<table>
<thead>
<tr>
<th>Title</th>
<th>An explicit formula for the zeros of the Rankin-Selberg $L$-function (New Aspects of Analytic Number Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Noda, Takumi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2009), 1639: 164-171</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2009-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/140547">http://hdl.handle.net/2433/140547</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
An explicit formula for the zeros of the Rankin-Selberg $L$-function

Takumi Noda  野田 工
College of Engineering Nihon University  日本大学 工学部

Abstract

In this report, we describe one explicit formula for the zeros of the Rankin-Selberg $L$-function by using the projection of the $C^\infty$-automorphic forms [Noda, (Kodai. Math. J. 2008)]. The projection was introduced by [Sturm (Duke Math. J. 1981)] in the study of the special values of automorphic $L$-functions. Combining the idea of [Zagier (Springer, 1981, Proposition 3)] and the integral transformation of the confluent hypergeometric function, we derive an explicit formula which correlates the zeros of the zeta-function and the Hecke eigenvalues. The main theorem contains the case of the symmetric square $L$-function, that first appeared in author's previous paper [Noda, (Acta. Arith. 1995)].

1 Rankin-Selberg $L$-function

Let $k$ and $l$ ($k \leq l$) be positive even integers and $S_k$ (resp. $S_l$) be the space of cusp forms of weight $k$ (resp. $l$) on $SL_2(\mathbb{Z})$. Let $f(z) \in S_k$ and $g(z) \in S_l$ be normalized Hecke eigenforms with the Fourier expansions $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ and $g(z) = \sum_{n=1}^{\infty} b(n)e^{2\pi inz}$. For each prime $p$, we take $\alpha_p$ and $\beta_p$ such that $\alpha_p + \beta_p = a(p)$ and $\alpha_p\beta_p = p^{k-1}$, and define

$$M_p(f) = \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}.$$ 

The Rankin-Selberg $L$-function attached to $f(z)$ and $g(z)$ is defined by

$$L(s, f \otimes g) = \prod_{p \text{ prime}} \det(I_4 - M_p(f) \otimes M_p(g)p^{-s})^{-1}.$$ 

Here the product is taken over all rational primes, and $I_n$ is the unit matrix of size $n$. 
2 Fundamental properties

1. Dirichlet series
\[ L(s, f \otimes g) = \zeta(2s + 2 - k - l) \sum_{n=1}^{\infty} a(n)b(n)n^{-s} \]

2. Inner product (Rankin, Selberg)
\[ L(s, f \otimes g) \zeta(2s + 2 - k - l)^{-1} = \frac{(4\pi)^s}{\Gamma(s)} \int_{SL_2(\mathbb{Z}) \backslash H} f(z)\overline{g(z)}E_{l-k}(z, s-l+1)y^{l-2}dxdy \]

3. Analytic continuation
For \( l > k \), \( \Gamma(s)\Gamma(s-k+1)L(s, f \otimes g) \) is an entire function in \( s \). The functional equation is also known.

4. Others
(1) The critical strip is \((k+l-2)/2 < \text{Re}(s) < (k+l)/2\).
(2) For \( l = k \), \((\Gamma-\text{factor})\zeta(s-k+1)^{-1}L(s, f \otimes f) \) is an entire function in \( s \) (Shimura, Zagier).

