<table>
<thead>
<tr>
<th>Title</th>
<th>An Application Model of a Nonlinear Difference Equation whose all Eigenvalues are 1 (Dynamics of Functional Equations and Mathematical Models)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Suzuki, Mami</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2009), 1637: 57-65</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2009-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/140521">http://hdl.handle.net/2433/140521</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
An Application Model of a Nonlinear Difference Equation whose the all Eigenvalues are 1

Department of Mathematics and Science, College of Liberal Arts, J. F. Oberlin University.

1 Introduction

In the previous work, we consider the following second order nonlinear difference equations,

\[
\begin{aligned}
x(t + 1) &= X(x(t), y(t)), \\
y(t + 1) &= Y(x(t), y(t)).
\end{aligned}
\]

(1.1)

Here \(X(x, y), Y(x, y)\) are holomorphic functions and expanded in a neighborhood of \((0, 0)\) in the following form

\[
\begin{aligned}
X(x, y) &= x + y + \sum_{i+j \geq 2} c_{ij}x^i y^j = x + X_1(x, y), \\
Y(x, y) &= y + \sum_{i+j \geq 2} d_{ij}x^i y^j = y + Y_1(x, y),
\end{aligned}
\]

(1.2)

where \(X_1(x, y) \not\equiv 0\) or \(Y_1(x, y) \not\equiv 0\).

Hereafter we consider \(t\) to be a complex variable. We define domain \(D_1(\kappa_0, R_0)\) by

\[
D_1(\kappa_0, R_0) = \{ t : |t| > R_0, |\arg[t]| < \kappa_0 \},
\]

(1.3)

where \(\kappa_0\) is any constant such that \(0 < \kappa_0 \leq \frac{\pi}{4}\), and \(R_0\) is sufficiently large number which may depend on \(X\) and \(Y\). Further we define domain \(D^*(\kappa, \delta)\) by

\[
D^*(\kappa, \delta) = \{ x : |\arg[x]| < \kappa, 0 < |x| < \delta \},
\]

(1.4)

where \(\delta\) is a small constant, and \(\kappa\) is a constant such that \(\kappa = 2\kappa_0\), i.e., \(0 < \kappa \leq \frac{\pi}{2}\).

Here we defined \(g_0^\pm\) as follows for the coefficients of \(X(x, y)\) and \(Y(x, y)\)

\[
\begin{aligned}
g_0^+(c_{20}, d_{11}, d_{30}) &= \frac{-(2c_{20} - d_{11}) + \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4}, \\
g_0^-(c_{20}, d_{11}, d_{30}) &= \frac{-(2c_{20} - d_{11}) - \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4},
\end{aligned}
\]

(1.5)
respectively, and
\[ A_2 = g_0^+(c_{20}, d_{11}, d_{30}) + c_{20}, \quad A_1 = g_0^-(c_{20}, d_{11}, d_{30}) + c_{20}, \]

We have proved the following Theorem 1 in previous works in [7].

**Theorem 1.** Suppose \( X(x, y) \) and \( Y(x, y) \) are expanded in the following forms
\[
\begin{align*}
X(x, y) &= x + y + \sum_{i+j \geq 2} c_{ij} x^i y^j = x + X_1(x, y), \\
Y(x, y) &= y + \sum_{i+j \geq 2} d_{ij} x^i y^j = y + Y_1(x, y).
\end{align*}
\] (1.2)

(1) Suppose \( d_{20} = 0 \), and we assume the following conditions,
\[
\begin{align*}
A_2 n &\neq c_{20} - d_{11} - g_0^+(c_{20}, d_{11}, d_{30}) \\
A_1 n &\neq c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30})
\end{align*}
\] (1.7) (1.8)

for all \( n \in \mathbb{N}, \ (n \geq 4) \), then we have formal solutions \( x(t) \) of the difference system (1.1) the following forms
\[
-\frac{1}{A_2 t} \left( 1 + \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right)^{-1}, \quad -\frac{1}{A_1 t} \left( 1 + \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right)^{-1}
\] (1.9)

where \( \hat{q}_{jk} \) are constants defined by \( X \) and \( Y \).

