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積分方程式の基本定理とその応用
Some Fundametal Theorems for Integral Equation and Its Applications

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Abstract

この論文は非線形項を含む人口論に応用することができる連立関数積分方程式を扱っている。特に、全人口が出生率や死亡率に対して、本質的に影響を与える場合を考えている。解の存在定理を証明するためには、縮小写像定理と、シャウダー・ティコノフの不動点定理を使うことができることは、以前にも発表している。今回の発表は、特に縮小写像定理について、不用意に逐次近似のステップを作ると、解の初期分布を大きさにおいて、範囲を制限するような仮定が必要になってしまうという現実に注目した。人口論のモデルを考えるときは、初期分布を積分したときに、必ずしも現実の初期値となる全人口に一致するという仮定をしない。そのときに、初期値となる全人口を制限して解の存在を証明するのは、一意性があるとしても非現実であることは否めない。その点に注目し今回、一般的な連立関数積分方程式の解の存在定理に注目してみた。

1.Introduction

In this paper we shall investigate the basic theory some functional integral equation which occur in the theory of populational problems. This type integral equation was first treated by Gurtin and MacCamy.Their model included the parameters which were death rate and birth rate depended on the total number of the population. Usual equations for mathematical population model can be solves along the characterlistic line. At last those equations will be some shape of integral equations. Also Gurtin and MacCamy made the integral equation which had the functional depended on the integration of the populational distribution.

$$\begin{aligned}&\frac{\partial n}{\partial a}+\frac{\partial n}{\partial t}+\mu(a,N(t))n(a,t)=0, \quad a>0,\, 0<t<T\\ &n(0,t)=\int_0^\infty m(a,N(t))n(a,t)da, \quad 0<t\le T,\\ &n(a,0)=\varphi(a), \quad a\ge 0.\end{aligned} \tag{1}$$

where n is the distribution of the population and N is the total number of the population, that is,

$$N(t)=\int_0^\infty n(a,t)da. \tag{2}$$

As in the previous case the birth process B satisfies the equation,

$$B(t)=n(0,t).$$

For we considering the population model, $\varphi \in L^1(R_+), \mu(a,N), m(a,N)$ are all nonnegative function. Especially μ, m have the integral term of n, so μ, m are the functional of n. In the paper of Gurtin, MacMamy they putted the hypotheses on μ, m that those functional have the continuous patial derivative with respect to N. We can remove this assumption instead of the Lipshitz continuous. Then we can get the same theorem with Gurtin and MacCamy under the following two assumptions, that is, under these assumption there exists only one positive solution $n(a,t)$ for the equaton(1).
$(H1)$ φ is piecewise continuous,
$(H2)$ $\mu, m \in C(R^+ \times R^+)$ and with respect to N these functional are unifomly Lipshitz continuous.
The integral equation along the characteristic line is following.

$$\begin{aligned}
N(t) &= \int_0^t K(t-a;t;N)B(a)da + \int_0^\infty L(a,t;N)\varphi(a)da,\\
B(t) &= \int_0^t m(t-a,N(t))K(t-a,t;N)B(a)da\\
&\quad + \int_0^\infty m(t+a,N(t))L(a,t;N)\varphi(a)da,\\
K(\alpha,t;N) &= exp(-\int_{t-a}^t \mu(\alpha+\tau-t,N(\tau))d\tau),\\
L(\alpha,t;N) &= exp(-\int_0^t \mu(\tau+\alpha,N(\tau))d\tau).
\end{aligned}$$

By using iterational method, that is, using Banach contraction method, we can prove the exisetence of the unique solution on the nonnegative real half line.

2.The Existence Theorems

In this paper we shall consider the following functional integral equation, which is generalization of the integral equation appeared in Introduction.

$$x(t) = \int_0^t k(t-s,t;x)y(s)ds + \int_0^\infty L(t,s;x)\varphi(s)ds, \tag{3}$$

$$\begin{aligned}
y(t) &= \int_0^t \beta(t-s,x(t))k(t-s,t;x)y(s)ds\\
&\quad + \int_0^\infty \beta(t+s,x(t))L(t,s;x)\varphi(s)ds.
\end{aligned} \tag{4}$$

For this rather general integral equation, we put the next assumptions. Through this paper let us call these assumptions as basic hypotheses. We consider the funcion k and L are nonnegative function. In general integral equation theory we do not need this assumption. For the theory of populational problem, this nonnegative assumption must be set for the kernel.

