcal{C}$-MI) is the following schema:

$$\forall X \forall Y \forall x(X \subseteq Y \wedge \eta(x, X) \rightarrow \eta(x, X)) \rightarrow \exists W(W \text{ is the fixed point of } \eta),$$

where η is a $\mathcal{C}$ operator.

Recall that we only consider boldface formulae classes. In particular, $\mathcal{C}$ inductive definition is the boldface version, which allows any operator to have set parameters.

Lemma 3.3. Π^0_2-Det *and* Σ^1_1-MI *and* Σ^1_1-ID *are pairwise equivalent over* RCA_0.

Proof. See [3]. Although the original assertion in [3] might be a lightface version and it is over ATR_0, we can easily modify the proof for this lemma, the boldface version, and then the base theory can be replaced with RCA_0. □

We consider the iteration of inductive definition in the following sense: For a given wellordering $(Y, \prec)$, a sequence $\langle \eta(y, x, X) : y \in Y \rangle$ of operations along $(Y, \prec)$, W^y is a fixed point of $\eta(y, x, X)$ starting from $\bigcup_{z \prec y} W^z$.

Definition 3.4 ($\mathcal{C}$ transfinite inductive definition). Let $\mathcal{C}$ be a class of L_2 formulae. *$\mathcal{C}$ transfinite inductive definition* ($\mathcal{C}$-TID) asserts that for any wellordering $(Y, \prec)$ and any operator $\eta(y, x, X)$ in $\mathcal{C}$ with another distinguished number variable y, there exists a sequence $\langle W^y : y \in Y \rangle$ such that

1. $(W^y, <_y)$ is a pre-wellordering for each $y \in Y$.
2. $\forall y \in Y \forall z \in Y \forall v \in \text{field}(W^y) \forall w \in \text{field}(W^z)(z \prec y \to w \leq_y v)$.
3. $\forall v \forall y \in Y(v \in W^y \leftrightarrow \eta(y, v, W^y_{<_y v}))$
4. $\forall v \forall y \in Y(\eta(y, v, \text{field}(W^y)) \to v \in \text{field}(W^y))$.

For an operator $\eta(y, x, X)$ and a wellordering $(Y, \prec)$, the sequence $\langle W^y : y \in Y \rangle$ satisfying the above four conditions is the *sequence of fixed points along* $(Y, \prec)$.

In this paper, we consider the following determinacy schema.

Definition 3.5. $\mathsf{Sep}(\Delta^0_n, \Sigma^0_m)$ determinacy in $X^{\mathbb{N}}$ is the following schema:

$$\forall f \in X^{\mathbb{N}}(\psi(f) \leftrightarrow \xi(f)) \to \mathsf{Det}^X[(\psi \wedge \eta_0) \vee (\neg\psi \wedge \eta_1)],$$

where $\psi(f)$ is Σ^0_n, where $\xi(f)$ is Π^0_n, where $\eta_0(f)$ is Π^0_m and where $\eta_1(f)$ is Σ^0_m.

For the motivation behind the above determinacy schemata, see [4]. In [4] and [5], the strength of $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-$\mathsf{Det}^*$ is considered.

Now we consider the determinacy strength of $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-$\mathsf{Det}$.

We need several lemmata.

Lemma 3.6. *For any Σ^0_1 formula $\varphi(X)$, we can find a Π^0_0 formula $\theta(x)$ such that RCA_0 proves $\forall X(\varphi(X) \leftrightarrow \exists n \theta(X[n]))$*

Proof. See [6, Theorem II. 2.7]. □

In descriptive set theory, Hausdorff proved (cf. [2, §37. III. Theorem]) that a $\mathbf{\Delta}^0_{n+1}$ set can be represented as a boolean combination of transfinitely many $\mathbf{\Pi}^0_n$ sets, i. e., for any $\mathbf{\Delta}^0_{n+1}$ set A of Polish space $\mathcal{X}$, there exists an ordinal $\gamma < \omega_1$ and a decreasing sequence $\langle A_\alpha : \alpha < \gamma \rangle$ of $\mathbf{\Pi}^0_n$ sets such that

$$A = \{x \in A_0 : \min\{\alpha : x \notin A_\alpha\} \text{ is odd}\}.$$

The following lemma is a formalization of the case $n = 2$ in second order arithmetic.