(2) Further suppose \( R_1 = \max(R_0, 2/(|A_2| \delta)) \) and we assume \( A_2 < 0, \ A_1, A_2 \in \mathbb{R} \), there are two solutions \( x_1(t) \) and \( x_2(t) \) of (1.1) such that
(i) \( x_s(t) \) are holomorphic in \( D_1(\kappa_0, R_1) \), and \( x_s(t) \in D^*(\kappa, \delta) \) for \( t \in D_1(\kappa_0, R_1) \), \( s = 1, 2 \),
(ii) \( x_s(t) \) are expressible in the following form
\[
x_1(t) = -\frac{1}{A_1 t} \left( 1 + b_1(t, \frac{\log t}{t}) \right)^{-1}, \quad x_2(t) = -\frac{1}{A_2 t} \left( 1 + b_2(t, \frac{\log t}{t}) \right)^{-1}.
\] (1.10)

Here \( b_1(t, \log t/t), b_2(t, \log t/t) \) are asymptotically expanded in \( D_1(\kappa_0, R_1) \) such that
\[
b_1(t, \frac{\log t}{t}) \sim \sum_{j+k \geq 1} \hat{q}_{jk(1)} t^{-j} \left( \frac{\log t}{t} \right)^k,
\]
\[
b_2(t, \frac{\log t}{t}) \sim \sum_{j+k \geq 1} \hat{q}_{jk(2)} t^{-j} \left( \frac{\log t}{t} \right)^k,
\]
as \( t \to \infty \) through \( D_1(\kappa_0, R_1) \).
In this proof, we use results of T. Kimura [2] and the following Theorem C in [6], though we need some modifications of Kimura's results.

**Theorem C.** Suppose $X(x, y)$ and $Y(x, y)$ are defined in (1.2). We assume $d_{20} = 0$ and the following conditions,

\begin{align*}
A_2 n &\neq c_{20} - d_{11} - g^+_0(c_{20}, d_{11}, d_{30}), \\
A_1 n &\neq c_{20} - d_{11} - g^-_0(c_{20}, d_{11}, d_{30}),
\end{align*}

for all $n \in \mathbb{N}$, $(n \geq 4)$.

1. We have two formal solutions

\begin{align*}
\Psi^+(x) &= \sum_{n \geq 2}^{\infty} a^+_n x^n, \\
\Psi^-(x) &= \sum_{n \geq 2}^{\infty} a^-_n x^n
\end{align*}

of

\[ \Psi(X(x, \Psi(x))) = Y(x, \Psi(x)), \]

where $a^+_n$, $a^-_n$ are given by $X$ and $Y$.

2. Further we assume $A_1, A_2 \in \mathbb{R}$, $A_2 < 0$. For any $\kappa$ ($0 < \kappa \leq \frac{\pi}{2}$) and small $\delta > 0$, there is a constant $\delta$, and two solutions $\Psi^+(x)$, $\Psi^-(x)$ of

\[ \Psi(X(x, \Psi(x))) = Y(x, \Psi(x)), \]

which are holomorphic and can be expanded asymptotically in $D^*(\kappa, \delta)$ such that

\[ \Psi^+(x) \sim \sum_{n=2}^{\infty} a^+_n x^n, \quad \text{and} \quad \Psi^-(x) \sim \sum_{n=2}^{\infty} a^-_n x^n. \]

as $x \to 0$ through $D^*(\kappa, \delta)$.

When we assume $d_{20} \neq 0$, then there are no analytic solution of (1.11).

---

2 **An Application**

Next we will consider the following population model (P)

\[ u(t + 2) = \alpha u(t + 1) + \beta \frac{u(t + 1) - \alpha u(t)}{\alpha u(t)}, \]

where $\alpha = 1 + r$, $\beta$ are constants. This model is proposed by Prof. D. Dendrios [1]. Here $r$ is the net (births minus death) endogenous population (stock) growth rate. The second term, in the right side hand, is a function depicting net immigration at $t + 1$, which should be considered as a "momentum" to grow from $t$ to $t + 1$. We assume that $\alpha$ and $\beta$ are constants such that $\alpha > 0$, i.e., $r > -1$ and $\beta > 0$ in (P).
Let
\[ u(t + 2) = u_1(t + 2) + u_2(t + 2), \]
where
\[ u_1(t + 2) = \alpha u(t + 1), \]
\[ u_2(t + 2) = \frac{\beta (u(t + 1) - \alpha u(t))}{\alpha u(t)}. \]

