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A Solution to a Problem posed by S. Manni and the Related Topics

弘前大学・理工学部 小関 道夫 (Michio Ozeki)
Faculty of Science and Technology, Hirosaki University
29. October 2008

1 Introduction

Throughout this talk we consider only positive definite even unimodular lattices.
S. Manni [12] proved

Theorem 1.1. In 56 (resp. 76) dimensional even unimodular extremal lattices, the theta series associated to such lattices we can say that in degree 3 their difference is, up to a multiplicative, possibly 0, constant, and equal to $\chi_{36}$ (resp. $\chi_{32}$).

In 40 dimensional lattices, if two extremal theta series are equal in degree 2, then in degree 3 their difference is up to a multiplicative, possibly 0, constant, and equal to $\chi_{28}$.

He then wrote

Find two even unimodular extremal lattices $L_1$ and $L_2$ of rank 40 whose theta series coincide in degree 2 and differ in degree 3. Besides this he posed the problems in ranks 32, 48 and 56.

In the present report we show that there are 40 dimensional two even unimodular extremal lattices coming from two doubly even self-dual extremal codes, whose theta series of degree 2 coincide and theta series of degree 3 differ definitely. We also show an instance of two another even unimodular extremal lattices coming from another two doubly even self-dual extremal codes, whose theta series of degree 2 and degree 3 coincide. These are shown by computing some beginning Fourier coefficients of theta series of the lattices in question combined with some facts on the dimensions of the linear spaces of Siegel modular forms already proved by other people. S. Manni [12] also proved

Theorem 1.2. In 32 (resp. 48) dimensional even unimodular extremal lattices, about the theta series associated to such lattices we can say that

(i) it is unique in degree 3,
(ii) in degree 4 their difference is, up to a multiplicative (possibly 0) constant, equal to a power of Schottky's polynomial $J$.

He then wrote

Find two even unimodular extremal lattices $L_3$ and $L_4$ of rank 32 or 48 whose theta series differ in degree 4.

We discus some related trials to this problem.

2 A brief account

2.1 32 dimensional case

Erokhin [6] proved

Theorem 2.1. If two 32 dimensional even unimodular lattices have identical theta series of degree 1, then they have identical theta series of degrees up to 3.

Venkov [28], [29] gave a method to compute some Fourier coefficients of Siegel theta series of degree 3 associated with even unimodular extremal 32 dimensional lattices.
2.2 40 or higher dimensional cases

The 40 dimensional case is our present topic. There is not any explicit result for the 48 dimensional and 56 dimensional cases along with Manni’s questions. The reasons for this would be the facts that there are few explicit constructions of lattices and that they are constructed through ternary codes. In 32 dimensional case our present method will apply to Manni’s problem, but we have not pursued this case since the shapes of minimal vectors in an extremal 32 dimensional lattice are complicated.

3 Some Basics

3.1 Lattice

A lattice $L$ of rank $n$ (or dimension $n$) is a $\mathbb{Z}$-module generated by the vectors $x_1, \ldots, x_n$ in $\mathbb{R}^n$ that are linearly independent over $\mathbb{R}$. The vectors $x_1, \ldots, x_n$ are called the basis of $L$.

$L$ is integral if the inner product $(x, y)$ belongs to $\mathbb{Z}$ for all pairs $x$ and $y$ in $L$.

The dual lattice $L^\#$ of $L$ is defined to be

$$L^\# = \{ y \in \mathbb{R}^n | (x, y) \in \mathbb{Z}, \forall x \in L \}. $$

A lattice $L$ is unimodular if it holds that $L = L^\#$.

A lattice $L$ is even if any element $x$ of $L$ has even norm $(x, x)$.

Even unimodular lattices exist only when $n \equiv 0 \pmod{8}$.

$$\text{Min}(L) = \min_{0 \neq x \in L} (x, x)$$

When $L$ is even unimodular of rank $n$ it holds that

$$\text{Min}(L) \leq 2 \left\lfloor \frac{n}{24} \right\rfloor + 2.$$ 

A lattice which attains the above maximum is called an extremal lattice.

Let $L$ be an even unimodular lattice of rank $n$.

$L_{2m}(L)$ : The set of $x$ in $L$ with $(x, x) = 2m$ $(m \geq 1)$.

3.2 Siegel modular forms

The symplectic group $Sp_g(\mathbb{R})$ of degree $g$ over $\mathbb{R}$ is defined to be

$$Sp_g(\mathbb{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2g}(\mathbb{R}) | {}^tMJM = J, J = \begin{pmatrix} O & -I_g \\ -I_g & O \end{pmatrix} \right\}.$$

Siegel modular group $Sp_g(\mathbb{Z})$ of degree $g$ is a subgroup of $Sp_g(\mathbb{R})$ consisting of elements in $Sp_g(\mathbb{R})$ whose entries are in $\mathbb{Z}$. Let $\mathbb{H}_g$ be the Siegel upper half-space of degree $g$:

$$\mathbb{H}_g = \{ \tau \mid \tau = X + Yi \in M_g(\mathbb{C}), \mid \tau = \tau, Y \text{is positive definite} \}.$$ 

A Siegel modular form of degree $g$ $(g \geq 2)$ and weight $k$ is a holomorphic complex valued function $f(\tau)$ defined on $\mathbb{H}_g$ satisfying the condition:

$$f((Ar + B)(C\tau + D)^{-1}) = (\det(C\tau + D))^k f(\tau) \quad \forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_g(\mathbb{Z}).$$
Note that when $g = 1$ an additional condition of the holomorphicity of $f$ at the cusp is necessary.

