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Kyoto University
Differential equations and rational approximations of polylogarithms

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Abstract

• The monodromy's study of Fuchsian hypergeometric differential equation provides a natural framework for the explicit determination of rational approximations of polylogarithmic functions. Thus, we can obtain almost without calculation explicit determination of many polynomials and hypergeometric power series related to their Padé approximations.

From now on, using a classical way, one can study the arithmetic nature of numbers related to the values taken by these functions.

It is an expanded version of the conference given at the symposium on New aspects of analytic Number theory, held at the RIMS of the university of Kyoto in october. 27-29 , 2008.
I would like to thank the organizer Professor Takao Komatsu and also N.Hirata Kohno for their invitation to come to Japan.

1 Introduction

♦ In this paper I want to explain the origin of many formulas which are related to the simultaneous rational approximations of polylogarithmic functions.

Let us recall that:

Definition 1

\[ Li_q(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^q} \]
For $q = 1$, one recognizes the power series expansion to $-\log(1 - z)$.
For $q = 2$ this function is called the dilogarithmic function.

1.1 Arithmetic motivations

\[ \ell \]

The arithmetic motivation for searching such effective rational approximations comes from proving irrationality or transcendence of numbers arising as values of polylogarithmic functions, such as $Li_q(1/p), p \in \mathbb{Z}$,

\[ Li_q(1) = \zeta(q), \]

(for $q$ integer $q \geq 2$).

$\zeta(2), \zeta(3), \cdots$ etc,

- We shall now describe the preliminaries for the main result of this paper.
  (Marc Huttner: Israel Math Journal 2006), [Hu].

1.2 Riemann-Hilbert problem

\[ S \]

Find a very natural way to the explicit construction of functional linear forms in polylogarithmic functions using the construction of a fuchsian "hypergeometric" differential equations with prescribed singular points $0, 1, \infty$ and prescribed monodromy.

We solve in this particular case a "Riemann-Hilbert problem".

Remark 1 Let us recall that the Riemann-Hilbert problem is: Prove that there always exists a Fuchsian linear differential equation of order $q + 1$ such that its singular points and monodromy operator are given.

In general this fuchsian equation involves on accessory parameters and apparent singularities (i.e singular points for the differential equation but not for the solutions!)

For our special case there exists a solution, we shall prove that this equation does not involve accessory parameters and apparent singularities. This operator is thus unique!

We use a new explicit construction which replaces and generalizes many constructions often given without proofs by many authors. (See Apéry [Ap], Nesterenko, [Ne], Gutnik [Gu], Ball-Rivoal [Ba, Ri], Zudilin, [Zu].

1.3 Pochhammer symbol, hypergeometric power series

Definition 2 In the following if $\alpha \in \mathbb{C}$ we put $(\alpha)_0 = 1$ and if $n \geq 1$,

\[ (\alpha)_n = \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + n - 1) \]
Definition 3

\[ q+1F_q \left( \begin{array}{c} a_0, a_1, \ldots, a_q \\ b_1, b_2, \ldots, b_q \end{array} \mid z \right) \]

\[ = \sum_{n=0}^{\infty} \frac{\prod_{j=0}^{q}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{z^n}{n!} \]

\[ \propto \binom{\infty}{a_0} a_0 \binom{1}{1-b_1 a_1} 1 \binom{1- b_2}{a_2} 2 \cdots \binom{q-1}{a_{q-1}} \binom{d}{a_q} \]

denotes the hypergeometric power series.

1.4 Hypergeometric differential equation, Levelt's construction

\[ \bullet \] The hypergeometric power series is the holomorphic solution at 0 of the following differential equation of order \( q + 1 \),

\[ \mathcal{H}yp((a)_{i}, (b)_{i}) \]

\[ ((\theta + b_1 - 1)(\theta + b_2 - 1) \cdots (\theta + b_q - 1) - z(\theta + a_0)(\theta + a_2) \cdots (\theta + a_q))y(z) = 0. \]

The natural domain of definition of the solutions of the ordinary differential equation (ODE) is the Riemann \(-sphere\) \( \mathbb{CP}_1 \).

By examination the ODE \( \mathcal{H}yp((a)_{i}, (b)_{i}) \) has 0, 1, \( \infty \) at its only regular singular points.