Here, \( u_1(t + 2) \) is a term for endogenous population growth rate from \( t + 1 \) to \( t + 2 \), and \( u_2(t + 2) \) is a term for net in-migration rate. We have
\[ u_1(t + 2) = \alpha u(t + 1) = \alpha (u_1(t + 1) + u_2(t + 1)), \]
\[ u_2(t + 2) = \frac{\beta (u(t + 1) - \alpha u(t))}{\alpha u(t)} = \frac{\beta u_2(t + 1)}{u_1(t + 1)}. \]

where \( u_1(t + 1) \) is the endogenous population growth rate from \( t \) to \( t + 1 \), and \( u_2(t + 1) \) due to net in-migration rate. We may write (P) as:
\[ u(t + 2) - \alpha u(t + 1) = \frac{c}{u(t)} \{u(t + 1) - \alpha u(t)\}, \quad c = \frac{\beta}{\alpha}. \]

When \( \alpha \neq 1 \), (P) admits the unique equilibrium value \( c = \frac{\beta}{\alpha} \), and we can have general analytic solutions such that \( u(t + n) \to c \), as \( n \to \infty \) \( (n \in \mathbb{N}) \), making use of theorems of previous studies [7] and [8].

When \( \alpha = 1 \), we note that, any value can be equilibrium point of (P). Suppose the equation (P) has a solution \( u(t) \) such that \( u(t + n) \to u_0 > 0 \), as \( n \to \infty \), in the case \( \alpha = 1 \). From [6], we have the following three cases.

1) \( u(t_0 + n) \downarrow u_0 \geq c \) as \( n \to \infty \),
2) \( u(t_0 + n) \uparrow u_0 > c \), as \( n \to \infty \), or
3) there is a \( n_0 \), such that \( u(t_0 + n_0) \leq 0 \), (extermination).

However we have not been able to prove the existence of a solution of (P) in [6]. Now we will show a solution of (P). Here we have the following Proposition 6, analogous to Theorem C.

**Proposition 6.** Suppose \( X(x, y) \) and \( Y(x, y) \) are defined \( d_{20} = 0 \), and we assume the following condition,
\[
A_1 n \neq c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30}), \quad (2.13)
\]
for all \( n \in \mathbb{N}, \ (n \geq 4) \).

1) We have a formal solution \( \Psi^-(x) = \sum_{n \geq 2} a_n^- x^n \) of
\[
\Psi(X(x, \Psi(x))) = Y(x, \Psi(x)), \quad (1.11)
\]
where \( a_n^- \) are given by \( X \) and \( Y \).
(2) We assume \( A_1, A_2 \in \mathbb{R}, (A_1 \leq A_2) \) and \( A_1 < 0 \). For any \( \kappa \) \((0 < \kappa \leq \frac{\pi}{2})\) and small \( \delta > 0 \), there is a constant \( \delta \), and a solution \( \Psi^{-}(x) \) of (1.11) which is holomorphic and can be expanded asymptotically in \( D^*(\kappa, \delta) \) such that

\[
\Psi^{-}(x) \sim \sum_{n=2}^{\infty} a_n x^n,
\]

as \( x \to 0 \) through \( D^*(\kappa, \delta) \).

When we assume \( d_{20} \neq 0 \), then there are no analytic solution of (1.11).

From Proposition 6, we have the following lemma 7, analogous to Theorem 1.

lemma 7. Suppose \( X(x, y) \) and \( Y(x, y) \) are expanded in the following forms

\[
\begin{aligned}
X(x, y) &= x + y + \sum_{i+j\geq 2} c_{ij}x^iy^j = X_1(x, y), \\
Y(x, y) &= y + \sum_{i+j\geq 2} d_{ij}x^iy^j = Y_1(x, y).
\end{aligned}
\]

(1) Suppose \( d_{20} = 0 \) and we assume the following conditions,

\[ A_1 n \neq c_{20} - d_{11} - g_0^{-}(c_{20}, d_{11}, d_{30}) \]

for all \( n \in \mathbb{N}, (n \geq 4) \). Then the difference system

\[
\begin{cases}
x(t+1) = X(x(t), y(t)), \\
y(t+1) = Y(x(t), y(t)),
\end{cases}
\]

has a formal solution \( x(t) \) of the following form

\[
-\frac{1}{A_1 t} \left( 1 + \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right)^{-1},
\]

\( \hat{q}_{jk} \) : constants defined by \( X \) and \( Y \).

