ON A LOCAL MODEL FOR
FINDING 4-DIM DUCK SOLUTIONS

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ABSTRACT. A slow-fast system in R^{2+2} has 4-dimensional duck solutions under some conditions which will be described in this paper. A time scaled reduced system for the original system gives some essential conditions for having such singular solutions. Through blowing up the system, it is transformed to a local model approximated as square-linear equations. The local model has exact solutions with no singularity, and it ensures the existence of the singular solutions as an approximation.

1.INTRODUCTION

In the coupled FitzHugh-Nagumo system, S.A.Campbell and the author have already proved the existence of the winding duck solutions using an indirect method. As the associated slow-fast system (or singular perturbation problem) has a 2-dimensional slow manifold (constrained surface), the system can be reduced to the slow-fast one in R^3. It turns to have two kinds of projected slow-fast systems in R^3.

In a direct method, giving a generalized transversality condition and also doing a condition on the fast vector field, it will be shown that there exists a duck solution in R^4. Under these conditions, two local models with an explicit duck solution will be provided.

2.SLOW-FAST SYSTEM IN R^4

Now, let us consider a slow-fast system(2.1):

$$(2.1) \quad \begin{aligned} \epsilon dx_1/dt &= h_1(x_1, x_2, y_1, y_2, \epsilon), \\ \epsilon dx_2/dt &= h_2(x_1, x_2, y_1, y_2, \epsilon), \\ dy_1/dt &= f_1(x_1, x_2, y_1, y_2, \epsilon), \\ dy_2/dt &= f_2(x_1, x_2, y_1, y_2, \epsilon), \end{aligned}$$

where $f = (f_1, f_2)$ and $h = (h_1, h_2)$ are standard defined on $R^4 \times R^1$ and ϵ is infinit small.

First, we assume the following condition $(A1)$ to get an explicit solution.

($A1$) f is of class C^1 and h is of class C^2.

Furthermore, we assume that the system(2.1) satisfies the following generic conditions ($A2$) − ($A5$):

($A2$) The set $S_2 = \{(x,y) \in R^4 | h(x,y,0) = 0\}$ is a 2-dimensional differentiable manifold and the set S_2 intersects the set $T_2 = \{(x,y) \in R^4 | det[\partial h(x,y,0)/\partial x] = 0\}$, which is a 3-dimensional differentiable manifold, transversely so that the generalized pli set $GPL = \{(x,y) \in S_2 \cap T_2\}$ is a 1-dimensional differentiable manifold.

($A3$) The value of f is nonzero at any point $p \in GPL$.

($A4$) The $rank[\partial h(x,y,0)/\partial x] = 2$ for any $(x,y) \in S_2 \setminus GPL$, and the $rank[\partial h(x, y,0)/\partial y] = 2$ for any $(x,y) \in S_2$. Then, the surface S_2 can be expressed as $y = \varphi(x)$ in the neighborhood of GPL. On the set GPL, $\partial h_1(x,y,0)/\partial x_2 \neq 0$ or $\partial h_2(x,y,0)/\partial x_1 \neq 0$, then $x_2 = \psi_2(x_1,y)$ and $x_1 = \psi_1(x_2,y)$, where we use the notations $x = (x_1,x_2)$, and $y = (y_1,y_2)$.

Let the latter of ($A4$) be satisfied, then the following two projected systems (2.2), (2.3) in R^3 are induced under the condition. We assume that $dx_1/dt, dx_2/dt$ are limited, that is, $\epsilon|dx_1/dt - dx_2/dt|$ tends to zero as ϵ tends to zero.

$$
\begin{aligned}
\epsilon dx_1/dt &= h_2(x_1, \psi_2(x_1,y), y, \epsilon), \\
dy_1/dt &= f_1(x_1, \psi_2(x_1,y), y, \epsilon), \\
dy_2/dt &= f_2(x_1, \psi_2(x_1,y), y, \epsilon),
\end{aligned}
$$
(2.2)

since the relation $x_2 = \psi_2(x_1, y)$ is established from the above assumption. First, we can analyze the vector field of the system(3.2) on the constrained surface. Then, we use $h_2(x_1, x_2, y_1, y_2, \epsilon)$ instead of $h_1(x_1, \psi_2(x_1, y_1, y_2), y_1, y_2, \epsilon)$ as an approximation. Because, we have to avoid redundancy for the system as is using h_1. Actually, we need the above condition: $dx_1/dt, dx_2/dt$ are limited, in such a case. Therefore, this approach is called *an indirect method*. In the case, we can use $h_2(x_1, x_2, y_1, y_2, \epsilon)$ itself, see *Remark* in the Section5.