### 3.3 Siegel theta series

Siegel theta series of degree $g$ attached to the lattice $L$ is defined by

$$
\theta_g(\tau, L) = \sum_{x_1, \ldots, x_g \in L} \exp(\pi i \sigma([x_1, \ldots, x_g]\tau)),
$$

where $\tau$ is the variable on Siegel upper-half space of degree $g$, $[x_1, \ldots, x_g]$ is a $g \times g$ square matrix whose $(i, j)$ entry is $(x_i, x_j)$ and $\sigma$ is the trace of the matrix.

Siegel theta series of degree $g$ can be expanded to

$$
\theta_g(\tau, L) = \sum_T a(T, L) e^{2\pi i \sigma(T\tau)}.
$$

Here $T$ runs over the set of positive semi-definite semi-integral symmetric square matrices of degree $g$, and $a(T, L) = \#\{(x_1, \ldots, x_g) \in L^g | [x_1, \ldots, x_g] = 2T\}$. Fact: Siegel theta series of degree $g$ associated with an even integral unimodular lattice $L$ of rank $2k$ ($2k$ is a multiple of 8) is a modular form of degree $g$ and weight $k$.

### 3.4 Theta Functions with characteristics

$$
\theta \left[ \begin{array}{c} \epsilon_1 \\ \epsilon_1' \end{array} \right] (\tau, Z) = \sum_{N \in \mathbb{Z}^g} \exp \left\{ 2\pi i \left[ \frac{1}{2} (N + \frac{\epsilon_1}{2}) \tau (N + \frac{\epsilon_1}{2}) + (N + \frac{\epsilon_1}{2}) (Z + \frac{\epsilon_1'}{2}) \right] \right\}
$$

Here $\epsilon, \epsilon'$ are integral vectors of length $g$ with entries 0 or 1, $Z$ is a variable on $\mathbb{C}^g$, and $\tau$ is a variable on $\mathbb{H}_g$, the Siegel upper half space of genus $g$. For $g = 2$ case

$$
\theta \left[ \begin{array}{c} \epsilon_1 \\ \epsilon_1' \end{array} \right] \left( \begin{array}{c} \epsilon_2 \\ \epsilon_2' \end{array} \right) (\tau, Z) = \sum_{n=(n_1, n_2) \in \mathbb{Z}^2} \exp \left\{ \frac{\pi i}{2} \left( \sum_{i,j=1}^{2} \tau_{ij} (n_i + \frac{\epsilon_i}{2}) (n_j + \frac{\epsilon_j}{2}) + \sum_{i=1}^{2} (n_i + \frac{\epsilon_i}{2}) (x_i + \frac{\epsilon_i'}{2}) \right) \right\}
$$

Here $\tau_{ij}$ is the $ij$ entry of $\tau$, $Z = (z_1, z_2) \in \mathbb{C}^2$, $q_1 = e^{\pi i z_1}$, $q_2 = e^{\pi i z_2}$, $q_3 = e^{\pi i z_3}$, $\zeta_1 = e^{\pi i z_1}$, $\zeta_2 = e^{\pi i z_2}$.
Two instances.

\[
\theta \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] (\tau, Z) \\
= \sum_{n=(n_{1},n_{2})\in \mathbb{Z}^{2}} q_{1}^{n_{1}^{2}n_{2}^{2}} q_{3}^{2n_{1}n_{2}} \zeta_{1}^{2n_{1}} \zeta_{2}^{2n_{2}} \\
\theta \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] (\tau, Z) \\
= \sum_{n=(n_{1},n_{2})\in \mathbb{Z}^{2}} q_{1}^{(n_{1}+1/2)^{2}n_{2}^{2}} q_{3}^{2(n_{1}+1/2)n_{2}} \zeta_{1}^{2(n_{1}+1/2)} \zeta_{2}^{2n_{2}}
\]

3.5 Binary linear code

Let \( \mathbb{F}_2 = GF(2) \) be the field of 2 elements. Let \( V = \mathbb{F}_2^n \) be the vector space of dimension \( n \) over \( \mathbb{F}_2 \). A linear \([n,k] \) code \( C \) is a vector subspace of \( V \) of dimension \( k \). An element \( x \in C \) is called a codeword of \( C \).