(2) Further suppose \( R_1 = \max(R_0, 2/(|A_1| \delta)) \), and we assume \( A_1 < 0 \), there is a solution \( x_1(t) \) of (1.1) such that

(i) \( x_1(t) \) is holomorphic and \( x_1(t) \in D^*(\kappa, \delta) \) for \( t \in D_1(\kappa_0, R_1) \),

(ii) \( x_1(t) \) are expressible in the following form

\[
x_1(t) = -\frac{1}{A_1 t} \left( 1 + b_1 \left( \frac{\log t}{t} \right) \right)^{-1}.
\]
Here \( b_1(t, \log t/t) \) is asymptotically expanded in \( D_1(\kappa_0, R_1) \) such that

\[
b_1(t, \frac{\log t}{t}) \sim \sum_{j+k \geq 1} \hat{q}_{jk(1)} t^{-j} \left( \frac{\log t}{t} \right)^k,
\]

as \( t \to \infty \) through \( D_1(\kappa_0, R_1) \).

3 The Population Model (P)

In the equation (P),

\[
u(t + 2) = \alpha u(t + 1) + \beta \frac{u(t + 1) - \alpha u(t)}{\alpha u(t)},
\]

we put \( u(t) = v(t) + \frac{\beta}{\alpha} \). We have

\[v(t + 2) = (1 + \alpha)v(t + 1) - v(t) + F(v(t), v(t + 1)),\]

where \( F(v(t), v(t + 1)) \) is defined by

\[
F(v(t), v(t + 1)) = -\frac{\alpha}{\beta} v(t)v(t + 1) + \frac{\alpha^2}{\beta} v(t)^2
+ (v(t + 1) - \alpha v(t) + \frac{\beta}{\alpha} - \beta) \sum_{i \geq 2} \left( -\frac{\alpha}{\beta} \right)^i v(t)^i.
\]

(3.1)

Next put \( v(t + 1) = \xi(t), v(t) = \eta(t) \), we have

\[
\begin{pmatrix} \xi(t + 1) \\ \eta(t + 1) \end{pmatrix} = \begin{pmatrix} \alpha + 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} + \begin{pmatrix} F(\eta(t), \xi(t)) \\ 0 \end{pmatrix}.
\]

When \( \alpha = 1 \), we have eigenvalues \( \lambda \) and \( \mu \) of \( M = \begin{pmatrix} \alpha + 1 & -1 \\ 1 & 0 \end{pmatrix} \) such that \( \lambda = \mu = 1 \).

Further put \( P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \) and \( \begin{pmatrix} \xi \\ \eta \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix} \), we have the following difference equation

\[
\begin{pmatrix} x(t + 1) \\ y(t + 1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + P^{-1} \begin{pmatrix} F(x(t) + y(t), x(t) + 2y(t)) \\ 0 \end{pmatrix}.
\]

(3.2)

Since \( P^{-1} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \) we can write the equations (3.2) as follows,

\[
\begin{cases}
x(t + 1) = X(x(t), y(t)), \\
y(t + 1) = Y(x(t), y(t)),
\end{cases}
\]

(1.3)
such that

\[
\begin{aligned}
X(x, y) &= x + y - F(x + y, x + 2y) = x + \left( y + \sum_{i+j \geq 2} c_{ij} x^i y^j \right) \\
&= x + X_1(x, y), \\
Y(x, y) &= y + F(x + y, x + 2y) = y + \left( \sum_{i+j \geq 2} d_{ij} x^i y^j \right) \\
&= y + Y_1(x, y),
\end{aligned}
\] (1.2')

where \( d_{ij} = -c_{ij} \).

From the definition of the function \( F \) by (3.1), when \( \alpha = 1 \), we have

\[
F(x + y, x + 2y) = -\frac{1}{\beta} (xy + y^2) + \frac{1}{\beta^2} (x^2y + 2xy^2 + y^3) + \sum_{i+j \geq 4, j \geq 1} \gamma_{ij} x^i y^j,
\] (3.3)

where \( \gamma_{ij} \) are constants which consist of \( \beta \). From (3.3), we have the coefficients of \( X \) and \( Y \) in (1.2') as follows,

\[
c_{20} = d_{20} = 0, \quad c_{n0} = d_{n0} = 0, (n \geq 3),
\]

\[
d_{11} = -\frac{1}{\beta} < 0, \quad d_{02} = -\frac{1}{\beta} < 0, \quad d_{21} = \frac{1}{\beta^2},
\]

and have

\[
A_1 = g_0^- (c_{20}, d_{11}, d_{30}) + c_{20} = \frac{-(0 + \frac{1}{\beta}) - \sqrt{(0 + \frac{1}{\beta})^2 + 0}}{4} + 0 = -\frac{1}{2\beta} < 0,
\]

\[
A_2 = g_0^+ (c_{20}, d_{11}, d_{30}) + c_{20} = \frac{-(0 + \frac{1}{\beta}) + \sqrt{(0 + \frac{1}{\beta})^2 + 0}}{4} + 0 = 0,
\] (3.4)

