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Kyoto University
On Floating Body Problem

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This paper consists of two parts. In Sections 1, 2 and 3, I give my recent work \cite{8} of 2-dimensional floating bodies. In Section 4, I give two important formulas of 3-dimensional floating bodies. These formulas are already known, maybe, but I do not know where the statements and the proofs are given. In Section 5, we apply the result of Section 4 to Ulam's problem.

1 Ulam's Floating Body Problem of Two Dimension

S. M. Ulam posed a problem: If a body of uniform density floats in water in equilibrium in every direction, must it be a sphere? See \cite{3} or \cite{9} for detail. The problem is still open. However, in two dimensional case of the problem, Auerbach \cite{1} gives a counter-example.

Theorem 1. \((\cite{1})\) There is a non-circular figure \(D \subset \mathbb{R}^2\) of density \(\rho = 1/2\) which floats in equilibrium in every direction.

Before we state our result, we define some terminology of two-dimensional floating bodies. Consider a figure \(D \subset \mathbb{R}^2\) whose perimeter \(\partial D\) is a simple closed curve, and take a number \(0 < \rho < 1\). For a given angle \(0 \leq \theta < 2\pi\), there is a directed line \(L_{\theta}\) of slope angle \(\theta\) which divides the area of \(D\) in the ratio \(\rho : 1 - \rho\). In this paper, we assume the following three conditions:

(C1) \(\partial D\) is of class \(C^1\).

(C2) \(L_{\theta}\) meets \(\partial D\) at exactly two points, say, \(P\) and \(Q\).

(C3) Neither the tangent at \(P\) nor at \(Q\) is not parallel to the line \(PQ\).

We call \(\rho\) the density of \(D\), and the segment \(PQ\) the water line of slope angle \(\theta\). We denote by \(D_u\) and \(D_a\) the divided figures of area ratio \(\rho : 1 - \rho\). We call \(D_u\) and \(D_a\) the underwater and above-water parts of \(D\), respectively. We denote by \(G_u\) and \(G_a\) the centroids of \(D_u\) and \(D_a\), respectively. We say that \(D\) floats in equilibrium in direction \(e_2(\theta) = (-\sin \theta, \cos \theta)\) if the line \(G_u, G_a\) is parallel to \(e_2(\theta)\).

If the figure \(D\) of density \(\rho\) floats in equilibrium in every direction, we call \(D \subset \mathbb{R}^2\) an Auerbach figure of an Auerbach density \(\rho\). It is known that, if \(D \subset \mathbb{R}^2\) is an Auerbach figure, then the water surface divides \(\partial D\) in constant ratio, say, \(\sigma : 1 - \sigma\). See (ii) of Corollary 7. We call \(\sigma\) the perimetal density of the Auerbach figure \(D\).

If \(D\) is an Auerbach figure of density \(\rho = 1/2\), then the water lines \(L_{\theta}\) and \(L_{\theta+\pi}\) are the same but opposite directed lines. Thus it is of perimetal density \(\sigma = 1/2\). In the proof of Theorem 1, the condition \(\rho = 1/2\) is essential. It is difficult to make an Auerbach figures of density \(\rho \neq 1/2\). So a question arises: Is there a non-circular Auerbach figure of density \(\rho \neq 1/2\)?

Recently, Wegner \cite{10} gave an positive answer to this question. Wegner's examples exhibit more interesting fact. That is, for given integer \(p \geq 3\), one of his examples has \((p-2)\) different Auerbach densities. So one Auerbach figure can have many perimetal densities.

On the other hand, Bracho, Montejano and Olibes \cite{2} gave a following result.

Theorem 2. \((\cite{2})\) If there is an Auerbach figure \(D \subset \mathbb{R}^2\) of perimetal density \(\sigma = 1/3\) or \(1/4\), then it is a circle.

The purpose of the first part of this paper is to prove the following theorem.
Theorem 3. (1) If an Auerbach figure $D \subset \mathbb{R}^2$ has three perimetral densities $\sigma_1$, $\sigma_2$, and $\sigma_3$, and if $\sigma_1 + \sigma_2 + \sigma_3 = 1$, then it is a circle. (These $\sigma_i$'s are not necessarily different.)