3 Statement of the results

Theorem 1 Let \( k \) and \( l \) be positive even integers such that \( k, l = 12, 16, 18, 20, 22, \) and 26 respectively. Suppose \( k \leq l \). Let \( \Delta_k(z) = \sum_{n=1}^{\infty} \tau_k(n)e^{2\pi inz} \in S_k \) be the unique normalized Hecke eigenform, and let \( \rho \) be a zero of \( L(s - 1 + (k+l)/2, \Delta_k \otimes \Delta_l) \) in the critical strip \( 0 < \text{Re}(s) < 1 \). Assume that \( \zeta(2\rho) \neq 0 \). Then for each positive integer \( n \),
\[
-\tau_k(n) \left\{ \frac{n^{1-2\rho}(-1)^{\frac{l}{2}} \zeta(2\rho)}{(2\pi)^{2\rho}\Gamma(-\rho + \frac{k+l}{2})} + \frac{\zeta(2\rho-1)\Gamma(2\rho-1)}{\Gamma(\rho-1 + \frac{k+l}{2})\Gamma(\rho + \frac{k-l}{2})\Gamma(\rho-\frac{k-l}{2})} \right\}
\]
\[
= \frac{1}{\Gamma(k)\Gamma(\rho - \frac{k-l}{2})} \sum_{m=1}^{n-1} \tau_k(m)\sigma_{1-2\rho}(n-m)F(1-\rho + \frac{k-l}{2}, -\rho + \frac{k+l}{2}; k; \frac{m}{n})
\]
\[
+ \frac{1}{\Gamma(l)\Gamma(\rho - \frac{k-l}{2})} \sum_{m=n+1}^{\infty} \left( \frac{n}{m} \right)^{-\rho + \frac{k+l}{2}} \tau_k(m)\sigma_{1-2\rho}(m-n)
\]
\[
\times F(1-\rho - \frac{k-l}{2}, -\rho + \frac{k+l}{2}; l; \frac{n}{m}).
\]
4 Corollary and Remarks

**Corollary 1** Let $T(n, \rho; k; l)$ be the right-hand side of the equality in Theorem 1. Then, the following equivalence holds:

\[ \Re(\rho) = \frac{1}{2} \iff T(n, \rho; k; l) \asymp \tau_k(n) \quad (as \ n \to \infty). \]

**Remark 1.** By Shimura (1976, 77), it is known that the periods of the modular form for $L(s, f \otimes g)$ are dominated by the cusp form of large weight, whereas our theorem is expressed by using the Fourier coefficients of the cusp form of small weight.

**Remark 2.** The Theorem 1 includes the formula for the symmetric square $L$-function $L_{Q}(s, f)$ and the Riemann zeta function $\zeta(s)$, that first appeared in author’s previous paper [5].

5 Eisenstein series

Let $k \geq 0$ be an even integer, Let $i$ be the imaginary unit, $s$ be a complex number whose real part $\sigma$ (sigma) and imaginary part $t$. As usual, $H$ is the upper half plane. The non-holomorphic Eisenstein series for $SL_2(\mathbb{Z})$ is defined by

\[ E_k(z,s) = \gamma^s \sum_{\{c,d\}} (cz+d)^{-k} |cz+d|^{-2s}. \]  
(1)

Here $z$ is a point of $H$, $s$ is a complex variable and the summation is taken over $(c,d)$, a complete system of representation of $\{(0,*) \in SL_2(\mathbb{Z}) \} \setminus SL_2(\mathbb{Z})$. The right-hand side of (1) converges absolutely and locally uniformly on $\{(z,s) \mid z \in H, \Re(s) > 1 - \frac{k}{2}\}$, and $E_k(z,s)$ has a meromorphic continuation to the whole $s$-plane. It is also well-known the functional equation:

\[ \pi^{-s} \Gamma(s) \zeta(2s) E_k(z,s) = \pi^{-1+s+k} \Gamma(1-s-k) \zeta(2-2s-2k) E_k(z,1-s-k). \]
6 Projection to the space of cusp forms

The $C^\infty$-automorphic forms of bounded growth are introduced by Sturm in the study of zeta-functions of Rankin type. The function $F$ is called a $C^\infty$-modular form of weight $k$, if $F$ satisfies the following conditions:

(A.1) $F$ is a $C^\infty$-function from $H$ to $\mathbb{C}$,

(A.2) $F((az+b)(cz+d)^{-1})=(cz+d)^k F(z)$ for all $(a,b,c,d)\in SL_2(\mathbb{Z})$.

We denote by $\mathfrak{M}_k$ the set of all $C^\infty$-modular forms of weight $k$. The function $F \in \mathfrak{M}_k$ is called of bounded growth if for every $\epsilon > 0$

$$\int_0^\infty \int_0^\infty |F(z)|y^{k-2}e^{-\epsilon y}dydx < \infty.$$ 

Let $k$ be a positive even integer and $S_k$ be the space of cusp forms of weight $k$ on $SL_2(\mathbb{Z})$. For $F \in \mathfrak{M}_k$ and $f \in S_k$, we define the Petersson inner product as usual

$$(f,F) = \int_{SL_2(\mathbb{Z}) \backslash H} f(z)\overline{F(z)} y^{k-2}dxdy.$$ 