In \( V \), the inner product, which is denoted by \( x \cdot y \) for \( x,y \in V \), is defined as usual. Two codes \( C_1 \) and \( C_2 \) are said to be equivalent if and only if after a suitable change of coordinate positions of \( C_1 \) all the codewords in both codes coincide.

The dual code \( C^\perp \) of \( C \) is defined by

\[
C^\perp = \{ u \in V \mid u \cdot v = 0 \ \forall v \in C \}.
\]

The code \( C \) is called self-orthogonal if it satisfies \( C \subseteq C^\perp \), and the code \( C \) is called self-dual if it satisfies \( C = C^\perp \).

Self-dual codes exist only if \( n \equiv 0 \ (mod \ 2) \) and \( k = \frac{n}{2} \).

Let

\[
x = (x_1, x_2, \ldots, x_n)
\]

be a vector in \( V \), then the Hamming weight \( wt(x) \) of the vector \( x \) is defined to be the number of \( i \)'s such that \( x_i \neq 0 \). The Hamming distance \( d \) on \( V \) is also defined by \( d(x,y) = wt(x - y) \). Let \( C \) be a code, then the minimum distance \( d \) of the code \( C \) is defined by

\[
d = \text{Min}_{x,y \in C, x \neq y} d(x,y) = \text{Min}_{x \in C, x \neq 0} wt(x).
\]

Let \( C \) be a self-dual binary \([n, \frac{n}{2}] \) code, then the weight \( wt(x) \) of each codeword \( x \) in \( C \) is an even number. Further, if the weight of each codeword \( x \) in \( C \) is divisible by 4, then the code is called a doubly even binary code. It is known that doubly even self-dual binary codes \( C \) exist only when the length \( n \) of \( C \) is a multiple of 8.

Let \( C \) be a self-dual doubly even code of length \( n \), which are embedded in \( \mathbb{F}_2^n \). Let \( u = (u_1, u_2, \ldots, u_n) \), \( v = (v_1, v_2, \ldots, v_n) \) be any pair of vectors in \( \mathbb{F}_2^n \), then the number of common 1's of the corresponding coordinates for \( u \) and \( v \) is denoted by \( u \cdot v \). This is called the intersection number of \( u \) and \( v \), and \( u \cdot u \) is nothing else \( wt(u) \).

3.6 Multiple weight enumerator

Let \( C \) be a doubly even self-dual code of length \( n \), and \( g \) be a positive integer and we let \( \alpha \) run the set \( \mathbb{F}_2^g \) of \( g \)-tuple vectors. The \( 2^g \) algebraically independent over \( \mathbb{C} \) variables \( x_\alpha \) are parametrized by \( \alpha \in \mathbb{F}_2^g \). Let
\[ u_1 = (u_1^1, u_1^2, \ldots, u_1^n), u_2 = (u_2^1, u_2^2, \ldots, u_2^n), \ldots, u_g = (u_g^1, u_g^2, \ldots, u_g^n) \] be the \( g \)-tuple codewords of \( C \). For each \( \alpha \in \mathbb{F}^g \) a generalized weight
\[ wt_\alpha(u_1, u_2, \ldots, u_g) \alpha \] is defined to be the number of coordinates \( j \) \( (1 \leq j \leq n) \) such that the equation \( \alpha = (u_1^j, u_2^j, \ldots, u_g^j) \) holds.

The multiple weight enumerator \( W_g(x_\alpha; C) \) of genus \( g \) for the code \( C \) is defined by
\[
W_g(x_\alpha; C) = \sum_{(u_1, u_2, \ldots, u_g) \in C} \prod x_\alpha^{wt_\alpha(u_1, u_2, \ldots, u_g)}.
\]

The multiple weight enumerator of second degree is called a biweight enumerator, and the multiple weight enumerator of third degree is called a triweight enumerator.

### 3.7 From binary codes to lattices

**C**: binary self-orthogonal \([n, k]\) code

Construction \( A_2 \)

\[
\rho : \mathbb{Z}^n \rightarrow \mathbb{F}_2^n \\
\cup \cup \\
x \mapsto x \mod 2
\]

\[
L(C) = \frac{1}{\sqrt{2}} \rho^{-1}(C).
\]

Construction \( B_2 \)

\[
\rho : \mathbb{Z}^n \rightarrow \mathbb{F}_2^n \\
\cup \cup \\
x \mapsto x \mod 2
\]

\[
M(C) = \frac{1}{\sqrt{2}} \left\{ x = (x_1, x_2, \ldots, x_n) \in \rho^{-1}(C) | \sum_{i=1}^{n} x_i \equiv 0 \pmod{4} \right\}
\]

Doubling the density process:

Suppose that \( C \) is a doubly even self-dual binary \([n, n/2]\) code. Put

\[
\gamma = \begin{cases} 
\frac{1}{\sqrt{2}} (1, \ldots, 1, -3) & \text{if } n \equiv 8 \pmod{16}, \\
\frac{1}{\sqrt{2}} (1, \ldots, 1, 1) & \text{if } n \equiv 0 \pmod{16}.
\end{cases}
\]

\[
\mathcal{N}(C) = M(C) \cup (\gamma + M(C))
\]