(2) If an Auerbach figure $D \subset \mathbb{R}^2$ has four perimetral densities $\sigma_1$, $\sigma_2$, $\sigma_3$, and $\sigma_4$, and if $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 1$, then it is a circle. (These $\sigma_i$'s are not necessarily different.)

The above theorem is a generalization of Theorem 2. Certainly, putting $\sigma_1 = \sigma_2 = \sigma_3 = 1/3$ gives the 1/3 case of Theorem 2, and putting $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1/4$ gives the 1/4 case of Theorem 2.

2 Auerbach Figures

In this section, we give a short survey of Auerbach figures.

Theorem 4. ([1], [10]) If a figure $D \subset \mathbb{R}^2$ is Auerbach, then the water line is of constant length.

We give a proof of the above theorem in Section 5.

Theorem 5. If a figure $D \subset \mathbb{R}^2$ is Auerbach, and if $PQ$ is the water line of slope angle $\theta$, then there is a 2$\pi$-periodic function $f$ of class $C^2$ such that the position vectors of $P$ and $Q$ are given by

$$p(\theta) = -f(\theta)e_2(\theta) + (f'(\theta) - l)e_1(\theta), \quad q(\theta) = -f(\theta)e_2(\theta) + (f'(\theta) + l)e_1(\theta),$$

(1) where $e_1(\theta) = (\cos \theta, \sin \theta)$, $e_2(\theta) = (-\sin \theta, \cos \theta)$, and $l$ is half the length of $PQ$.

Proof. Assume that $D$ is an Auerbach figure. Then by Theorem 4, the waterline is of constant length. Since $\{e_1(\theta), e_2(\theta)\}$ is a basis of $\mathbb{R}^2$, we can represent the position vectors of the points $P$ and $Q$ as follows:

$$p(\theta) = -f(\theta)e_2(\theta) + g(\theta)e_1(\theta), \quad q(\theta) = -f(\theta)e_2(\theta) + (g(\theta) + 2l)e_1(\theta).$$

(2)

Suppose that the chord $P^*Q^*$ of $C$ is the water line of slope angle $\theta + h$. Then the position vector of the intersection $H$ of the chords $PQ$ and $P^*Q^*$ are given by

$$\overrightarrow{OH} = -f(\theta)e_2(\theta) + \lambda e_1(\theta) = -f(\theta + h)e_2(\theta + h) + \mu e_1(\theta + h).$$

(3)

By taking the inner product of (3) and $e_2(\theta + h)$, we have that $f(\theta + h) = \lambda \sin h + f(\theta) \cos h$. Thus we obtain that

$$f'(\theta) = \frac{f(\theta + h) - f(\theta)}{h} + o(1) = \lambda \frac{\sin h}{h} - f(\theta) \frac{1 - \cos h}{h} + o(1) = \lambda + o(1).$$

(4)

We can evaluate the areas of the sectors $HPP^*$ and $HQQ^*$ by

$$\frac{1}{2} HP^2 h + o(h) = \frac{1}{2} | g(\theta) - f'(\theta) |^2 h + o(h) \quad \text{and}$$

$$\frac{1}{2} HQ^2 h + o(h) = \frac{1}{2} | g(\theta) - f'(\theta) + 2l |^2 h + o(h),$$

(5)

respectively. Since these two areas are equal, we obtain that $g(\theta) = f'(\theta) - l$. Hence we have proved (1).

By taking the inner product of (1) and $e_1(\theta)$, we have that $f'(\theta) = p(\theta) \cdot e_1(\theta) + l$. Thus the function $f(\theta)$ is of class $C^2$.

The following result is a "proof" of Theorem 1.

Example 6. Put $f(\theta) = -k \cos 3\theta$ in Equation (1). Then the curve is rotational symmetric with respect to the angle $2\pi/3$. So it surrounds an Auerbach figure of density $1/2$. See Theorem 11. The figures of $k/l = 0.03$ and $k/l = 0.1$ are drawn as follows:
The following result gives geometric properties of Auerbach figures.