\( q+1F_q \) can be continued to a meromorphic function on \( Z = \mathbb{CP}_1 - \{0, 1, \infty\} \) which is generally multivalued.

\[ \bullet \] The solution space of any order ODE on \( \mathbb{CP}_1 \) is determined by the characteristic exponents associated to a symbol called Riemann-P-scheme (see for example [AAR] ), which indicates the location of the singular points, and the exponents relative to each singularity.

(These exponents do not depend of the basis of solution choose!)

\[ \bullet \] The equation \( \mathcal{H}yp((a)_{i}, (b)_{i}) \) is free of accessory parameters and the Riemann-P-symbol related to this equation is

Theorem 1

\[ q+1F_q \left( \begin{array}{c} a_0, a_1, \ldots, a_q \\ b_1, b_2, \ldots, b_q \end{array} \mid z \right) \propto \binom{0}{0} \binom{\infty}{a_0} \binom{1}{1-b_1 a_1} 1 \binom{1- b_2}{a_2} 2 \cdots \binom{q-1}{a_{q-1}} \binom{d}{a_q} \]

\[ \begin{pmatrix} 0 & \infty & 1 \\ 0 & a_0 & 0 \\ 1 - b_1 & a_1 & 1 \\ 1 - b_2 & a_2 & 2 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & q - 1 \\ 1 - b_q & a_q & d \end{pmatrix} \]
$$d = \sum_{j=1}^{q} b_j - \sum_{j=0}^{q} a_j$$

The notation \( \propto \) indicates that the unique analytic solution \( f(z) \) (hypergeometric power series) \( f(z) = q+1 F_q(z) \) belongs to the zero exponent at \( z = 0 \) and satisfies \( f(0) = 1 \).

The main point is that at \( z = 1 \) there exist \( q \) holomorphic linearly independent solutions of \( \mathcal{H}yp((a)_i,(b)_i) \). This result is very important and is characteristic of the hypergeometric ODE , (Levelt) [Le].

**Remark 2** When \( d \in \mathbb{Z} \), one solution at \( z = 1 \) is in general logarithmic i.e can be written \( \psi(z) = u(z) + (1 - z)^d[v(z) \log(1 - z) + w(z)] \) where \( u \) is a polynomial of degree \( q - 1 \) and \( v \) resp \( w \) are analytic functions at \( z = 1 \).

### 1.5 Padé problem

\( \clubsuit \) Find the \( \sigma = q(n+1) \) coefficients of the polynomials \( A_k(z) \), of degree \( n \) (\( 1 \leq k \leq q \)) and the remainder \( R_\infty(z) \) such that for given \( \sigma_\infty \geq n+1 \), the linear form:

$$R_\infty(z) = A_0(z) + \sum_{k=1}^{q} A_k(z) Li_k(1/z).$$

satisfies \( \text{Ord}_{\infty} R_\infty(z) = \sigma_\infty \).

### 1.6 Rivoal's problem

\( \bullet \) Recall that : \( \text{Ord}_{\infty} R_\infty(z) = \sigma_\infty \). i.e

$$R_\infty(z) = \frac{1}{z^{\sigma_\infty}}(c_0 + c_1 \frac{1}{z} + \cdots)$$

with \( c_0 \neq 0 \). (The polynomial \( A_0(z) \) is completely determined and of degree \( \leq n - 1 \).)

Construct (if possible) these polynomials such that \( A_1(1) = 0 \) i.e \( R_\infty(1) \) exists. and also \( A_{q-1}(1) = A_{q-3}(1) = \cdots = A_2(1) = 0 \).

The following assumption :

$$\sigma = \sigma_\infty + \sigma_1 + \sigma_0 \quad (4)$$

where \( \sigma_1 \) and \( \sigma_0 \) are positive integer (related to analytic continuation of \( R_\infty(z) \) at \( z = 0 \) rep \( z = 1 \) ) will be needed to prove the following theorem :

[Huttner.Israel Math Journal,[Hu]]
Theorem 2 (Main theorem) ♦ Under the assumption (3), the polynomial $A_q(z)$ and the remainder $R_\infty(z)$ are solutions of the Fuchsian differential equation:

$$\theta^q(\theta + 1 - \sigma_0) - z(\theta + \sigma_\infty)(\theta - n)^q = 0$$