We pick up peculiar codes. We denote the codes \( C_1 \) (respectively \( C_2, C_3, C_4 \)) the second code in [17], Yorgov's \( C_5 \), Yorgov's code \( C_2 \) and Yorgov's code \( C_4 \) [?] respectively. The lattices constructed by the above process are denoted by \( M_{11} = \mathcal{N}(C_1), M_{12} = \mathcal{N}(C_2), M_{13} = \mathcal{N}(C_3) \) and \( M_{14} = \mathcal{N}(C_4) \) respectively.
3.7.1 40 dimensional case

We are particularly concerned with the set of minimal vectors $\Lambda_4(N(C))$ in an extremal even unimodular lattice constructed from binary self-dual extremal $[40, 20, 8]$ code. When $C$ is a doubly even self-dual binary $[40, 20, 8]$ code, $\Lambda_4 = \Lambda_4(N(C))$ consists of two kinds of vectors:

$$\Lambda_4^1 = \left\{ \frac{1}{\sqrt{2}}((\pm 2)^2, 0^{38}) \right\} \quad \text{number} = 3120$$

$$\Lambda_4^2 = \left\{ \frac{1}{\sqrt{2}}((\pm 1)^8, 0^{32}) \right\} \quad \text{number} = 36480$$

The set $\Lambda_4^1$ forms a root system of type $D_{40}$ scaled by a factor $\sqrt{2}$, and the vectors in the set $\Lambda_4^2$ come from codewords of weight 8 in the code $C$.

To each $y \in \Lambda_4$ we associate a binary vector $v = \text{supp}(y) \in \mathbb{F}_2^{40}$ which corresponds to non zero positions of $y$.

3.8 Duke-Runge map

We explain the map by using the case $g = 2$.

We put

$$\varphi_e(\tau) = \theta \begin{bmatrix} \epsilon \\ 0 \end{bmatrix} (2\tau, 0).$$

These are theta zero values with the variable $\tau$ multiplied by 2. There are $2^g$ functions $\varphi_e(\tau)$.

$$\varphi_{00}(\tau) = \sum_{n=(n_1, n_2) \in \mathbb{Z}^2} q_1^{2n_1^2} q_2^{2n_2^2} q_3^{4n_1 n_2}$$

$$= 1 + 2q_1^2 + 2q_2^2 + 2q_1^2 q_2^2 (q_3^4 + q_3^{-4}) + 2q_1^8 + 2q_2^8$$

$$+ 2q_1^6 q_2^6 (q_3^6 + q_3^{-6}) + 2q_1^8 q_2^8 (q_3^8 + q_3^{-8}) + \cdots$$

$$\varphi_{10}(\tau) = \sum_{n=(n_1, n_2) \in \mathbb{Z}^2} q_1^{2(n_1+1/2)^2} q_2^{2n_2^2} q_3^{4(n_1+1/2)n_2}$$

$$= 2q_1^{3/2} + 2q_2^{3/2} (q_3^2 + q_3^{-2})$$

$$+ 2q_1^6 q_2^6 (q_3^6 + q_3^{-6}) + 2q_1^{1/2} q_2^{1/2} (q_3^2 + q_3^{-2})$$

$$+ 2q_1^6 q_2^6 (q_3^{12} + q_3^{-12}) + \cdots$$

Likewise $\varphi_{01}(\tau), \varphi_{11}(\tau)$ can be expanded. Let $W_g(x_\alpha; C)$ be a multiple weight enumerator of genus $g$ for a doubly even self-dual code $C$, then $W_g(\varphi_e; C)$ is proved to be Siegel theta series of degree $g$ that is associated
with the lattice constructed by using Construction $A_2$ in Section 3.7.

For instance

\[
W_2(x_{00}, x_{01}, x_{10}, x_{11}; \text{Ham}) = x_{00}^2 + x_{01}^2 + x_{10}^2 + x_{11}^2 + 14(x_{00}x_{01} + x_{00}x_{10} + x_{00}x_{11} + x_{01}x_{10} + x_{01}x_{11} + x_{10}x_{11}) + 168x_{11}^2x_{10}x_{01}x_{00}
\]

is the biweight enumerator of the Hamming $[8,4,4]$ code. And

\[
W_2(\varphi_{00}(\tau), \varphi_{01}(\tau), \varphi_{10}(\tau), \varphi_{11}(\tau); \text{Ham}) = 1 + 240q_2^2 + 2160q_2^4 + 6720q_2^6 + 17520q_2^8 + 30240q_2^{10}
\]

\[
+ q_1^2[240(240 + 240q_2^2 + 13440q_2^2 + 13440q_2^2 + 181440q_2^2 + 181440q_2^2 + 138240q_2^2 + 138240q_2^2 + 13440q_2^2 + 138240q_2^2 + 13440q_2^2] + 362880q_2^{12} + 725760q_2^{14} + 30240q_2^{16}
\]

is the Siegel theta series of degree 2 for the root lattice $E_8$.