**Corollary 7.** If a figure $D \subset \mathbb{R}^2$ is Auerbach, and if $PQ$ is the water line of slope angle $\theta$, then:

(i) The vectors $p'(\theta)$ and $q'(\theta)$ are symmetric with respect to the line $PQ$.

(ii) The arc $PQ$ of $\partial D$ is of constant length.

**Proof.** By differentiating (1), we have that

$$p'(\theta) = s(\theta)e_1(\theta) - le_2(\theta), \quad q'(\theta) = s(\theta)e_1(\theta) + le_2(\theta), \quad (6)$$

where $s(\theta) = f(\theta) + f''(\theta)$. Since the line $PQ$ is parallel to the vector $e_1(\theta)$, we have proved (i).

(ii) By (6), we have that $|p'(\theta)| = |q'(\theta)| = \sqrt{s(\theta)^2 + l^2}$. This implies that the points $P$ and $Q$ move at the same speed along $\partial D$. Thus we have proved (ii). \(\blacksquare\)

**Remark.** By integrating (6), we have that

$$p(\theta) = c + \int_0^\theta s(\phi)e_1(\phi) \, d\phi - le_1(\theta), \quad q(\theta) = c + \int_0^\theta s(\phi)e_1(\phi) \, d\phi + le_1(\theta), \quad (7)$$

where $c$ is a constant vector. These formulas are same as those given in Section 2 of [10].

### 3 Proof of Theorem 3

**Proof of Theorem 3.** (i) Let $P_1$, $P_2$, and $P_3$ be three points of $\partial D$ such that for each $i = 1, 2, 3$, the line $P_iP_{i+1}$ can be a water surface of perimetral density $\sigma_i$. (The indices are taken cyclic in modulo 3.) For each $i = 1, 2, 3$, we denote by $p_i(\theta)$ the position vector of $P_i$, by $x_i$ the angle $\angle P_{i-1}P_iP_{i+1}$ and by $\alpha_i$ the angle between $p_i'(\theta)$ and $P_iP_{i+1}$. By (i) of Corollary 7, the angle between $P_iP_{i-1}$ and $p_i'(\theta)$ is equal to $\alpha_i$. So we obtain that $x_1 + \alpha_3 + \alpha_1 = \pi$, $x_2 + \alpha_1 + \alpha_2 = \pi$ and $x_3 + \alpha_2 + \alpha_3 = \pi$. Since $x_1 + x_2 + x_3 = \pi$, we have that $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. See the figure below left. So we obtain that $\alpha_1 = x_3$. By the converse of Alternate Segment Theorem, $p_i'(\theta)$ tangents to the circumcircle of the triangle $P_1P_2P_3$. Thus $P_1$ varies on the circumcircle. Hence $D$ is a circle.

(ii) Let $P_1$, $P_2$, $P_3$ and $P_4$ be four points of $\partial D$ such that for each $i = 1, 2, 3, 4$, the line $P_iP_{i+1}$ can be a water surface of perimetral density $\sigma_i$. (The indices are taken cyclic in modulo 4.) By the same notation and argument used in (i) of this theorem, we obtain that $x_1 + \alpha_4 + \alpha_1 = \pi$, $x_2 + \alpha_1 + \alpha_2 = \pi$, $x_3 + \alpha_2 + \alpha_3 = \pi$ and $x_4 + \alpha_3 + \alpha_4 = \pi$. See the figure below right. Since $x_1 + x_2 + x_3 + x_4 = 2\pi$, we have that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \pi$. So we obtain that $x_1 + x_3 = \pi$. By the converse of Inscribed Quadrangle Theorem, the quadrangle $P_1P_2P_3P_4$ inscribes to a circle. Thus $P_3P_1$ is of constant length, and therefore, it can be a water line of perimetral density $\sigma_3 + \sigma_4$. Hence, by (i) of this theorem, $D$ is a circle. \(\blacksquare\)
4 On formulas for 3-dimensional floating bodies