(5) $R_\infty(z)$ is analytic in the vicinity of $z = \infty$ and belongs to the exponent $\sigma_\infty$ at $z = \infty$. As usual, we put $\theta = z \frac{d}{dz}$. $R_\infty(z)$ is an hypergeometric power series!

$$R_\infty(z) = C_\infty(n)(1/z)^{\sigma_\infty} \times$$

$$q+1 F_q \left( \begin{array}{c} \sigma_\infty, \cdots, \sigma_\infty, \sigma_\infty + \sigma_0 \\ \sigma_\infty + n, \cdots, \sigma_\infty + n \end{array} \right| \frac{1}{z} \right)$$

(6) where $C_\infty(n)$ denotes a constant which depends on $\sigma_\infty$ and $\sigma_0$.

$$A_q(z) = q+1 \ F_q \left( \begin{array}{c} -n, -n, \cdots, -n, \sigma_\infty \\ 1, \cdots, 1, 1 - \sigma_0 \end{array} \right| z \right)$$

(7)

- To obtain a polynomial (hypergeometric) solution at $z = 0$, we must suppose that $\sigma_0 = 0$. (In this case the polynomial $A_q(z) \in \mathbb{Z}[z]$ or $1 - \sigma_0 \leq -n$. i.e, $\sigma_0 > 1 + n.$) (See the well-poised-case where we have the relation:

$$\sigma_\infty + 1 - \sigma_0 = 1 - n$$

In particular the study of this differential equation gives the rational approximation related to Rivoal’s theorem.

Theorem 3 (Rivoal’s Theorem) For any even $q \geq 4$,

$$\text{dim}_\mathbb{Q}(\mathbb{Q} + \mathbb{Q}\zeta(3) + \mathbb{Q}\zeta(5) + \cdots + \mathbb{Q}\zeta(q-1)) \geq \frac{1 + o(1)}{1 + \log 2} \log(q)$$

2 Polylogarithmic functions and local systems

♦ Now we review the necessary mathematical background which allows to understand this lecture:
In the following we put:

$$Z = \mathbb{P}_1(\mathbb{C}) - \{0, 1, \infty\}$$

Let us recall that for $q$ integer, $q \geq 1$

$$Li_q(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^q}, z \in \mathbb{C}, \ |z| \leq 1$$

(8)
(If $q \geq 2$ and $< 1$ if $q = 1$).

has an analytic continuation to the cut plane $X = \mathbb{C} - [1, +\infty[$.

For $q \geq 2$, we have

$$\theta(Li_q(z)) = Li_{q-1}(z)$$

In this case, $y(z) = Li_q(z)$ is the holomorphic solution of the non-homogeneous differential equation:

$$(1 - z) \frac{d}{dz} (\theta^{q-1})(y) = 1$$

**Remark 3** A basis of solutions at $z = 0$ of this equation is

$$1, \log z, \frac{(\log z)^2}{2}, \cdots, \frac{(\log z)^{q-1}}{(q - 1)!}, Li_q(z)$$

The polylogarithmic functions, $Li_q(z)$ has an analytic continuation to $X$ and may be conceived of as a 'multivalued' function on $Z$ (i.e. function on $W$ the universal covering of $Z$).

Let us recall also the following integral formulæ

$$Li_1(z) := -\log(1 - z) = \int_0^z \frac{dt}{1 - t}$$

and for the higher logarithm:

$$Li_{q+1}(z) := \int_0^z \frac{Li_q(t)}{t} dt.$$ 

We use the analytic continuation of $Li_1(z), Li_2(z) \cdots, Li_q(z)$ along loops $\gamma_1$ circling $z = 1$, and $\gamma_0$ circling $z = 0$.

- Analytic continuation along $\gamma_1$ gives:

$$Li_k(z) \rightarrow Li_k(z) + \frac{(2i\pi)^{k-1}}{(k - 1)!}(\log z)^{k-1}$$

Using monodromy, it is easy to see that the $q + 1$ functions

$$1, \log(1 - z), Li_2(z) \cdots, Li_q(z)$$

are $\mathbb{Q}(z)$ linearly independent. Thus, we obtain a local system

$$\mathcal{P}L_i(q) =: \mathbb{C}(z)\{\log(1 - z), \cdots, Li_q(z)\}$$

which is of rank $q + 1$ over $\mathbb{C}(z)$.