Here we can not have the condition \( A_1 \leq A_2 < 0 \), however we have \( A_1 < A_2 = 0 \). Thus we put \( a_2 = g_0^- (c_{20}, d_{11}, d_{30}) \), \( A_1 = a_2 + c_{20} < 0 \). Since

\[
c_{20} - d_{11} - g_0^- (c_{20}, d_{11}, d_{30}) = \frac{3}{2\beta} > 0,
\]

we have

\[
A_1 n \neq c_{20} - d_{11} - g_0^- (c_{20}, d_{11}, d_{30}),
\]

for all \( n \in \mathbb{N} \).

By Proposition 6, the functional equation

\[
\Psi(X(x, \Psi(x))) = Y(x, \Psi(x)),
\] (1.11)

which \( X \) and \( Y \) are defined by (1.2') has a formal solution \( \Psi^-(x) = \sum_{n \geq 2} a_n^- x^n \), where \( a_n^- \) are defined by \( X \) and \( Y \).
Further, for any $\kappa, 0 < \kappa \leq \frac{\pi}{2}$, there are a $\delta > 0$ and a solutions $\Psi^{-}(x)$ of (1.11), which are holomorphic and can be expanded asymptotically such that

$$
\Psi^{-}(x) \sim \sum_{n=2}^{\infty} a_{n}^{-} x^{n},
$$
as $x \to 0$ through

$$
D^{*}(\kappa, \delta) = \{x ; |\arg[x]| < \kappa, 0 < |x| < \delta\}. \tag{1.4}
$$

From Lemma 7, we have a formal solution $x(t)$ of (3.2) such that

$$
-\frac{1}{A_{1}t} \left(1 + \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^{k}\right)^{-1} = \frac{2\beta}{t} \left(1 + \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^{k}\right)^{-1}, \tag{3.5}
$$

where $\hat{q}_{jk}$ are constants defined by $X$ and $Y$ in (1.2').

Further suppose $R_{1} = \max(R_{0}, 2/(|A_{1}|\delta))$, since $A_{1} = -\frac{1}{2\beta} < A_{2} = 0$, there is a solution $x(t)$ of (3.2) such that

(i) $x(t)$ are holomorphic and $x(t) \in D^{*}(\kappa, \delta)$ for $t \in D_{1}(\kappa_{0}, R_{1})$,

(ii) $x(t)$ is expressible in the following form

$$
x(t) = -\frac{1}{A_{1}t} \left(1 + b(t, \frac{\log t}{t})\right)^{-1} = \frac{2\beta}{t} \left(1 + b(t, \frac{\log t}{t})\right)^{-1}. \tag{3.6}
$$

Here $b(t, \log t/t)$ are asymptotically expanded in $D_{1}(\kappa_{0}, R_{1})$ such that

$$
b(t, \frac{\log t}{t}) \sim \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^{k},
$$
as $t \to \infty$ through $D_{1}(\kappa_{0}, R_{1})$.

By the definition (1.7), we have $y(t) = \Psi(x(t))$. Since

$$
\begin{pmatrix}
u(t + 1) - \frac{\beta}{\alpha} \\
u(t) - \frac{\beta}{\alpha}
\end{pmatrix} = 
\begin{pmatrix}
u(t + 1) \\
u(t)
\end{pmatrix} = 
\begin{pmatrix}\xi \\
\eta
\end{pmatrix} = P \begin{pmatrix}x \\ y\end{pmatrix} = 
\begin{pmatrix}1 & 2 \\ 1 & 1\end{pmatrix} \begin{pmatrix}x \\ y\end{pmatrix},
$$
we have a solution $u(t)$ of the (P), such that

$$
u(t) = x(t) + y(t) + \frac{\beta}{\alpha} = x(t) + \Psi(x(t)) + \frac{\beta}{\alpha} \sim x(t) + \sum_{n \geq 2} a_{n}^{-} x(t)^{n} + \frac{\beta}{\alpha}
$$
where $x(t)$ is given in the equation (3.6) as $t \to \infty$ through $D_{1}(\kappa_{0}, R_{1})$.

Here we have proved existence a solution of the Population Model (P).
References