The Poincaré series are defined by

$$P_m(z) = \sum_{\{a,b,c,d\}} e\left(m \cdot \frac{az+b}{cz+d}\right) (cz+d)^{-k}$$

for $k \geq 4$, $m \in \mathbb{Z}_{\geq 0}$ and $z = x + iy \in H$. Here the summation is taken over as in the definition of the Eisenstein series. In 1981, Sturm constructed a certain kernel function by using Poincaré series, and showed the following theorem:

**Theorem 2** (Sturm 1981) Assume that $k > 2$. Let $F \in \mathfrak{M}_k$ be of bounded growth with the Fourier expansion $F(z) = \sum_{n=-\infty}^\infty a(n,y)e^{2\pi inz}$. Let

$$c(n) = (2\pi n)^{k-1} \Gamma(k-1)^{-1} \int_0^\infty a(n,y)e^{-2\pi ny}y^{k-2}dy.$$ 

Then $h(z) = \sum_{n=1}^\infty c(n)e^{2\pi inz} \in S_k$ and

$$(g,F) = (g,h)$$

for all $g \in S_k$.  

7  Fourier expansion of the Eisenstein series

Let $e(u) := \exp(2\pi i u)$ for $u \in \mathbb{C}$. For $z \in H$ and Re$(s) > 1 - \frac{k}{2}$, $E_k(z, s)$ has an expansion:

$$E_k(z, s) = y^{s} + a_0(s)y^{1-k-s} + \frac{y^s}{\zeta(k+2s)} \sum_{m \neq 0} \sigma_{1-k-2s}(m)a_m(y, s)e(mx),$$

(2)

where

$$a_0(s) = (-1)^{\frac{k}{2}} 2\pi \cdot 2^{1-k-2s} \frac{\zeta(k+2s-1) \Gamma(k+2s-1)}{\zeta(k+2s) \Gamma(s) \Gamma(k+s)},$$

$$\sigma_s(m) = \sum_{d|m, d>0} d^s,$$

$$a_m(y, s) = \int_{-\infty}^{\infty} e(-mu)(u+iy)^{-k}|u+iy|^{-2s}du.$$  

(3)

and

$$a_m(y, s) = \begin{cases} 
(1)^{\frac{k}{2}} \frac{(2\pi)^k 2^k s^2 m^{k+2s-1}}{\Gamma(k+s)} e^{-2\pi ym} \Psi(s, k+2s; 4\pi ym) & (m > 0), \\
(1)^{\frac{k}{2}} \frac{(2\pi)^k 2^k s^2 |m|^{k+2s-1}}{\Gamma(s)} e^{-2\pi |m|} \Psi(k+s, k+2s; 4\pi |m|) & (m < 0). 
\end{cases}$$

Here $\Psi(\alpha, \beta; z)$ is the confluent hypergeometric function defined for Re$(z) > 0$ and Re$(\alpha) > 0$ by the following

$$\Psi(\alpha, \beta; z) := \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} e^{-zu} u^{\alpha-1}(1+u)^{\beta-\alpha-1}du.$$  

We call the first two terms of (2) the constant term of $E(z, s)$. The integral (3) is entire function in $s$ and of exponential decay in $y|m|$. This fact gives the meromorphical continuation and the $y$-aspect of $E(z, s)$ when $y$ tends to $\infty$. Namely, there exist positive constants $A_1$ and $A_2$ depending only on $k$ and $s$ such that

$$|E_k(z, s)| \leq A_1 y^{|\text{Re}(s)|} + A_2 y^{1-\text{Re}(s)-k} \quad (y \rightarrow \infty),$$

except on the poles. Further, the modularity for $y^\frac{k}{2} E_k(z, s)$ gives the following:

**Proposition 1** Assume $E_k(z, s)$ is holomorphic at $s \in \mathbb{C}$. Then, there exist positive constants $A_1$ and $A_2$ depending only on $k$ and $s$ such that

$$|E_k(z, s)| \leq \begin{cases} 
A_1 (y^{-\text{Re}(s)-k} + y^{\text{Re}(s)}) & (\text{Re}(s) > \frac{1-k}{2}) \\
A_2 (y^{1+\text{Re}(s)} + y^{1-\text{Re}(s)-k}) & (\text{Re}(s) \leq \frac{1-k}{2}) 
\end{cases}$$

for every $y > 0$.  


