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Asymptotic forms of slowly decaying solutions of quasilinear ordinary differential equations with critical exponents

臨界的な場合における準線型常微分方程式の緩減衰解の漸近形

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1 Introduction and statement of the main result

Let us consider the quasilinear ordinary differential equation

$$(t^{\beta}|u'|^{\alpha-1}u')' + t^{\sigma}(1+o(1))|u|^{\lambda-1}u = 0, \quad \text{near } +\infty, \tag{A}$$

where $o(1)$ denotes a continuous function going to 0 as $t \to \infty$. Furthermore we assume that $\beta > \alpha > 0$, $\lambda > \alpha$ and $\sigma \in \mathbf{R}$. In what follows a positive C^1-function u defined near $+\infty$ is called positive solution of (A) if $t^{\beta}|u'|^{\alpha-1}u'$ is continuously differentiable and it satisfies (A).

Let u be a positive solution of (A). Since $t^{\beta}|u'|^{\alpha-1}u'$ is decreasing, it is shown [8, p.133] that every positive solution u of (A) satisfies one of the following three asymptotic properties as $t \to \infty$:

$$u(t) \sim c_1 \quad \text{for some constant } c_1 > 0; \tag{1.1}$$

$$u(t) \sim c_2 t^{-(\beta-\alpha)/\alpha} \quad \text{for some constant } c_2 > 0; \tag{1.2}$$

and

$$u(t) \to 0 \quad \text{and} \quad \frac{u(t)}{t^{-(\beta-\alpha)/\alpha}} \to \infty. \tag{1.3}$$

(Here and in the sequel the symbol "$f(t) \sim g(t)$ as $t \to \infty$" means that $\lim_{t\to\infty} f(t)/g(t) = 1$.) Qualitative properties of solutions satisfying (1.1) or (1.2) have been deeply investigated, because asymptotic forms of such solutions are explicitly given by definition. On the other hand, as far as the author knows, very little is known about asymptotic forms of solutions satisfying (1.3). Motivated by this fact, we have been studying on this subject. In the talk, we refer positive solutions u satisfying (1.3) as *slowly decaying solutions.*

When

$$(\beta-\alpha) - 1 < \sigma < \frac{\lambda}{\alpha}(\beta-\alpha) - 1,$$

it is shown [5] that, under suitable conditions on the term "$o(1)$", every slowly decaying solutions u of (A) has the asymptotic form

$$u(t) \sim Ct^{-\gamma} \quad \text{as} \quad t \to \infty$$

for some constants $C = C(\alpha, \beta, \lambda, \sigma) > 0$ and $\gamma = \gamma(\alpha, \beta, \lambda, \sigma)$, $0 < \gamma < (\beta - \alpha)/\alpha$. (These constants can be written down explicitly; see [5].) Accordingly, in this talk we consider equation (A) for the critical case $\sigma = \beta - \alpha - 1$; that is, we will treat the equation:

$$(t^{\beta}|u'|^{\alpha-1}u')' + t^{\beta-\alpha-1}(1+\varepsilon(t))|u|^{\lambda-1}u = 0, \quad \text{near } +\infty. \tag{E}$$

The following conditions are assumed throughout the talk:

(A_1) $\lambda > \alpha > 0$ and $\beta > \alpha$ are positive constants;
(A_2) $\varepsilon(t)$ is a C^1-function satisfying $\lim_{t\to\infty} \varepsilon(t) = 0$.

Remark 1.1. When condition (A_1) is replaced by $0 < \lambda < \alpha$ and $\beta > \alpha$, asymptotic forms of slowly decaying solutions of (E) have been obtained completely [6].

Equations of the form (E) appear in the study of quasilinear elliptic equations as seen below.