Suppose that a solid $\mathcal{F} \subset \mathbb{R}^3$ has a volume $V$ and a uniform density $\rho$ ($0 < \rho < 1$) and that it floats on water so that a unit vector $\mathbf{n} \in \mathbb{R}^3$ is the vertical and upward direction. The water surface is a plane which is orthogonal to $\mathbf{n}$ and cuts $\mathcal{F}$ in ratio $1 - \rho : \rho$. By Archimedes Principle, the volume of the undersurface part of $\mathcal{F}$ is equal to $\rho V$. In the paper, we assume that the boundary of $\mathcal{F}$ is sufficiently differentiable and do not tangent to the water surface. From now, we consider that $\mathcal{F}$ is fixed and $\mathbf{n}$ varies, that is, we consider that $\mathcal{F}$ is not inclined but the water surface is inclined. We call $\mathbf{n}$ the vertical vector and orthogonal vectors to $\mathbf{n}$ horizontal vectors.

We can solve the stability problem of floating bodies by analyzing the relation between the positions of its center of gravity and its center of buoyancy. The center of buoyancy is the center of gravity of the underwater part of the floating body. The center of gravity $G$ does not depend on $\mathbf{n}$, but the center of buoyancy $B$ is a function of $\mathbf{n}$.

The gravity acts vertically downward on $G$ and the buoyancy acts vertically upward on $B$. If $G$ and $B$ do not lie on same vertical line, then the floating body rotates. We say that $\mathcal{F}$ is in equilibrium state with respect to the direction $\mathbf{n}$ if $G$ and $B$ lie on the same vertical line, that is, $(B - G) \cdot \mathbf{u} = 0$ for every horizontal vector $\mathbf{u}$.

Suppose that the floating body $\mathcal{F}$ is in equilibrium state with respect to $\mathbf{n}_0$. Then fix a horizontal vector $\mathbf{t}$ and consider the vector $\mathbf{u}_0$ which is made by the vector product $\mathbf{u}_0 = \mathbf{n}_0 \times \mathbf{t}$. Rotate the vectors $\mathbf{u}_0$ and $\mathbf{n}_0$ counterclockwise with respect to $\mathbf{t}$, that is,

$$\mathbf{u} = \mathbf{u}_0 \cos \theta + \mathbf{n}_0 \sin \theta, \quad \mathbf{n} = -\mathbf{u}_0 \sin \theta + \mathbf{n}_0 \cos \theta. \quad (8)$$

It means that the water line inclines counterclockwise by the angle $\theta$, that is, the floating body inclines clockwise by the angle $\theta$. We call $\mathbf{t}$ the vector of rotation axis.

Assume that the floating body inclines by a sufficiently small angle $\theta$. If the function $F(\theta) = (B - G) \cdot \mathbf{u}$ is monotone increasing, then the gravity and buoyancy act as the floating body returns to the original state. If the function $F(\theta)$ is monotone decreasing, then the gravity and buoyancy act as the floating body inclines more. So we define as follows:

**Definition.** We say that a floating body $\mathcal{F} \subset \mathbb{R}^3$ in equilibrium is stable (unstable) with respect to a rotation axis $\mathbf{t}$ if $F(\theta)$ is monotone increasing (decreasing). We say that $\mathcal{F}$ is stable if it is stable with respect to every rotation axis $\mathbf{t}$.

We denote by $E$ the cross section of $\mathcal{F}$ cut by the water surface. We make a coordinate plane whose origin is the center of gravity of $E$ and $x$- and $z$-axis have the same direction as $\mathbf{u}$ and $\mathbf{t}$, respectively. Then we can regard $E$ as the figure in the $xz$-plane. Consider the following quantity:

$$I(\mathbf{n}, \mathbf{t}) = \int_E x^2 \, dx \, dz. \quad (9)$$
We call it the moment of inertia with respect to \( \mathbf{n} \) and \( t \). From now, we fix vectors \( \mathbf{n}_0 \) and \( t \), and consider the moment of inertia as a function of \( \theta \), that is, we denote \( I(\theta) = I(n, t) \).