The multiple weight enumerators for the class of doubly even self-dual codes are invariant under the action of certain finite group $G$ of linear transformations. Runge discussed the ring $\mathcal{R}$ of invariants under a special subgroup $H$ of $G$ and extended the mapping $\Phi$ to $\mathcal{R}$.

## 4 Preliminary results

Table 1 The dimensions of the linear space of Siegel modular forms of degree $g$ and weight $k$.

<table>
<thead>
<tr>
<th>$n$</th>
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<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
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Proposition 4.1. Siegel theta series $\theta_g(Z, L)$ of degree $g$ associated with an even unimodular lattice of rank $2k$ $(k \equiv 0 \pmod{2})$ is determined uniquely if the Fourier coefficients $a(T, L)$ are known for $T$'s given in the Table 2-1 $- 2-5$.

Table 2-1 $g = 1$ case

<table>
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<tr>
<th>$2k$</th>
<th>8</th>
<th>16</th>
<th>24</th>
<th>32</th>
<th>40</th>
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<tr>
<td>$T$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>1</td>
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Table 2-2 $g = 2$ case

<table>
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<th>$T$ = ($^{11}<em>{12}$,$^{11}</em>{12}$,$^{11}_{12}$)</th>
<th>$8$</th>
<th>$16$</th>
<th>$24$</th>
<th>$32$</th>
<th>$40$</th>
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<td>($^{0}<em>{0}$, $^{0}</em>{0}$)</td>
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<td>($^{0}<em>{0}$, $^{0}</em>{0}$)</td>
<td>($^{0}<em>{0}$, $^{0}</em>{0}$)</td>
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Table 2-3 $g = 3$ case

<table>
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</table>

even unimodular lattices $K_i$ of rank 32, whose underlining root lattices are denoted below:
$K_1:3E_8, K_2:D_{24}, K_3:A_{24}, K_4:A_{17}\oplus E_7$

<table>
<thead>
<tr>
<th>$m, j$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>5</th>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>720</td>
<td>436320</td>
<td>219024000</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1104</td>
<td>1022304</td>
<td>781393536</td>
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<tr>
<td>3</td>
<td>1</td>
<td>600</td>
<td>303600</td>
<td>127512000</td>
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<tr>
<td>4</td>
<td>1</td>
<td>432</td>
<td>158112</td>
<td>48263040</td>
</tr>
</tbody>
</table>

even unimodular lattices $L_i$ of rank 32, whose underlining root lattices are denoted below:
$L_1:4E_8, L_2:D_{24}\oplus E_8, L_3:A_{24}\oplus E_8, L_4:E_7\oplus A_{17}+E_8, L_5:D_{32}, L_6:A_1\oplus A_{31}, L_7:A_{16}\oplus A_{16}$

Table of the Fourier coefficients of Siegel theta series of degree 3
$\phi_3(Z, L_m)$ (1 $\leq m \leq 8$)

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<tr>
<th>$m, j$</th>
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<td>158112</td>
<td>15760</td>
<td>45879920</td>
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<tr>
<td>3</td>
<td>1</td>
<td>1564</td>
<td>1589544</td>
<td>119092</td>
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<td>4</td>
<td>1</td>
<td>672</td>
<td>989713</td>
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<td>5</td>
<td>1</td>
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<td>2566328</td>
<td>236800</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>2664</td>
<td>2566328</td>
<td>685280</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>444</td>
<td>293208</td>
<td>142120</td>
</tr>
</tbody>
</table>

even unimodular lattices $M_i$ of rank 40, whose underlining root lattices are denoted below:
$M_1:E_8^6, M_2:D_{24}\oplus E_8, M_3:A_{24}\oplus E_8^2, M_4:E_7\oplus A_{17}\oplus E_8^2, M_5:D_{32}\oplus E_8, M_6:A_1\oplus A_{31}\oplus E_8, M_7:A_{17}^2\oplus E_8, M_8:D_{20}, M_9:D_{40}, M_{10}:D_{28}\oplus D_{12}$. The lattices $M_{11}, M_{12}$ and $M_{13}$ respectively are the ones coming from doubly even self-dual [40, 20, 8] codes: Iorgov’s $C_2$ [9], a code in [17], Iorgov’s code $C_3$ [9] respectively.

Table of the Fourier coefficients of Siegel theta series of degree 3
$\phi_3(Z, M_m)$ (1 $\leq m \leq 13$)
Theorem 5.1. There is a pair of even unimodular 40 dimensional lattices $L_1$ and $L_2$ such that their Siegel theta series of degrees 1 and 2 coincide and their theta series of degree 3 differ.