Example 1.2. Let $N > m > 1$ and $\lambda > m - 1$. Consider the following elliptic equation near ∞ of $\mathbf{R}^N$:

$$\operatorname{div}(|Du|^{m-2}Du) + |x|^{-m}(1 + o(1))u^{\lambda} = 0.$$

Here $o(1)$ denotes a radial smooth function going to 0 at ∞. Radial solutions $u = u(r), r = |x|$, satisfy the ODE

$$(r^{N-1}|u_r|^{m-2}u_r)_r + r^{N-m-1}(1 + o(1))u^{\lambda} = 0 \quad \text{near} \quad +\infty, \tag{1.4}$$

which is of the form (E). A solution $u(r)$ is a slowly decaying solution if

$$u(r) \to 0 \quad \text{and} \quad r^{\frac{N-m}{m-1}} u(r) \to \infty \quad \text{as} \quad r \to +\infty.$$

To state the results we must introduce some notation. Put $\rho = \alpha/(\lambda - \alpha)$ and $A = [\rho^{\alpha}(\beta - \alpha)]^{1/(\lambda-\alpha)}$. Define

$$u_0(t) = A(\log t)^{-\rho}. \tag{1.5}$$

We note that u_0 is a slowly decaying solution of an ODE of the form (E) with some $\varepsilon(t)$. Hence we conjecture that slowly decaying solutions of equation (E) may behave like $u_0(t)$ under suitable conditions on $\varepsilon(t)$. We can answer this conjecture affirmatively. This is the main result of the talk:

Theorem 1.3. *Let $\alpha \geq 1$ and*

$$\int^{\infty} |\varepsilon'(t)| dt < \infty. \tag{1.6}$$

Then every slowly decaying solution u of equation (E) *has the asymptotic form $u(t) \sim u_0(t)$ as $t \to +\infty$, where u_0 is given by* (1.5).

Theorem 1.3 enable us to determine the asymptotic form of slowly decaying solution of equation (1.4):

Example 1.4. Consider equation (1.4) with "$o(1) = r^{-\tau}, \tau = \text{const} > 0$." and $m \geq 2$:

$$(r^{N-1}|u_r|^{m-2}u_r)_r + r^{N-m-1}(1 + r^{-\tau})u^{\lambda} = 0 \quad \text{near} \quad +\infty.$$

Theorem 1.3 asserts that every slowly decaying solution u of this equation has the asymptotic form

$$u(r) \sim B(\log r)^{-\frac{m-1}{\lambda-m+1}} \quad \text{as} \quad r \to \infty,$$

where $B = B(N, m, \lambda) > 0$ is a constant.

Remark 1.5. (i) Existence results of slowly decaying solutions of equation (1.4) are discussed in [7].

(ii) When $m = 2$, the asymptotic forms of slowly decaying solutions of equation (1.4) are obtained in [9].

(iii) The reader may have a question: For the case $\sigma = \frac{\lambda}{\alpha}(\beta - \alpha) - 1$, the other critical case, how do slowly decaying solutions of equation (E) behave? However, in this case equation (A) does not have positive solutions at all [8].

(iv) Related results are found in [1, 2, 3, 4, 8].

2 Proof of the main results

To see Theorem 1.3 we must give several preparatory considerations.

Lemma 2.1. *Let u be a slowly decaying solution of equation* (E). *Then the following statements hold:*

(i) $\lim_{t\to\infty} t^{\beta}|u'(t)|^{\alpha} = \infty$;

(ii) $u(t) = O(u_0(t))$ *and* $u'(t) = O(|u_0'(t)|)$ *as* $t \to \infty$.

Proof. (i) Since $t^{\beta}|u'|^{\alpha-1}u'$ decreases, $\lim_{t\to\infty} t^{\beta}|u'|^{\alpha-1}u' \in [-\infty, \infty)$ exists. If this limit is finite, it is easily seen that u satisfies (1.1) or (1.2).

(ii) An integration of the both sides of equation (E) on $[t_0, t]$ gives

$$t^\beta(-u'(t))^\alpha \geq \int_{t_0}^t r^{\beta-\alpha-1}(1+\varepsilon(r))u^\lambda dr,$$

where t_0 is a sufficiently large number. Since u is a decreasing function, we have

$$t^\beta(-u'(t))^\alpha \geq u(t)^\lambda \int_{t_0}^t r^{\beta-\alpha-1}(1+\varepsilon(r))dr;$$

that is,

$$-u'(t)u(t)^{-\lambda/\alpha} \geq \left(t^{-\beta}\int_{t_0}^t r^{\beta-\alpha-1}(1+\varepsilon(r))dr\right)^{1/\alpha}.$$

One more integration of the both sides gives the estimates for u.