Using the other relation $x_1 = \psi_1(x_2, y)$, we can get the following:

$$
\begin{aligned}
\epsilon dx_2/dt &= h_1(\psi_1(x_2, y), x_2, y, \epsilon). \\
dy_1/dt &= f_1(\psi_1(x_2, y), x_2, y, \epsilon), \\
dy_2/dt &= f_2(\psi_1(x_2, y), x_2, y, \epsilon).
\end{aligned}
$$
(2.3)

Assume $y = \varphi(x)$. On the set S_2, differentiating both sides of $h(x, \varphi(x), 0) = 0$ with respect to x,

$$[h_x] + [h_y]D\varphi = 0,$$
(2.4)

where $D\varphi$ is a derivative with respect to x, thus the following (2.5) is established:

$$D\varphi(x) = -[h_y]^{-1}[h_x].$$
(2.5)

On the other hand,

$$dy/dt = D\varphi(x)dx/dt,$$
(2.6)

because of $y = \varphi(x)$. We can reduce the slow system to the following:

$$D\varphi(x)dx/dt = f(x, \varphi(x)).$$
(2.7)

Using (2.5), the system (2.7) is described by

$$(2.8) \qquad [h_x]dx/dt = -[h_y]f(x,\varphi(x)).$$

Put $A = [h_x] = [h_{ij}]$ simply, then

$$(2.9) \qquad dx/dt = -B[h_y]f(x,\varphi(x)),$$

where B is a cofactor matrix of A, that is, $B = [A_{ji}]$. A_{ij} is a *cofactor* of h_{ij}.

The system(2.9) is the time scaled reduced system projected into R^2. Again, we assume the set $T_2 = \{(x,y) \in R^4 | detA = 0\} \neq \phi$.

($A5$) All the singular points of the system(2.9) are nondegenerate, that is, the matrix induced from the linearized system of (2.9) at a singular point has distinct nonzero eigenvalues.

Remark. All these points are contained in the set $GPS = \{(x,y) \in GPL | detA = 0\}$, which is called the set of *generalized pseudo singular points.*

As this approach transforms the original system to the time scaled reduced system directly, it is called a *direct method.*

Definition2.1. Let $p \in GPS$ and μ_1, μ_2 be two eigenvalues of the matrix associated with the linearized system of (2.9) at $p \in R^4$. The point p is called *generalized pseudo singular saddle* if $\mu_1 < 0 < \mu_2$ and called *generalized pseudo singular node* if $\mu_1 < \mu_2 < 0$ or $\mu_1 > \mu_2 > 0$. It is called *generalized pseudo singular focus* if they are compex conjugate.

Definition2.2. If there exists a duck in the both systems (2.2) and (2.3) at the common pseudo singular point in R^4, it is called a *total duck in R^4*. If there exists a duck in only one of the above systems, it is called *a partial duck* in R^4.

Now, we have to give a description on the definition of the duck solution in R^4 along the direct method.

Definition2.3. Let a point p be in GPS. If a trajectory follows first the attractive surface before this point and the saddle one at the point p, and then it goes along the slow manifold, which is not infinit small, it is called a *duck solution in R^4*.

Furthermore, we assume that the following.

($A6$) We assume that there exists the set co-GPL, which may contain GPS and then the transversality condition is also established on co-GPL. In the situation, we assume that the invariant manifold through GPS intersects GPL and co-GPL transversely.