Theorem 5.2. There is a pair of even unimodular 40 dimensional non-isomorphic lattices $L_3$ and $L_4$ such that their Siegel theta series of degrees 1, 2 and 3 coincide.

6 A brief sketch of computing the Fourier coefficients of $\theta_3(Z, L)$

We compute

$$a(T, L) = \# \{(x, y, z) \in L^3 \mid [x, y, z] = 2T\},$$

for the case when $L$ is an even unimodular 40-dimensional extremal lattice constructed from binary code. This quantity is expressed as

$$a(T, L) = \sum_{(x, y) \in L^2, [x, y] = (\begin{array}{ll} t_1 & t_12 \\ t_12 & t_2 \end{array})} \mu(x, y; t_1, t_{13}, t_{23}),$$

where

$$\mu(x, y; t_1, t_{13}, t_{23}) = \# \{z \in \Lambda_{2\ell_S} \mid (x, z) = 2t_{13}, (y, z) = 2t_{23}\}.$$ 

We need to compute $a(T, L)$ for particular $T$'s given in the Table 2-3. For

$$T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

We see that

$$a(T, L) = \sum_{(x, y) \in L^2, [x, y] = (\begin{array}{ll} 4 & 0 \\ 0 & 4 \end{array})} \mu(x, y; 2, 0, 0),$$
and

\[ \mu(x, y; 2, 0, 0) = \#\{z \in \Lambda_4(L) | (x, z) = 0, (y, z) = 0\} = \mu_A(x, y; 2, 0, 0) + \mu_B(x, y; 2, 0, 0), \]

where

\[ \mu_A(x, y; 2, 0, 0) = \#\{z \in A | (x, z) = 0, (y, z) = 0\}, \]
\[ \mu_B(x, y; 2, 0, 0) = \#\{z \in B | (x, z) = 0, (y, z) = 0\}. \]

Further we get

\[ a(T, L) = \sum_{x \in A, y \in A} \{ \mu_A(x, y; 2, 0, 0) + \sum_{x \in A, y \in A} \mu_B(x, y; 2, 0, 0) \} \]
\[ + \sum_{x \in A, y \in B} \{ \mu_A(x, y; 2, 0, 0) + \sum_{x \in A, y \in A} \mu_B(x, y; 2, 0, 0) \} \]
\[ + \sum_{x \in B, y \in A} \{ \mu_A(x, y; 2, 0, 0) + \sum_{x \in A, y \in A} \mu_B(x, y; 2, 0, 0) \} \]
\[ + \sum_{x \in B, y \in B} \{ \mu_A(x, y; 2, 0, 0) + \sum_{x \in A, y \in A} \mu_B(x, y; 2, 0, 0) \} \]

We can easily prove that

Proposition 6.1. It holds that

\[ \sum_{x \in A, y \in A} \mu_B(x, y; 2, 0, 0) = \sum_{x \in A, y \in B} \mu_A(x, y; 2, 0, 0) = \sum_{x \in B, y \in A} \mu_A(x, y; 2, 0, 0), \]

and

\[ \sum_{x \in A, y \in B} \mu_B(x, y; 2, 0, 0) = \sum_{x \in B, y \in A} \mu_B(x, y; 2, 0, 0) = \sum_{x \in B, y \in B} \mu_A(x, y; 2, 0, 0). \]

By the above proposition we get an expression:

\[ a(T, L) = \sum_{x \in A, y \in A} \{ \mu_A(x, y; 2, 0, 0) + 3 \sum_{x \in B, y \in A} \mu_A(x, y; 2, 0, 0) \} \]
\[ + 3 \sum_{x \in B, y \in B} \{ \mu_A(x, y; 2, 0, 0) + \sum_{x \in A, y \in B} \mu_B(x, y; 2, 0, 0) \} \]
\[ + \sum_{x \in B, y \in B} \{ \mu_A(x, y; 2, 0, 0) + \sum_{x \in A, y \in A} \mu_B(x, y; 2, 0, 0) \} \]
\[ + \sum_{x \in B, y \in B} \{ \mu_A(x, y; 2, 0, 0) + \sum_{x \in A, y \in A} \mu_B(x, y; 2, 0, 0) \} \]
Computation of $\sum_{x \in A, y \in A} \mu_A(x, y; 2, 0, 0)$

We get

$$\sum_{x \in A, y \in A} \mu_A(x, y; 2, 0, 0) = 3120(2 \cdot 2812 + 2812 \cdot 2524) = 22161709440.$$

Computation of $\sum_{x \in B, y \in A} \mu_A(x, y; 2, 0, 0)$

We get

$$\sum_{x \in B, y \in A} \mu_A(x, y; 2, 0, 0) = 36480 \cdot (56 \cdot 2014 + 1984 \cdot 1798) = 134246983680.$$