To get the estimates for u', it suffices to notice the inequality

$$t^\beta(-u'(t))^\alpha \leq C_1 \int_{t_0}^t r^{\beta-\alpha-1}u(r)^\lambda dr,$$

where $C_1 > 0$ is a constant. Note that, to get this inequality, we must use (i).

Lemma 2.2. *Let u be a slowly decaying solution of equation* (E). *Introduce the change of variables $t = e^s$ and $v(s) = (\log t)^\rho u(t)$, and put $\delta(s) = \varepsilon(e^s)$ and $\cdot = d/ds$. Then, we have the following statements near $+\infty$:*

(i) $\rho v - s\dot{v} > 0$;

(ii) $v(s) = O(1)$, *and* $\dot{v}(s) = O(s^{-1})$ *as* $s \to \infty$;

(iii) $v(s)$ *satisfies the ODE*

$$\alpha s\ddot{v} + \{(\beta-\alpha)s - 2\alpha\}\dot{v} - \left(\rho(\beta-\alpha) - \frac{\alpha\rho(\rho+1)}{s}\right)v$$
$$+(\rho v - s\dot{v})^{1-\alpha}(1+\delta(s))v^\lambda = 0. \tag{2.1}$$

In the sequel, for simplicity, we often rewrite (2.1) as

$$\alpha s\ddot{v} + (A_1 s - A_2)\dot{v} - \left(B_1 - \frac{B_2}{s}\right)v + (\rho v - s\dot{v})^{1-\alpha}(1+\delta(s))v^\lambda = 0. \tag{2.2}$$

Here A_1, A_1, B_1 and $B_2 > 0$ are appropriate constants defined by (2.1) and (2.2).

The statement of (i) of Lemma 2.2 is equivalent to $u'(t) < 0$. The estimates in (ii) of Lemma 2.2 are direct consequences of (ii) of Lemma 2.1.

Lemma 2.3. *Let $\alpha \geq 1$ and $v(s)$ be as in Lemma 2.2. Then, $\int^\infty s\dot{v}(s)^2 ds < \infty$.*

Proof. Since $\alpha \geq 1$, we get $(\rho v - s\dot{v})^{1-\alpha}\dot{v} \geq \rho^{1-\alpha}v^{1-\alpha}\dot{v}$. Therefore, multiplying equation (E) by $\dot{v}$, we have

$$\alpha s\ddot{v}\dot{v} + (A_1 s - A_2)\dot{v}^2 - B_1 v\dot{v} + \frac{B_2}{s}v\dot{v} + \rho^{1-\alpha}(1+\delta(s))v^{1+\lambda-\alpha}\dot{v} \leq 0. \tag{2.3}$$

Note that condition (1.6) is equivalent to the condition $\int^\infty |\dot{\delta}(s)|ds < \infty$. An integration of (2.3) on the interval $[s_0, s]$ with s_0 being a constant, gives

$$\frac{\alpha}{2}s\dot{v}^2 + \int_{s_0}^{s}\left(A_1 r - A_2 - \frac{\alpha}{2}\right)\dot{v}^2 dr - \frac{B_1}{2}v^2 + B_2\int_{s_0}^{s}\frac{v\dot{v}}{r}dr + \frac{\rho^{1-\alpha}}{2+\lambda-\alpha}v^{2+\lambda-\alpha} \leq \text{Const.}$$

By using (ii) of Lemma 2.2, we can show this lemma.

Outline of the proof of Theorem 1.3. Let $v(s)$ be as in Lemma 2.2. It suffices to show that $\lim_{t\to\infty} v(s) = A$. Introduce the auxiliary function $\Phi(s)$ by

$$\Phi(s) = \left[\frac{\rho^{\alpha-1}}{1+\delta(s)}\left(B_1 - \frac{B_2}{s}\right)\right]^{1/(\lambda-\alpha)}.$$

Then, $\lim_{s\to\infty}\Phi(s) = A$, and $\Phi(s)$ has the following important properties:

In the region $0 < v < \Phi(s)$ [*resp.* $v > \Phi(s)$], *the solution curve*

$v = v(s)$ *attains only local minimums* [*resp. local maximums*]. (2.4)