Definition2.4. If the trajectory near the point of GPS passes through along the slow manifold with not infinit small and after that it jumps away, it is called *a single duck solution*. If there exists a co-GPL in ($A6$) within the interval, it is called *a double duck solution.*

Remark. The first part of Definition2.4 ensures that only one of the eigenvalues of the matrix $[h_x(x,\phi(x))]$ on the slow manifold takes zero on GPS, because the fast vector field has saddle after GPS. On another GPL, however, the other eigenvalue takes zero. Note that these two eigenvalues of $[h_x(x,\phi(x))]$ are negative when the fast vector field is attractive, and are positive when it is repulsive. It occurs such a state satisfying the assumption ($A6$). When they have different sign, it is saddle.

3.Theorems

In this section, we shall give the following two theorems through a local model in R^{2+2}. See [8].

Theorem3.1. Let $0 \in GPS$ be saddle or node. If the matrix $[h_x(0, \phi(0))]$ has one zero eigenvalue and the other one has negative with a local model satisfying the conditions: (1)$\partial h_1(0)/\partial x_2 = 0, \partial h_2(0)/\partial x_2 = 0$, (2)$f_1(0) \not\equiv 0, f_2(0) \not\equiv 0$, there exists a duck solution in R^4.

(Proof) As only one of the eigenvalues of the matrix $[h_x(x, \phi(x))]$ on the slow manifold takes zero on GPS, the assumptions $(A2)$, $(A4)$ ensure that two eigenvalues of $[h_x(x, \phi(x))]$ are negative in the fast vector field before GPS. They are maybe it is meant negative, respectively positive after GPS. When each coefficient on GPS is limitted, a local model shows a precise structure as an approximation of the original system. Then, the property on GPS reflects directly the whole system. It can be shown that the time scaled reduced system ($\epsilon = 0$) is an apprximated one with a singular solution of the whole system ($\epsilon \neq 0$), because the corresponding solutions are very close each other under the only two conditions. Therefore, we can cnclude that there exists a duck solution.

Let $0 \in GPS$ be saddle or node. When changing the variables correspond to microscopes ($\alpha \simeq 0$): $x_1 = \alpha^p u_1, x_2 = \alpha^q u_2, y_1 = \alpha^r v_1, y_2 = \alpha^s v_2$, $p, q, r, s \in N$, the original system is reduced to the system with variables u_1, u_2, v_1, v_2. Then there exist local models which describe the 4-dimensional duck solutions.

Theorem3.2. If the system has a square-linear solution in a local model, for any $p, q, r, s \in N$, there exist essentially two local models describing the explicit duck solutions, .

(Proof)
In the case $p = 2$, $q = 1$, $r = 2$, $s = 2$, changing variables:

$$(3.1) \qquad x_1 = \alpha^2 u_1, x_2 = \alpha u_2, y_1 = \alpha^2 v_1, y_2 = \alpha^2 v_2,$$

we reduce the system as well in (3.2) as well in (3.3).

$$(3.2) \qquad \begin{aligned} \epsilon du_1/dt &= h_1(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \epsilon)/\alpha^2, \\ \epsilon du_2/dt &= h_2(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \epsilon)/\alpha, \\ dv_1/dt &= f_1(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \epsilon)/\alpha^2, \\ dv_2/dt &= f_2(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \epsilon)/\alpha^2. \end{aligned}$$

Multiplying the right hand side of the system(3.2) by α^2,

$$(3.3) \qquad \begin{aligned} (\epsilon/\alpha^2)du_1/dt &= h_1(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \epsilon)/\alpha^2 \\ (\epsilon/\alpha^2)du_2/dt &= h_2(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \epsilon)/\alpha, \\ dv_1/dt &= f_1(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \epsilon), \\ dv_2/dt &= f_2(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \epsilon). \end{aligned}$$

In fact, doing time scaling $t = \alpha^2 \tau$, then $dt = \alpha^2 d\tau$. It is easily shown that the formula(3.3) is equivalent to (3.2).

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By using the assumptions $(A1)$ and $(A4)$, we construct a local model under the most simple conditions:

(3.4)
$$(1)\partial h_1(0)/\partial x_2 = 0, \partial h_2(0)/\partial x_2 = 0,$$
$$(2)f_1(0) \ne 0, f_2(0) \ne 0.$$

Putting ϵ/α^2 infinit small to ϵ simply, the local model reduced from the system(1.1) is obtained.