Computation of $\sum_{x \in B, y \in B} \mu_A(x, y; 2, 0, 0)$

The biweight enumerator of a linear code of length $n$ is defined to be

$$BW(C, X_{11}, X_{10}, X_{01}, X_{00}) = \sum_{u, v \in C} X_{11}^{w_{11}(u, v)}X_{10}^{w_{10}(u, v)}X_{01}^{w_{01}(u, v)}X_{00}^{w_{00}(u, v)},$$

where $X_{11}, X_{10}, X_{01}$ and $X_{00}$ are algebraically independent variables over the field of complex numbers, and $w_{ij}(u, v) (0 \leq i, j \leq 1)$ is the number of the coordinates $k (1 \leq k \leq n)$ such that the $k$th component of $u$ takes the value $i$ and the $k$th component of $v$ takes the value $j$. We exhibit the biweight enumerators of the codes $C_i (1 \leq i \leq 4)$:

$$BW(C_1, X_{11}, X_{10}, X_{01}, X_{00}) = \cdots + 285X_{11}^8X_{00}^{32} + 5040X_{11}^4X_{10}^4X_{01}^4X_{00}^{28} +$$
$$+53760X_{11}^2X_{10}^6X_{01}^6X_{00}^{26} + 22140X_{10}^8X_{01}^8X_{00}^{24} + \cdots$$

$$BW(C_2, X_{11}, X_{10}, X_{01}, X_{00}) = \cdots + 285X_{11}^8X_{00}^{32} + 11760X_{11}^4X_{10}^4X_{01}^4X_{00}^{28} +$$
$$+40320X_{11}^2X_{10}^6X_{01}^6X_{00}^{26} + 28860X_{10}^8X_{01}^8X_{00}^{24} + \cdots$$

In the above we display all the terms for both $u$ and $v$ are of weight 8.

After all we get

$$\sum_{x \in B, y \in B} \mu_A(x, y; 2, 0, 0) = 27 \cdot (285 \cdot 70 \cdot 2008 + 5040 \cdot 48 \cdot 1540 + 53760 \cdot 64 \cdot 1360 + 22140 \cdot 128 \cdot 1216)$$
$$= 1092855490560 \text{ for codes } C_1, C_2$$
$$= 27 \cdot (285 \cdot 70 \cdot 2008 + 11760 \cdot 48 \cdot 1540 + 40320 \cdot 64 \cdot 1360 + 28860 \cdot 128 \cdot 1216)$$
$$= 1140584048640 \text{ for codes } C_3, C_4$$

Computation of $\sum_{x \in B, y \in B} \mu_B(x, y; 2, 0, 0)$

If we dare to explain every detail of the computation, it may take too much space, therefore we only describe
the inner product relations of the vectors $x, y$ and $z$ in $B$. The description is well-controlled by some terms of the triweight enumerator of a code $C$:

$$TW(C, X_{111}, X_{110}, X_{011}, X_{010}, X_{001}, X_{000}) = \sum_{u,v,w \in C} X_{111}^{u_{111}(u,v,w)} X_{110}^{u_{110}(u,v,w)} X_{101}^{u_{101}(u,v,w)} X_{011}^{u_{011}(u,v,w)} X_{100}^{u_{100}(u,v,w)} X_{010}^{u_{010}(u,v,w)} X_{001}^{u_{001}(u,v,w)},$$

where $X_{111}, X_{110}, X_{011}, X_{010}, X_{001}$ and $X_{000}$ are algebraically independent variables over the field of complex numbers, and $w_{ijh}(u, v, w) (0 \leq i, j, h \leq 1)$ is the number of the coordinates $k (1 \leq k \leq n)$ such that the $k$th component of $u$ takes the value $i$ and the $k$-th component $v$ takes the value $j$, and $k$-th component of $w$ takes the value $h$.

For our present computation we only need the terms coming from the codewords $u, v, w$ of weight 8. For instance, in case of $C_1$ terms such as $11760X_{111}^2 X_{110}^2 X_{101}^2 X_{011}^2 X_{100}^2 X_{010}^2 X_{01}^2 X_{\infty 0}^2$ and $42000X_{111}^4 X_{110} X_{011} X_{010} X_{001}$. There are 50 types of terms that correspond to triples of codewords of weight 8.

For a fixed $x \in B$ we want to count the vectors $y, z \in B$ such that $(x, y) = (x, z) = (y, z) = 0$. However the frequencies of the pairs $< y, z >$ vary according to the intersection relation among $suppx, suppy, suppz$. We omit the details.

After all we get

$$a\left(\begin{array}{lll}2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{array}\right), L(C_1) = 15596332778880,$$

$$a\left(\begin{array}{lll}2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{array}\right), L(C_2) = 15596205376896.$$

$$a\left(\begin{array}{lll}2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{array}\right), L(C_3) = a\left(\begin{array}{lll}2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{array}\right), L(C_4) = 17448486307200.$$

In the same way the values in the last table in Section 4 are determined. These values are the base of our Theorems in Section 5.

7 Further Research

7.1 Some Basic Difficulties

7.1.1 Graded Ring Structure

In genus (degree) 2 case the theory of Siegel modular forms has rich tools.