As the first step, we show that $\lim_{s\to\infty} v(s) \in [0,\infty)$ exists as a finite nonnegative number. Put $\underline{L} = \liminf_{s\to\infty} v(s)$ and $\overline{L} = \limsup_{s\to\infty} v(s)$. To see this claim we suppose to the contrary that $0 \leq \underline{L} < \overline{L}$. Let us introduce the auxiliary function $F(v), v \geq 0$, by

$$F(v) = \frac{\rho^{1-\alpha}}{2+\lambda-\alpha}v^{2+\lambda-\alpha} - \frac{B_1}{2}v^2.$$

On the interval $[0, A], F(v)$ decreases; on the interval $[A, \infty), F(v)$ increases. So the only global minimum of $F(v), v \geq 0$, is attained at $v = A$. The proof is divided into several cases according to the order relations between $\underline{L}, \overline{L}$ and A. For example, let us suppose that $A < \underline{L} < \overline{L}$. But it is impossible because of the property (2.4). Next, suppose that $\underline{L} \leq A < \overline{L}$. Then, we can find two sequences $\{\overline{s}_n\}$ and $\{\xi_n\}$ satisfying

$$\xi_n < \overline{s}_n < \xi_{n+1} < \overline{s}_{n+1} < \cdots, \lim_{n\to\infty}\overline{s}_n = \lim_{n\to\infty}\xi_n = \infty; \quad \text{and}$$

$$\dot{v}(\overline{s}_n) = 0; v(\xi_n) \equiv \Phi(\xi_n); \lim_{n\to\infty} v(\overline{s}_n) = \overline{L}(> A).$$

Integrating (2.3) on $[\xi_n, \overline{s}_n]$, we find that

$$F(v(\overline{s}_n)) \leq F(A) + o(1) \quad \text{as} \quad n \to \infty.$$

Here we have used Lemmas 2.2 and 2.3. Letting $n \to \infty$, we have $F(\overline{L}) \le F(A)$; that is, $\overline{L} = A$, a contradiction. The other cases can be treated similarly.

Secondly we claim that $\lim_{s\to\infty} v(s) > 0$. The proof of this fact is done by contradiction. Suppose to the contrary that $\lim_{s\to\infty} v(s) = 0$. Then $v(s)$ decreases to 0 by (2.4). We rewrite equation (2.2) in the form

$$\ddot{v} + \left(\frac{A_1}{\alpha} - \frac{A_2}{\alpha s}\right)\dot{v} \equiv h(s, v(s), \dot{v}(s)) \equiv h(s).$$

By the variation of constant formula we have

$$v(s) = c_1 + c_2\psi(s) + \int_{s_0}^{s} r^{-\frac{A_2}{\alpha}} e^{\frac{A_1}{\alpha}r}\psi(r)h(r)dr - \psi(s)\int_{s_0}^{s} r^{-\frac{A_2}{\alpha}} e^{\frac{A_1}{\alpha}r} h(r)dr, \tag{2.5}$$

where c_1 and c_2 are some constants, and $\psi(s) = \int_s^\infty r^{A_2/\alpha} e^{-A_1 r/\alpha} dr$. We find that $\psi(s) \sim c_3 s^{A_2/\alpha} e^{-A_1 s/\alpha}$ for some constant $c_3 > 0$. Furthermore, by using condition $\alpha \ge 1$, we can estimate $h(s)$ as follows:

$$\alpha h(s) \ge \frac{v}{s}\left\{\left(B_1 - \frac{B_2}{s}\right) - \rho^{1-\alpha}(1+\delta(s))v^{\lambda-\alpha}\right\}.$$