Lemma 3.7. *For any pair of a Σ^0_2 formula $\psi_0(f)$ and a Π^0_2 formula $\psi_1(f)$, we can find a Π^0_0 formula $\theta(x, i, y)$ such that ACA_0 proves the following.*

$$\forall f \in 2^{\mathbb{N}}(\psi_0(f) \leftrightarrow \psi_1(f)) \rightarrow$$
$$\begin{bmatrix} \exists Y((Y, \prec) \text{ is a wellordering}) \wedge \\ (\forall f \in 2^{\mathbb{N}})(((y,j) \prec^* (x,i) \wedge \forall n\theta(x,i,f[n])) \rightarrow \forall n\theta(y,j,f[n])) \wedge \\ (\forall f \in 2^{\mathbb{N}})(\psi_0(f) \leftrightarrow \exists x \in Y(\forall n\theta(x,0,f[n]) \wedge \neg\forall n\theta(x,1,f[n]))) \wedge \\ (\forall f \in 2^{\mathbb{N}})(\neg\psi_0(f) \leftrightarrow \exists x \in Y(\forall n\theta(x,1,f[n]) \wedge \neg\forall n\theta(x',0,f[n]))) \end{bmatrix} \quad (\star)$$

where x' is the $\prec$-successor of x, and where $(Y \times 2, \prec Y^*)$ is a wellordering defined by

$$(x,i) \prec^* (y,j) \leftrightarrow x \prec y \vee (x = y \wedge i < j).$$

Proof. See Theorem 3.5 of [3]. □

Remark 3.8. For any Π^0_1 (or Σ^0_1) game $\varphi(f)$ in the Baire space, the assertion that player I (resp. II) has a winning strategy in $\varphi(f)$ is equivalent to a Σ^1_1 formula RCA_0, since the assertion can be written as $\exists\sigma \in \mathcal{S}^{\mathbb{N}}_I \underline{\forall\tau \in \mathcal{S}^{\mathbb{N}}_{II}\varphi(\sigma \otimes \tau)}$ (resp. $\exists\sigma \in \mathcal{S}^{\mathbb{N}}_{II}\underline{\forall\tau \in \mathcal{S}^{\mathbb{N}}_I\neg\varphi(\tau \otimes \sigma)}$) and since the underlined part are equivalent to some Π^0_1 formula. Similarly, for any Π^0_1 (or Σ^0_1) game $\varphi(f)$ in the Baire space, the assertion that player I (resp. II) has a winning s-strategy in $\varphi(f)$ is equivalent to a Σ^1_1 formula over RCA_0.

For a given arithmetical game $\varphi(f)$, even if we know that player I wins at each $s \in W$, it seems that we need Π^1_1 axiom of choice to take a sequence

$\langle \sigma_s : s \in W \rangle$ of winning s-strategies for player I because, in general, the assertion "σ is a winning s-strategy for player I in $\varphi(f)$" is Π^1_1. The following lemma proves that actually we do not need it.

Lemma 3.9. *Let $1 \leq n < \omega$. Let $A \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$. Let $\varphi(f)$ be a game in the Baire space. In RCA_0^*, the following is provable: If, for each $(s, x) \in A$, player I (or II) wins Σ^0_n game $\varphi(x, f)$ at s, then Σ^0_n-Det yields a sequence $\langle \sigma_{s,x} : (s, x) \in A \rangle$ of winning s-strategies for player I (resp. II) in $\varphi(x, f)$.*

Proof. We work in RCA_0^*. Let $\varphi(f)$ be a Σ^0_n game in $\mathbb{N}^{\mathbb{N}}$. Assume that, for each $(s, x) \in A$, player I wins $\varphi(x, f)$ at s. Let $e : \mathbb{N} \to \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}$ be a fixed enumeration of $\mathbb{N}^{<\mathbb{N}} \times \mathbb{N}$. Let $e(m) = (m_0, m_1) \in \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}$.

Consider the following Σ^0_n game $\varphi'(f)$:

- Player II chooses $m \in \mathbb{N}$ at her first turn. If $e(m) \notin A$, player I wins.
- If $e(m) \in A$ and $|e(m)|$ is even, then player I wins if $\varphi(m_1, m_0 * (f \ominus 2))$.
- If $e(m) \in A$ and $|e(m)|$ is odd, then player I wins if $\varphi(m_1, m_0 * (f \ominus 3))$.

Σ^0_n-Det implies that one of the player has a winning strategy in $\varphi'(f)$. We can check that player II has no winning strategy in $\varphi'(f)$. For contradiction, suppose that player II has a winning strategy τ. Consider such a play f:

- Player I first play 0, i. e., $f(0) = 0$.
- Player II play m, following τ.