(3.5)
$$\epsilon du_1/dt = Au_1 + Bv_1 + Cv_2 + Du_2^2/2 + O(\epsilon),$$
$$\epsilon du_2/dt = Eu_2 + O(\epsilon),$$
$$dv_1/dt = f_1(0) + O(\epsilon),$$
$$dv_2/dt = f_2(0) + O(\epsilon),$$

where $A = \partial h_1(0)/\partial x_1$, $B = \partial h_1(0)/\partial y_1$, $C = \partial h_1(0)/\partial y_2$, $D = \partial^2 h_1(0)/\partial x_2^2$, $E = \partial h_2(0)/\partial x_2$.

Note that the conditions $A = \partial h_1(0)/\partial x_1 < 0$ and $E = \partial h_2(0)/\partial x_2 = 0$ imply that $0 \in GPS$ is saddle. See Definition3.3. The corresponding solutions in the local model are as follows: when $\epsilon = 0$,

(3.6)
$$u_1 = -(Bf_1(0) + Cf_2(0))t/A - Dt^2/(2A), u_2 = t,$$
$$v_1 = f_1(0)t, v_2 = f_2(0)t,$$

when $\epsilon \ne 0$,

(3.7)
$$u_1 = -(Bf_1(0) + Cf_2(0))t/A - Dt^2/(2A) + O(\epsilon), u_2 = t + O(\epsilon),$$
$$v_1 = f_1(0)t + O(\epsilon), v_2 = f_2(0)t + O(\epsilon).$$

In the case $p = 2$, $q = 1$, $r = 3$, $s = 2$, changing variables:

(3.8)
$$x_1 = \alpha^2 u_1, x_2 = \alpha u_2, y_1 = \alpha^2 v_1, y_2 = \alpha^2 v_2,$$

we construct a local model under the conditions:

(3.9)
$$(1)\partial h_1(0)/\partial x_2 = 0, \partial h_2(0)/\partial x_2 = 0,$$
$$(2)f_1(0) = 0, f_2(0) \ne 0.$$

The corresponding local model is

(3.10)
$$\epsilon du_1/dt = Au_1 + Bv_2 + Cu_2^2/2 + O(\epsilon),$$
$$\epsilon du_2/dt = Du_2 + O(\epsilon),$$
$$dv_1/dt = Eu_2 + O(\epsilon),$$
$$dv_2/dt = f_2(0) + O(\epsilon),$$

where $A = \partial h_1(0)/\partial x_1$, $B = \partial h_1(0)/\partial y_2$, $C = \partial^2 h_1(0)/\partial x_2^2$, $D = \partial h_2(0)/\partial x_2$, $E = \partial f_1(0)/\partial x_2$.

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Notice that we assume again that $A < 0$ and $D = 0$, because the fast vector field has one zero eigenvalue and the other one is negative.. The corresponding solutions in the local model are as follows: when $\epsilon = 0$,

$$(3.11) \qquad \begin{aligned} u_1 &= -Bf_2(0)t/A - Ct^2/(2A), u_2 = t, \\ v_1 &= Et^2/2, v_2 = f_2(0)t, \end{aligned}$$

when $\epsilon \neq 0$,

$$(3.12) \qquad \begin{aligned} u_1 &= -Bf_2(0)t/A - Ct^2/(2A) + O(\epsilon), u_2 = t + O(\epsilon), \\ v_1 &= Et^2/2 + O(\epsilon), v_2 = f_2(0)t + O(\epsilon). \end{aligned}$$

In another case, it is impossible to get an explicit solution with a square-linear one but a cubic-linear (or much higher order) one.

In this approach, an invertible affine transformation must be needed for a general point $p \in GPS$, because the conditions (3.4), (3.9) are assumed at only $0 \in GPS$. These conditions may not be satisfied at the general pseudo singular point. We have to change the coordinates from the point p to 0. Notice that we do not know if the corresponding affine transformation keeps the conditions(3.4). In many cases, however, it is feasible.

Remark. It is easily to find that any solutions (u_1, u_2, v_1, v_2) at the same time t in (3.6) and (3.7) are very near. This fact implies that the time scaled reduced system is an approximated one.

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