In genus 3 case thanks to Tsuyumine the graded ring structure of Siegel modular forms is available. However if we fix the weight $k$ we seem not to have the explicit method to determine the linear basis of the space of Siegel modular forms of genus 3 and weight $k$, although we could know the dimension of the space. We do not have the way to compute the Fourier expansion of those Siegel modular forms.

In genus 4 case the graded ring structure is not determined. Oura, Poor and Yuen [13] initiate to study this case.

7.1.2 Computational Difficulties

Duke-Runge map does not directly produce the Siegel theta series of even unimodular extremal lattice from the multiple weight enumerator of doubly even self-dual extremal binary code.

The weight enumerator of [24, 12, 8] binary Golay code is given by

$$W_{G_{24}}(x, y) = x^{24} + 759x^{18}y^{8} + 2576x^{12}y^{12} + 759x^{8}y^{16} + y^{24}.$$
In $g=1$ case the mapping from weight enumerators to modular forms is known as Broué-Enguehard map (cf. [2])

Example 1. In 24 dimension case.

$$W_{G_{24}}(\varphi_0(\tau), \varphi_1(\tau)) = 1 + 48q_1^2 + 195408q_1^4 + 16785216q_1^6 + 397963344q_1^8 + 4629612960q_1^{10} + \cdots$$

This is theta series of degree 1 associated with even unimodular lattice of root type $24 \times A_1$. The polynomial

$$\hat{W}_{G_{24}} = x^{24} + 759x^{16}y^8 + 2576x^{12}y^{12} + 759x^8y^{16} + y^{24} - 3(x^{20}y^4 - 4x^{16}y^8 + 6x^{12}y^{12} - 4x^8y^{16} + x^4y^{20})$$

leads to theta series of degree 1 associated with the Leech lattice:

$$W_{G_{24}}(\varphi_0(\tau), \varphi_1(\tau)) = 1 + 196560q_1^4 + 16773120q_1^6 + 398034000q_1^8 + 4629381120q_1^{10} + \cdots$$

Example 2. In 32 dimension there are five classes of doubly even self-dual binary linear codes, and they have identical weight enumerator:

$$W_{C_{32}}(x,y) = x^{32} + 620x^{24}y^8 + 13888x^{20}y^{12} + 36518x^{16}y^{16} + 13888x^{12}y^{20} + 620x^8y^{24} + y^{32}$$

The image of this polynomial under Broué-Enguehard map is

$$W_{C_{32}}(\varphi_0(\tau), \varphi_1(\tau)) = 1 + 64q_1^2 + 160704q_1^4 + 64543488q_1^6 + 4845725632q_1^8 + 137699222400q_1^{10} + \cdots$$

which is theta series of degree 1 associated with an even unimodular 32 dimensional extremal lattice.

Another polynomial:

$$\hat{W}_{C_{32}} = x^{32} + 620x^{24}y^8 + 13888x^{20}y^{12} + 36518x^{16}y^{16} + 13888x^{12}y^{20} + 620x^8y^{24} + y^{32}$$

leads to

$$W_{C_{32}}^\#(\varphi_0(\tau), \varphi_1(\tau)) = 1 + 167360q_1^4 + 65740800q_1^6 + 4867610560q_1^8 + 138035363840q_1^{10} + \cdots$$

which is theta series of degree 1 associated with an even unimodular 32 dimensional extremal lattice.

In $g=2$ case. We utilize the polynomials $P_8, P_{12}, P_{30}, P_{24}$ that are described in [14]. The biweight enumerator of extremal binary self-dual doubly even self-dual [32, 16, 8] code is

$$\frac{182}{729}P_8P_{24} + \frac{13}{27}P_8^4 + \frac{49}{729}P_8P_{12}^2 + \frac{49}{243}P_{12}P_{20}$$
The image under the Duke-Runge map is the Siegel theta series of degree 2 for the even unimodular lattice of root lattice type $32 \times A_1$.

The polynomial which corresponds to the Siegel theta series for even unimodular extremal lattice constructed from the above extremal code is

$$\frac{20}{81} P_8 P_{24} + \frac{4}{9} P_8^4 + \frac{25}{324} P_8 P_{12}^2 + \frac{25}{108} P_{12} P_{20},$$

which is not the biweight enumerator of a code, since it has negative coefficients.

A last remark: the reporter has downsized the total report, since he realizes the strong constraint that the number of pages should be under 16 posed by the organizer. The reader who wants to read this report more precisely may take enlarged copies.

References


[19] M. Ozeki, On the relation between the invariants of a doubly even self-dual binary code C and the invariants of the even unimodular lattices L(C) defined from the code C. Meeting on algebraic combinatorics Proc. RIMS No.671 (1988) 126-139


[26] E. Witt, Eine Identität zwischen Modulformen zweiten Grades, Abhandlungen aus Mathm. Seminar Hamburg, 14 (1941), 323-337,