Note that $e(m) \in A$, otherwise player II loses. Then τ yields a winning m_0-strategy for player II in $\varphi(m_1, f)$, which contradicts the assumption that player I wins $\varphi(f)$ at $e(m)$. Hence player I has a winning strategy σ in $\varphi'(f)$.

For $(s, x) \in \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}$, let $\overline{e}(s, x)$ be the least k with $e(k) = (s, x)$. Let s_x be the sequence $\langle \sigma(\langle\rangle), \overline{e}(s, x) \rangle$ if $|s|$ is even and $\langle \sigma(\langle\rangle), \overline{e}(s, x), \sigma(\langle \sigma(\langle\rangle), \overline{e}(s, x) \rangle) \rangle$ if $|s|$ is odd. Then define $\sigma_{s,x}$ by $\sigma_{s,x}(t) = \sigma(s_x * (t \ominus |s|))$ for each $(s, x) \in A$. Clearly $\sigma_{s,x}$ is a winning s-strategy for player I in $\varphi(x, f)$ for each $(s, x) \in A$.

The statement for player II can be proved similarly. □

Theorem 3.10. RCA_0 *proves that* Σ^1_1-TID *implies* $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-$\mathsf{Det}$.

Proof. Here we give only a rough sketch of the proof.

Suppose Σ^1_1-TID. Since Σ^1_1-TID implies Π^1_1 comprehension over RCA^*_0, we work in Π^1_1-CA_0, the system RCA_0 plus Π^1_1 comprehension:

$$\exists X \forall n (n \in X \leftrightarrow \phi(n)),$$

where $\phi(n)$ is any Π^1_1 formula in which X does not occur freely.

Let $\varphi(f)$ be a game of the form $(\psi(f) \wedge \zeta_0(f)) \wedge (\neg\psi(f) \wedge \zeta_1(f))$, where $\psi(f)$ and $\zeta_1(f)$ are Σ^0_2 and $\zeta_0(f)$ is Π^0_2. Assume that there is a Π^0_2 formula such that $\forall f \in \mathbb{N}^\mathbb{N}(\psi(f) \leftrightarrow \psi'(f))$. By applying Lemma 3.7, letting $\psi(f)$ and $\psi'(f)$ be $\psi_0(f)$ and $\psi_1(f)$ respectively, we can find a Π^0_1 formula $\forall n\theta(x, i, f[n])$ and a wellordering $(Y, \prec)$ such that Π^1_1-CA_0 proves $(\star)$.

Define new games $\zeta'_i(y, f, X)$ by

$$\begin{aligned}
\zeta'_0(y, f, X) \equiv& (\forall n\theta(y, 0, f[n]) \wedge \zeta_0(f)) \vee \\
& (\exists m \exists (z, j) \preceq^*_Y (y, i)(\neg\theta(z', 0, f[m]) \wedge f[m] \in X)) \vee \\
& (\exists m \exists (z, j) \preceq^*_Y (y, i)(\neg\theta(z, 1, f[m]) \wedge f[m] \notin X)), \\
\zeta'_1(y, f, X) \equiv& (\forall n\theta(y, 1, f[n]) \wedge \zeta_1(f)) \vee \\
& (\exists m \exists (z, j) \preceq^*_Y (y, i)(\neg\theta(z', 0, f[m]) \wedge f[m] \in X)) \vee \\
& (\exists m \exists (z, j) \preceq^*_Y (y, i)(\neg\theta(z, 1, f[m]) \wedge f[m] \notin X)).
\end{aligned}$$

By Lemma 3.6, we can find Π^0_0 formulae $\theta_0(x, y)$ and $\theta_1(x, y)$ such that RCA_0 proves that, for all $f \in \mathbb{N}^\mathbb{N}$,

$$\begin{aligned}
\zeta'_0(y, f, X) &\leftrightarrow \forall n \exists m \theta_0(n, y, f[m], X[m]), \\
\zeta'_1(y, f, X) &\leftrightarrow \exists n \forall m \theta_1(n, y, f[m], X[m]).
\end{aligned}$$

Define an operator $\eta(\langle y, i\rangle, x, X)$ by

$$\begin{aligned}
\eta(\langle y, i\rangle, x, X) \equiv& (i = 0 \wedge \theta(y, 1, x) \wedge \exists n (\text{player II wins } \theta'_0(n, y, f, X))) \vee \\
& (i = 1 \wedge \theta(y', 0, x) \wedge \exists n (\text{player I wins } \theta'_1(n, y, f, X)))
\end{aligned}$$