$$\beta \in C(R^+ \times R), \tag{5}$$
$$k(t,s;x) : cont.on[0,T] \times [0,T] \times \Sigma, \tag{6}$$
$$L(t,s;x) : cont.on[0,T] \times R^+ \times \Sigma, \tag{7}$$
$$|L(t,s;x) - 1| \longrightarrow 0 as T \longrightarrow 0, on 0 \leq t,s \leq T, x \in \Sigma. \tag{8}$$

Σ is defined by the following.

$$\Sigma = \{f | f \in C^+[0,T], \|f - \Phi\| < r, on[0,T]\},$$

where,

$$\Phi = \int_0^\infty \varphi(s)ds.$$

Theorem1

For the equations (3)(4),asume the basic hypotheses, and put Lipschitz continuous on the functional k, L for x. Then there exists a positive number T such that on the interval $[0, T]$, only one solution for (3)(4) exists.

Theorem2

For the equations (3)(4),asume the basic hypotheses. Then there exists a positive number T such that on the interval $[0, T]$, the solutions for (3)(4) exist..

We shall sketch the prooves for these theorems. For concerned integral equations φ is a initial functions. Hence we must look for the solutions near by the value Φ.

From the integral equation(4), we put

$$y(t) = B(x)(t) = \int_0^t \beta(t-s, x(t))k(t-s, t; x)y(s)ds + \int_0^\infty \beta(t+s, x(t))L(t, s; x)\varphi(s)ds.$$

There exists a positive number M, such that the inequality,

$$|B(x)(t)| \leq M \int_0^t |B(x)(s)|ds + M\Phi,$$

is satisfied. By using Gronwall inequality we can prove the following inequality.

$$|B(x)(t)| \leq Me^{Mt}.$$

We can think that the integral equation (4) as one operator for the solution x. Dfine the operator X by the following equation,

$$X(x)(t) = \int_0^t k(t-s, t; x)y(s)ds + \int_0^\infty L(t, s; x)\varphi(s)ds.$$

For this operator, we can apply the contoraction or, Schauder-Tychonoff fixed point theorem. Hence Theorem 1or 2 are established.

For proving Theorem1, the operator,

$$X(x)(\cdot) : \Sigma \longrightarrow \Sigma; contractive$$

must be satisfied. For this prove we must establish the next two inequalities.

$$\|X(x)(\cdot) - \Phi\| \leq r, \|X(x) - X(x')\| \leq \kappa\|x - x'\|, 0 < \kappa < 1.$$

These two inequalities will be proved by the evaluation the following three inequality by using the basic hypotheses. The positive number r can be calculated by same process.

$$\int_0^t |k(t-s, t; x) - k(t-s, t, ; x')||B(x)(s)|ds,$$
$$\int_0^t k(t-s, t; x')|B(x)(s) - B(x')(s)|ds,$$
$$\int_0^\infty |L(t, s; x) - L(t, s; x')|\varphi(s)ds.$$

For the proof on Therorem 2, we use the Shauder-Tychonoff fixed point thorem. By evaluation on the following three inequalities we can prove that operator$X(x)(\cdot)$ maps Σ into the set of equicontinuous functions.

$$\begin{aligned}|X(x)(t) - X(x)(t')| &\le \int_0^t k(t-s,s;x) - k(t'-s,t';x)||B(x)(s)|ds \\ &+ \int_t^{t'} |k(t'-s,t';x)B(x)(s)|ds \\ &+ \int_0^\infty |L(t,s;x) - L(t',s;x)|\varphi(s)ds.\end{aligned}$$

3. Kneser Type Theorem

If Schauder-Tychonoff type is established, there is the possibility that the integral equations have more than one solution. In this case we can consider Kneser type theorem.

Theorem3 (*Kneser*)
Assume the basic hypotheses on the functional integral equation (3)(4). Call the set of the graph of the solution set from the point P which belongs to the domain of the functional equation as $R(P)$, and call the cross section of $R(P)$ by the hypersurface $x = \xi$ as $S_\xi(P)$. Then $S_\xi(P)$ is continuum.