Suppose that a floating body \( \mathcal{F} \) is in equilibrium with respect to the direction \( \mathbf{n}_0 \). Then the following formula holds:

\[
F(\theta) = \int_0^\theta \left( \frac{I(\phi)}{\rho V} - \overline{GB}_0 \right) \cos(\theta - \phi) \, d\phi,
\]

(\text{F1})

where \( B_0 \) is the center of buoyancy of \( \mathcal{F} \) with respect to the direction \( \mathbf{n}_0 \). Remark that \( \overline{GB}_0 \) is a distance between \( G \) and \( B_0 \), and so, a constant number. We can approximate (F1) as follows:

\[
F(\theta) = \left( \frac{I(0)}{\rho V} - \overline{GB}_0 \right) \theta + O(\theta^2).
\]

(10)

By using (10), we deduce the following corollary:

**Corollary 8.** If \( I(0) > \rho V \overline{GB}_0 \), then \( \mathcal{F} \) is stable with respect to the axis \( t \). If \( I(0) < \rho V \overline{GB}_0 \), then \( \mathcal{F} \) is unstable with respect to the axis \( t \).

Set \( U(\theta) = (G - B) \cdot \mathbf{n} \). We call it the potential function of the floating body \( \mathcal{F} \). Suppose that a floating body \( \mathcal{F} \) is in equilibrium with respect to the direction \( \mathbf{n}_0 \). Then the following formula holds:

\[
U(\theta) = \int_0^\theta \left( \frac{I(\phi)}{\rho V} - \overline{GB}_0 \right) \sin(\theta - \phi) \, d\phi
\]

(\text{F2})

By (F1) and (F2), we find that

\[
U'(\theta) = F(\theta), \quad F'(\theta) = -U(\theta) + \frac{I(\theta)}{\rho V} - \overline{GB}_0.
\]

(11)

By the first formula of (11), we obtain the following corollary:

**Corollary 9.** The floating body \( \mathcal{F} \) is stable with respect to the axis \( t \) if and only if the function \( U(\theta) \) takes a local minimum at \( \theta = 0 \). The floating body \( \mathcal{F} \) is unstable with respect to the axis \( t \) if and only if the function \( U(\theta) \) takes a local maximum at \( \theta = 0 \).

**Proof of (F1) and (F2).** Rotate \( u \) and \( n \) by a small angle \( \epsilon \) with respect to the axis \( t \). Denote them by \( u_\epsilon \) and \( n_\epsilon \), that is,

\[
u_\epsilon = u \cos \epsilon + n \sin \epsilon, \quad n_\epsilon = -u \sin \epsilon + n \cos \epsilon.
\]

(12)

We denote by \( \mathcal{W} \) and \( \mathcal{W}_\epsilon \) the under surface parts of \( \mathcal{F} \) with respect to the vertical vectors \( \mathbf{n} \) and \( \mathbf{n}_\epsilon \), respectively, and by \( B \) and \( B_\epsilon \) the centers of buoyancy of them. By the definition, \( B \) and \( B_\epsilon \) are the center of gravity of \( \mathcal{W} \) and \( \mathcal{W}_\epsilon \), respectively. Then put

\[
\mathcal{W}_3 = \mathcal{W} \cap \mathcal{W}_\epsilon, \quad \mathcal{W}_1 = \mathcal{W} \setminus \mathcal{W}_3 \text{ and } \mathcal{W}_2 = \mathcal{W}_\epsilon \setminus \mathcal{W}_3,
\]

(13)

and denote by \( B_3 \), \( B_1 \) and \( B_2 \) the centers of gravity of them, respectively. Since the volumes of \( \mathcal{W} \) and \( \mathcal{W}_\epsilon \) are equal to \( \rho V \), the volumes of \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) are equal, putting \( V_1 \). Then by the property of center of gravity, we obtain that

\[
B_\epsilon = \left( 1 - \frac{V_1}{\rho V} \right) B_3 + \frac{V_1}{\rho V} B_2, \quad B = \left( 1 - \frac{V_1}{\rho V} \right) B_3 + \frac{V_1}{\rho V} B_1.
\]