Here the reader recall that $\dot{v}(s) \le 0$ and the inequality $(\rho v - s\dot{v})^{1-\alpha} \le \rho^{1-\alpha}v^{1-\alpha}$. This estimate implies that $h(s) \ge 0$, and $\lim_{s\to\infty} h(s) = 0$. Therefore the last term on the right hand side of (2.5) tends to 0 as $s \to \infty$. From these facts and the boundedness of $v(s)$ and $\psi(s)$, we find that the first integral on the right hand side of (2.5) converges as $s \to \infty$. Therefore, (2.5) can be rewritten in the form

$$v(s) = c_4\psi(s) - \int_{s}^{\infty} r^{-\frac{A_2}{\alpha}} e^{\frac{A_1}{\alpha}r}\psi(r)h(r)dr - \psi(s)\int_{s_0}^{s} r^{-\frac{A_2}{\alpha}} e^{\frac{A_1}{\alpha}r} h(r)dr, \tag{2.6}$$

for some constant $c_4 > 0$. Since $h(s) \ge 0$, this formula implies that $v(s) = O(\psi(s))$ as $s \to \infty$. Returning to the original t-variable, this is equivalent to $u(t) = O(t^{(\beta-\alpha)/\alpha}(\log t)^\rho)$ as $t \to \infty$. Finally, returning to equation (E), we have $u(t) = O(t^{(\beta-\alpha)/\alpha})$ as $t \to \infty$. This is an obvious contradiction to the definition of slowly decaying solution. Hence $0 < \lim_{s\to\infty} v(s) < \infty$.

Finally, we must show that $\lim_{s\to\infty} v(s) = A$; or equivalently $\lim_{t\to\infty} u(t)/u_0(t) = 1$. Put $L = \lim_{t\to\infty} u(t)/u_0(t) \in (0,\infty)$. Since $u(t), u_0(t) \to 0$, L'Hôspital's rule implies that $L = \lim_{t\to\infty} u'(t)/u_0'(t)$. Repeating this procedure, we find that

$$\begin{aligned} L &= \lim_{t\to\infty}\left(\frac{t^\beta(-u'(t))^\alpha}{t^\beta(-u_0'(t))^\alpha}\right)^{1/\alpha} = \lim_{t\to\infty}\left(\frac{(t^\beta(-u'(t))^\alpha)'}{(t^\beta(-u_0'(t))^\alpha)'}\right)^{1/\alpha} \\ &= \lim_{t\to\infty}\left(\frac{-t^{\beta-\alpha-1}(1+\varepsilon(t))u(t)^\lambda}{-t^{\beta-\alpha-1}(1+o(1))u_0(t)^\lambda}\right)^{1/\alpha} = L^{\lambda/\alpha}. \end{aligned}$$

Here we have used the fact that $u_0(t)$ satisfies an ODE of the form (E). Since $L \ne 0$, we get $L = 1$. This completes the proof of Theorem 1.3.

References

[1] M.-F. Bidaut-Véron, Local and global behavior of solutions of quasilinear equations of Emden-Fowler type, Arch. Rat. Mech. Anal., 107 (1988), 293-324.

[2] T. A. Chanturia and I. T. Kiguradze, Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations, Kluwer Academic Publishers, Dordrecht, 1993.

[3] M. Guedda and L. Véron, Local and global properties of solutions of quasilinear elliptic equations, J. Differential Equations, 76 (1988), 159-189.

[4] K. Kamo and H. Usami, Asymptotic forms of weakly increasing positive solutions for quasilinear ordinary differential equations, Electron. J. Differential Equations, 2007(2007), No.126, 1-12.

[5] K. Kamo and H. Usami, Asymptotic forms of slowly decaying positive solutions of second-order quasilinear ordinary differential equations, RIMS Kôkyûroku 1582 (2008) *Modeling and Complex Analysis for Functional Equations*, 43-52. (数理解析研究所講究録 1582, 「関数方程式論におけるモデリングと複素解析」)

[6] K. Kamo and H. Usami, Characterization of slowly decaying positive solutions of second-order quasilinear ordinary differential equations with sub-homogeneity (preprint)

[7] N. Kawano, E. Yanagida and S. Yotsutani, Structure theorems for positive radial solutions to $\operatorname{div}(|Du|^{m-2}Du)+K(|x|)u^q=0$ in $\mathbf{R}^n$, J. Math. Soc. Japan, 45 (1993), 719-742.

[8] T. Kusano, A. Ogata and H. Usami, Oscillation theory for a class of second order quasilinear ordinary differential equations with application to partial differential equations, Japan. J. Math., 19 (1993), 131-147.

[9] Y. Li, Asymptotic behavior of positive solutions of equation $\Delta u + K(x)u^p = 0$ in $\mathbf{R}^n$, J. Differential Equations, 95 (1992), 304-330.

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