The proof of this theorem we esatablish that the solution set $F(P)$ with initial point P, which means the couple of the initial data for the solution (x, y), is continuum. This process is devided into four steps.
(1)$F(P)$ is totally compact and closed.
(2)Generally, for the decreasing series of compact and continuum set $\{C_\nu\}$, $C = \bigcup C_\nu$ is continuum.
(3)ϵ-asymptotic solution set $F(P;\epsilon)$ is continuum.
(4)$S_\xi(P)$ is continuum.
At first note that ϵ-approximate solution for the equation (3), (4), we can make the following process.

$$\begin{aligned}x_j(t) &= \Phi, 0 \le t \le \alpha/j, \\ y_j(t) &= \int_0^t \beta(t-s, x_j(t))k(t-s,s;x_j)y_j(s)ds \\ &+ \int_0^\infty \beta(t+s, x_j(t))L(t,s;x_j)\varphi(s)ds, 0 \le t \le \alpha/j, \\ x_j(t) &= \int_0^{t-\alpha/j} k(t-\alpha/j, s; x_j)y_j(s)ds \\ &+ \int_0^\infty L(t,s;x_j)\varphi(s)ds, \alpha/j < t \le \alpha, \\ y_j(t) &= \int_0^t \beta(t-s, x_j(t))k(t-s,s;x_j)y_j(s)ds \\ &+ \int_0^\infty \beta(t+s, x_j(t))L(t,s;x_j)\varphi(s)ds, \alpha/j < t \le \alpha.\end{aligned}$$

First step. Suppse that $(x_n, y_n) \in F(P)$ and $(x_n, y_n) \to (x, y)$, then from the hypotheses $(x, y) \in F(P)$. This fact proves that $F(P)$ is closed. Also We can prove that each series $\{(x_n, y_n)\} \subset F(P)$ is equicontinuous and equibounded. Then there exists a sub-sequence of $\{(x_n, y_n)\}$, which converges to one solution of $F(P)$. Hence first step was established. Second step is the general fact of topological theory.

Third step. We can make the ϵ-asymptotic solutions for every positive ϵ. The set of ϵ-asymptotic solutions are no empty. Note that $F(P) = \cap F(P;\epsilon_n)$. If $F(P;\epsilon_n)$ is continuum, by the step two $F(P)$ is also continuum. For every $\epsilon > 0$, choose sufficiently small $\delta > 0$ and choose $(x, y), (x', y') \in F(P;\epsilon)$ with $\rho((x, y), (x', y')) < \delta$, with supremun norm ρ. Let the interval $[0, T]$, where the solutions exist, divide into the subintervals on which we can make the ϵ-asymptotic solutions. Put $\xi \in [0, T]$, and call the point $(\xi, x(\xi), y(\xi)), (\xi, x'(\xi), y'(\xi))$ as Q and Q' respectively. Let (x_ξ, y_ξ) and (x'_ξ, y'_ξ) be ϵ-asymptotic solutions with initial points Q and Q' respectively. Dfine two ϵ-asymptotic solutions as follows. If $0 \le t \le \xi, (X_\xi(t), Y_\xi(t)) = (x(t), y(t)), (X'_\xi(t), Y'_\xi(t)) = (x'(t), y'(t))$, and if $\xi \le t \le T, (X_\xi(t), Y_\xi(t)) = (x_\xi(t), y_\xi(t)), (X'_\xi(t), Y'_\xi(t)) = (x'_\xi(t), y'_\xi(t))$. Then we can define the ϵ-asymptotic solution, $u_\xi(t) = (1-\lambda)X_\xi(t) + \lambda X'_\xi(t), v_\xi(t) = (1-\lambda)Y_\xi(t) + \lambda Y'_\xi(t), 0 \le \lambda \le 1$. If we change the value of λ from 0 to 1, (u_ξ, v_ξ) goes from (X_ξ, Y_ξ) to (X'_ξ, Y'_ξ) continuously. And if ξ moves from 0 to T, then (x, y) goes to (x', y') continuously. At last we can prove that the set of ϵ-asymptotic solutions is continuum.

The proof of the step four is same as usual thory of differential equation. Hence Kneser type theorem will be established.

Second Problem

4.Introduction

In the next part, we will present some stationary solution for nonlinear partial differential equation called Mullins Equation which is occered in the theory of grain boundary grooving.

$$u_t = -C_1^E(u)(1+u_x^2)^{1/2}exp(-C_2^E(u)\frac{u_{xx}}{(1+u_x^2)^{3/2}}) + C_1^C(u)(1+u_x^2)^{1/2}. \quad (1)$$

The main tool, which we can use, is the admissibility property between weighted continuous function spaces for the integral operator,as follows.

$$T_\xi x(t) = -\int_t^\infty e^{\zeta_1(t-s)}F(x(s), y(s))ds,$$

$$T_\xi y(t) = \xi e^{\zeta_2 t} + \int_0^t e^{\zeta_2(t-s)}F(x(s), y(s))ds. \quad (2)$$

From this admissibility we can prove the existence theorem for the special simultaneous differential equation. This existence theorem can be applied for the second order differential equation,

$$u'' = f(u, u') = \frac{kT(u)(1+u'^2)^{3/2}}{v\gamma}ln(\frac{P_0(u)}{P_c}). \quad (3)$$

The solution of this equation is one of the stationary solution for Mullins Equation.