(14)

By the above equality, we obtain that

\[
B_\epsilon - B = \frac{V_1}{\rho V} (B_2 - B_1).
\]

(15)
We cut the cross section $E$ into the following two parts:
\[ E_2 = E \cap W_2, \quad E_1 = E \cap W_1. \] (16)

We take the center of gravity of $E$ as the origin, and take $x$, $y$ and $z$-axis so that they have the same directions as $u$, $n$ and $t$, respectively. We denote by $z = \delta$ the border line of $E_2$ and $E_1$, and set
\[
\bar{E}_2 = \{(r \cos \theta + \delta, r \sin \theta, z) \mid (r + \delta, z) \in E_2, \, 0 \leq \theta \leq \epsilon\},
\]
\[
\bar{E}_1 = \{(-r \cos \theta + \delta, -r \sin \theta, z) \mid (-r + \delta, z) \in E_1, \, 0 \leq \theta \leq \epsilon\}. \] (17)

Then we obtain that
\[
V_1 = \iiint_{W_2} dxdydz = \iiint_{\bar{E}_2} rdrd\theta dz + O(\epsilon^2)
= \epsilon \iiint_{(r+\delta,z)\in E_2} rdrdz + O(\epsilon^2)
= \epsilon \iiint_{E_2} (x-\delta) dxdz + O(\epsilon^2). \] (18)

Similarly, we obtain that
\[
V_1 = \iiint_{W_2} dxdydz = \epsilon \iiint_{E_1} (\delta-x) dxdz + O(\epsilon^2). \] (19)

By (18) and (19), we obtain that
\[
\epsilon \left( \int_E x dxdz - \delta \int_E dxdz \right) = \epsilon \iiint_{E_2} (x-\delta) dxdz - \epsilon \iiint_{E_1} (\delta-x) dxdz = O(\epsilon^2). \] (20)

Since we take the center of gravity of $E$ as the origin, we obtain that $\int_E x dxdz = 0$. Putting it into (20), we obtain that $\delta = O(\epsilon)$. Since $u$ is the unit vector of direction $x$-axis, we obtain that
\[
V_1 B_2 \cdot u = \iiint_{W_2} x dxdydz = \iiint_{\bar{E}_2} r(r \cos \theta + \delta) drd\theta dz + O(\epsilon^2)
= \epsilon \int_{(r+\delta,x)\in B_3} r(r \cos \epsilon + \delta \epsilon) drdz + O(\epsilon^2)
= \epsilon \int_{E_2} x(x-\delta) dxdz + O(\epsilon^2) = \epsilon \int_{E_2} x^2 dxdz + O(\epsilon^2). \] (21)

Similarly, we obtain that
\[
V_1 B_1 \cdot u = -\epsilon \int_{E_1} x^2 dxdz + O(\epsilon^2), \quad V_1 B_2 \cdot n = O(\epsilon^2), \quad V_1 B_1 \cdot n = O(\epsilon^2). \] (22)

By using (15), (21) and (22), we obtain that
\[
(B_\epsilon - B) \cdot u = \frac{\epsilon}{\rho V} \int_E x^2 dxdz + O(\epsilon^2) = \frac{\epsilon I(\theta)}{\rho V} + O(\epsilon^2). \] (24)

Similarly, we obtain that
\[
(B_\epsilon - B) \cdot n = O(\epsilon^2). \] (25)

By dividing (24) and (25) by $\epsilon$, and taking $\epsilon \to 0$, we obtain that
\[
B'(\theta) \cdot u = \frac{I(\theta)}{\rho V}, \quad B'(\theta) \cdot n = 0. \] (26)
By putting (8) into (26), we obtain that
\[ B' (\theta) \cdot u_0 = \frac{I(\theta)}{\rho V} \cos \theta, \quad B' (\theta) \cdot n_0 = \frac{I(\theta)}{\rho V} \sin \theta. \] (27)