8 Proof of Theorem 1

By Proposition 1, it is easy to see the Eisenstein series $E_k(z,s)$ is a $C^\infty$-modular form of weight $k$, and of bounded growth for $2-k < \text{Re}(s) < -1$ except on the poles. Therefore

Lemma 1 For $f(z) \in S_k$ and $s \in \mathbb{C}$ in $k/2 - l + 2 < \text{Re}(s) < k/2 - 1$, $f(z)E_{l-k}(z,s)$ is a $C^\infty$-modular form of weight $l$ and of bounded growth.

We have also the following:

Lemma 2 Let $f(z) \in S_k$ and $g(z) \in S_l$ be normalized Hecke eigenforms. Let $\rho$ be a zero of $L(s-1+(k+l)/2, f \otimes g)$ in the critical strip $0 < \text{Re}(s) < 1$. Assume $\zeta(2\rho) \neq 0$. Then

$$\langle f(z)E_{l-k}(z, \rho + \frac{k-l}{2}), g(z) \rangle = 0.$$ 

To evaluate the Laplace-Mellin transform of the Fourier coefficient of the product of the Eisenstein series and the Hecke eigenform, we use the following proposition.

Proposition 2 The integral transform

$$\int_0^\infty \Psi(a,c;y)y^{b-1}e^{-uy}dy = \frac{\Gamma(b)\Gamma(b-c+1)}{\Gamma(a+b-c+1)}u^{-b}$$

$$\times F(a,b;a+b-c+1;1-\frac{1}{u})$$

is valid when $\text{Re}(u) > 0$ and $\text{Re}(b-a) - M - N > 0$. Here $M$ and $N$ are non-negative integers so as $\text{Re}(a+M) > 0$ and $\text{Re}(c-a) \leq N + 1$ respectively.

Proof of Theorem 1 Let $\Delta_k(z)$ be the unique normalized Hecke eigenform for $k = 12, 16, 18, 20, 22, 26$. We write the Fourier expansion as follows:

$$\Delta_k(z) \cdot E_{l-k}(z,s) = \sum_{n=-\infty}^{\infty} b(n,y,s)e^{2\pi i n x}.$$ 

Using the notation $a_0(s)$ and $a_n(y,s)$ defined by (2) and (3),

$$b(n,y,s) = \{y^s + a_0(s)y^{l+k-s}\} \tau_k(n)e^{-2\pi ny}$$

$$\phantom{b(n,y,s)} + \frac{\zeta(2s+l-k)}{\zeta(2s)} \sum_{m=1 \text{ m \neq n}}^{\infty} \tau_k(m)\sigma_{l+k-2s}(n-m)a_{n-m}(y,s)e^{-2\pi ny}.$$
Here we regard $\tau_k(m)$ as 0 if $m \leqq 0$.

By Lemma 1 and Theorem 2, there exists $h(z, s) = \sum_{n=1}^{\infty} c(n, s) e^{2\pi inz} \in S_l$ such that $\langle f(z), E_{l-k}(z, s), g(z) \rangle = \langle h(z, s), g(z) \rangle$ for all $g(z) \in S_l$ in the region $k/2 - l + 2 < \text{Re}(s) < k/2 - 1$. The Fourier coefficients of $h(z, s)$ are given by

$$c(n, s) = (2\pi n)^{l-1} \Gamma(l-1)^{-1} \int_0^\infty b(n, y, s) e^{-2\pi ny} y^{l-2} dy,$$

for $n > 0$. We put $\gamma(n, l) = (2\pi n)^{l-1} \Gamma(l-1)^{-1}$. Then we have

$$c(n, s) = \frac{\gamma(n, l)}{\zeta(2s+l-k)} \sum_{m=1, m \neq n}^{\infty} \tau_k(m) \sigma_{1-l+k-2s}(n-m)$$

$$\times \int_0^\infty a_{n-m}(y, s) y^{l-2} e^{-2\pi(m+n)y} dy$$

$$+ \text{ (transformed constant terms)}.$$

Combining Lemma 2 and Proposition 2, we obtain the equation in the Theorem 1.

References