5.Theorems

On the equation (1), we are interested in the stational solution. So we shall consider the equation (3) which we can make by putting $u_t = 0$ for the equation (1). To prove the existence theorem for the stational solution, we use the next two theorems.

Theorem1

For the second oreder differential equation,

$$u'' = f(u, u'), \quad (4)$$

suppose that the following hypotheses.

$$f(u,p) \in C^1(R^2), \quad x > 0, \quad {}^\exists\lambda \in R^1 \quad s.t \quad .f(\lambda, 0) = 0, \quad f_u(\lambda, 0) > 0$$

.

Then there exits the solution on $(0, \infty)$ and it satisfies that

$${}^\exists D > 0 \quad s.t. \quad |u(x) - \lambda| \le Dexp(-\tau x),$$

where

$$0 < \tau < \left| \frac{f_p(\lambda, 0) - \sqrt{f_p(\lambda, 0)^2 + 4f_u(\lambda, 0)}}{2} \right|.$$

Theorem2

On the differential equation,

$$\omega_1' = \zeta_1\omega_1 + F(\omega_1, \omega_2), \quad \omega_2' = \zeta_2\omega_2 + F(\omega_1, \omega_2), \quad x > 0,$$

where,

$$f(\eta_1, \eta_2) \in C^1(R^2), \quad F(0,0) = 0, \quad F_{\eta_1}(0,0) = 0, \quad \zeta_1 > 0, \zeta_2 < 0,$$

there exists some global nontrivial solution

$$\omega(x) = (\omega_1(x), \omega\ _2(x)), \quad x > 0,$$

for every $\tau, 0 < \tau < |\zeta_2|$, and the next inequality is satisfied.

$$|e^{\tau x}\omega_1(x)| + |e^{\tau x}\omega_2(x)| < \infty, \quad x > 0.$$

At first we consider Theorem2. By using the addmissibility of the integral operator(2), we can establish the proof of Theorem2. Let consider the integral operator on the following function set B,

$$B = \omega(x) = (\omega_1(x), \omega_2(x)) \in C^0([0, \infty)); ||\omega|| \leq 2|\xi|,$$

$$||\omega|| = \sup_{x \geq 0}(e^{\tau x}\omega_1(x) + e^{\tau x}\omega_2(x)).$$

On this set the integral operator(2) satisfies the contraction princeple. Then the operator $T_\xi : B \longrightarrow B$ has the unique fixes point $\omega(x) = (\omega_1(x), \omega_2(x))$. Hence we can prove Theorem2. Next we treat Theorem1, by using the results of Theorem2. Let define the function $F(\omega_1, \omega_2)$ in Theorem2 by the next equation,

$$F(\eta_1, \eta_2) = f(\frac{\eta_1 - \eta_2}{\zeta_1 - \zeta_2} + \lambda, \frac{\zeta_1\eta_1 - \zeta_2\eta_2}{\zeta_1 - \zeta_2}) - \frac{\eta_1 - \eta_2}{\zeta_1 - \zeta_2} f_u(\lambda, 0) - \frac{\zeta_1\eta_1 - \zeta_2\eta_2}{\zeta_1 - \zeta_2} f_p(\lambda, 0),$$

where

$$\zeta_1 = \frac{f_p(\lambda, 0) + \sqrt{f_p(\lambda, 0)^2 + 4f_u(\lambda, 0)}}{2} > 0,$$

$$\zeta_2 = \frac{f_p(\lambda, 0) - \sqrt{f_p(\lambda, 0)^2 + 4f_u(\lambda, 0)}}{2} < 0,$$

where the function f as in Theorem1. By the result of Theorem2 there exists the solution $\omega(x) = (\omega_1(x), \omega_2(x))$. Define

$$u(x) = \frac{\omega_1(x) - \omega_2(x)}{\zeta_1 - \zeta_2} + \lambda, \quad x > 0.$$

This function u is the solution in Theorem1. At last, we can apply Theorem1 for the equation (3), we get the stational solution of (1).

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