By Lemma 3.8, $\eta_i(y, x, X)$ is a Σ^1_1 operator. Σ^1_1-TID yields the sequence $\langle W^{\langle y,i\rangle} : y \in Y, i < 2\rangle$ of fixed points of $\eta(y, x, Y)$ along $(Y, \prec_Y)$. Define $\widetilde{W}^{\langle y,i\rangle}$ and $\widetilde{V}^{\langle y,i\rangle}$ by

$$\widetilde{W}^{\langle y,0\rangle} = \{s : \neg\theta(y, 1, s) \wedge s \notin W^{\langle y,0\rangle}\} \quad \widetilde{W}^{\langle y,1\rangle} = W^{\langle y,1\rangle}$$
$$\widetilde{V}^{\langle y,0\rangle} = W^{\langle y,0\rangle} \qquad \widetilde{V}^{\langle y,1\rangle} = \{s : \neg\theta(y', 0, s) \wedge s \notin W^{\langle y,1\rangle}\}$$

In a similar way as in the proof of Theorem 3.1 of [7], we can prove that player I wins $\zeta''_i(y, f)$ at each $s \in \widetilde{W}^{\langle y,i\rangle}$ and that player II wins $\zeta''_i(y, f)$ at each $s \in \widetilde{V}^{\langle y,i\rangle}$, where $\zeta''_i(y, f)$ is defined by

$$\exists n \neg\theta(\bar{y}, 1 - i, f[n]) \wedge ((\forall n \theta(y, i, f[n]) \wedge \zeta_i(f)) \vee \exists n f[n] \in \bigcup_{\langle z,j\rangle <^*_Y \langle y,i\rangle} \widetilde{W}^{\langle z,j\rangle})$$

where $\bar{y}$ is y if $i = 0$ and y' if $i = 1$.

Claim 1.

1. *If $s \in \bigcup_{y\in Y, i<2} \widetilde{W}^{\langle y,i\rangle}$, player I wins $\varphi(f)$.*
2. *If $s \in \bigcup_{y\in Y, i<2} \widetilde{V}^{\langle y,i\rangle}$, player II wins $\varphi(f)$.*

Proof of the claim. We only prove 1, because 2 can be proved in a similar way.

By Lemma 3.9, we have a sequence $\langle \sigma^*_s : s \in \bigcup_{y\in Y} W^y\rangle$ of winning s-strategies for player I in $\zeta''_{i_s}(y_s, f)$, where $\langle y_s, i_s\rangle$ is the $<^*_Y$-least $\langle y, i\rangle$ with $s \in \widetilde{W}^{\langle y,i\rangle}$.

By arithmetical transfinite recursion (cf. [6, Chapter V]), which is proved in Π^1_1-CA_0, we can define a sequence σ_s defined by

$$\sigma_s(t) = \begin{cases} \sigma_u(t) & \text{if } u \text{ is the } \subseteq\text{-least initial segment of } t \\ & \text{with } t \in \bigcup_{\langle z,j\rangle < \langle y_s, i_s\rangle} \widetilde{W}^{\langle z,j\rangle} \\ \sigma^*_s(t) & \text{if there is no such } u \subseteq t \end{cases}$$

It is easy to prove σ_s is a winning s-strategy for player I in $\varphi(f)$. □

Then define a new game $\varphi^*(f)$ by $\exists n(\forall m < n(f[m] \notin \bigcup_{y\in Y, i<2} \widetilde{V}^{\langle y,i\rangle}) \wedge f[n] \in \bigcup_{y\in Y, i<2} \widetilde{W}^{\langle y,i\rangle})$. Σ^1_1-TID implies that $\varphi^*(f)$ is determinate. Then we can prove that the player who wins $\varphi^*(f)$ also wins $\varphi(f)$. □

We close this paper by making a conjecture. If it is true, then it turns out that actually there is no proper hierarchy of determinacy of Wadge classes between Σ_2^0-Det and $\Sigma_2^0 \wedge \Pi_2^0$-Det, since $\mathsf{Sep}(\Delta_2^0, \Sigma_2^0)$-Det implies all determinacy schemata corresponding to Wadge classes below $\Sigma_2^0 \wedge \Pi_2^0$. A predictably-effective approach is to modify the proof of Theorem 2.2 of [3], which shows that Σ_2^0 determinacy implies Σ_1^1-ID over RCA_0.

Conjecture 3.11. RCA_0 *proves that* Σ_2^0-Det *implies* Σ_1^1-TID.

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