By replacing \( \theta \) of (27) by \( \phi \), and by integrating it on the interval \( 0 \leq \phi \leq \theta \), we obtain that
\[ (B - B_0) \cdot u_0 = \frac{1}{\rho V} \int_0^\theta I(\phi) \cos \phi \, d\phi, \quad (B - B_0) \cdot n_0 = \frac{1}{\rho V} \int_0^\theta I(\phi) \sin \phi \, d\phi. \] (28)

By (8) and (28), we obtain that
\[ (B - B_0) \cdot u = \frac{1}{\rho V} \int_0^\theta I(\phi)(\cos \theta \cos \phi + \sin \theta \sin \phi) \, d\phi = \frac{1}{\rho V} \int_0^\theta I(\phi) \cos(\theta - \phi) \, d\phi. \] (29)

Similarly, we obtain that
\[ (B - B_0) \cdot n = -\frac{1}{\rho V} \int_0^\theta I(\phi)(\sin \theta \cos \phi - \cos \theta \sin \phi) \, d\phi = -\frac{1}{\rho V} \int_0^\theta I(\phi) \sin(\theta - \phi) \, d\phi. \] (30)

Hence we obtain that
\[ F(\theta) = \left(- (G - B_0) + (B - B_0) \right) \cdot u \\
= -(G - B_0) \cdot (u_0 \cos \theta + n_0 \sin \theta) + (B - B_0) \cdot u \\
= \overline{GB}_0 \sin \theta + \frac{1}{\rho V} \int_0^\theta I(\phi) \cos(\theta - \phi) \, d\phi \\
= \int_0^\theta \left( \frac{I(\phi)}{\rho V} - \overline{GB}_0 \right) \cos(\theta - \phi) \, d\phi. \] (31)

Similarly, we obtain that
\[ U(\theta) = \int_0^\theta \left( \frac{I(\phi)}{\rho V} - \overline{GB}_0 \right) \sin(\theta - \phi) \, d\phi. \] (32)

Hence we have proved (F1) and (F2). \( \square \)

5 Application to Ulam's problem

Firstly, we consider three dimensional case. Remark that a floating body \( \mathcal{F} \in \mathbb{R}^3 \) is in equilibrium in every direction if and only if \( F(\theta) = 0 \) for every rotation axis \( t \). If \( F(\theta) = 0 \), then by (11), we have that \( I(n, t) = I(\theta) \) is constant. Hence we have proved the following theorem:

Theorem 10. If a floating body \( \mathcal{F} \) is in equilibrium in every direction, then the moment of inertia \( I(n, t) \) is constant for every \( n \) and \( t \).

Secondly, we consider two dimensional case. Consider a rod of base \( D \subset \mathbb{R}^2 \) and height \( h \). Suppose that it floats so that the rotation axis \( t \) is parallel to the axis of the rod and fixed. In this case, denote by \( \ell = \ell(\theta) \) the length of the water line. Then we obtain that
\[ I(\theta) = \int_{-h/2}^{h/2} \int_{-\ell/2}^{\ell/2} x^2 \, dx \, dz = \frac{h}{12} \ell^3. \] (33)

By Theorem 10 and (33), we have proved Theorem 4.
The converse of Theorem 4 holds under a rotational symmetry condition.

**Theorem 11.** If a figure $D \subset \mathbb{R}^2$ is rotational symmetric with respect to some angle, and if the water line is of constant length, then it is Auerbach.

**Proof.** By (33), the moment of inertia $I(\theta)$ is constant. So by (28), we obtain that

$$B = G + \left( \frac{I(0)}{\rho V} - \overline{GB}_0 \right) n_0 + \frac{I(0)}{\rho V} (u_0 \sin \theta - n_0 \cos \theta).$$

So the locus of $B$ is a circle. Since $D$ is rotational symmetric, so is the locus of $B$. Thus we obtain that

$$B = G + \frac{I(0)}{\rho V} (u_0 \sin \theta - n_0 \cos \theta).$$

By (35), we obtain that $F(\theta) = 0$. Hence $D$ is Auerbach. □

**References**