**Remark 4** • The connections formulæ for the $Li_q(z)$ between $z = 0$ and $z = \infty$ involve Bernoulli polynomials in $\log z$. That give the analytic continuations of $R_\infty(z)$ at $z = 0$ and $z = 1$.

The monodromy group of this local system is well known; it is in particular unipotent.
3 Periods

We use the analytic continuation of $Li_1(z), Li_2(z) \cdots, Li_q(z)$ along loops $\gamma_1$ and $\gamma_0$.

The second row is a result of the monodromy transform of the first row along loop $\gamma_1$, the third row along loop $\gamma_0$, i.e analytic continuation along

$$\gamma_1, \gamma_0 \circ \gamma_1 \cdots, \gamma_0^{q-2} \circ \gamma_1, \gamma_0^{q-1} \circ \gamma_1.$$  

We obtain the following matrix of "periods":

**Theorem 4**

$$\Lambda(z) = \begin{pmatrix} 1 & Li_1(z) & \cdots & Li_q(z) \\ 0 & 2i\pi & \cdots & 2i\pi \log (q-1)/z/(q-1)! \\ 0 & 0 & \cdots & (2i\pi)^{q-1} \end{pmatrix}$$

3.1 Proofs: Analytic construction of linear forms of polylogarithmic functions

Let us recall the main steps of this proof which is almost the same as in [Hu]). (In this paper we study the approximation at infinity, i.e. $z$ is replaced by the local parameter $1/z$).

- Consider the linear form:

**Definition 4**

$$R_{\infty}(z) = A_0(z) + \sum_{k=1}^{q} A_k(z) Li_k(1/z).$$

Now this form gives rise to linear forms obtained by use of analytic continuation of $R_{\infty}(z)$ along loops based in a vicinity of $z = 1$ resp $z = 0$ (i.e. is monodromy around the points $z = 1$ and $z = 0$)

$$\begin{pmatrix} R_{\infty}(z) \\ R_1(z) \\ \vdots \\ R_q(z) \end{pmatrix} = \Lambda(z) \begin{pmatrix} A_0(z) \\ \vdots \\ A_k(z) \end{pmatrix}$$

Now, from a local system of rank $q+1$ we can construct a differential equation whose solutions are given by a basis of this local system.
Theorem 5 (Classical theorem) • Let \( f_1(z), \ldots, f_{q+1}(z) \) be a system of multivalued and regular holomorphic functions on \( Z \) such that its Wronskian \( \det(f_i^{(j)}) \neq 0 \) and such that the analytic continuations of the \( f_j \)'s along the loops \( \gamma_j \) define automorphisms of the space of functions spanned by \( f_k \)'s.

Then there exists a \((q+1)^{th}\) order differential equation with coefficients in \( \mathbb{C}(z) \) such that the system \( f_1(z), \ldots, f_{q+1}(z) \) of functions is its fundamental system. (The matrix of analytic continuations of the \( f_j \)'s along loops \( \gamma_j \) are called monodromy matrices).

Using this theorem, we obtain:

Theorem 6 • \( R_1(z), \ldots, R_q(z) = (2i\pi)^q A_q(z) \) satisfy the same Fuchsian differential equation of order \( q + 1 \) as \( R_\infty(z) \).

3.2 Applications of Levelt's construction to \textit{Padé} problem

As the analytic continuation of \( R_\infty(z) \) along \( \gamma_1 \) is

\[
R_\infty(z) \rightarrow R_\infty(z) + 2i\pi \sum_{k=1}^{q} A_k(z)(\frac{\log(1/z)^{k-1}}{(k-1)!})
\]

• We put now

\[
R_1(z) = 2i\pi \left( \sum_{k=1}^{q} A_k(z)(\frac{\log(1/z)^{k-1}}{(k-1)!}) \right)
\]

Using analytic continuation of \( R_1(z) \) along \( \gamma_0 \) gives

\[
R_2(z) = (2i\pi)^2 \sum_{k=2}^{q} A_k(z)(\frac{\log(1/z)^{k-2}}{(k-2)!}) ; R_3(z) = \ldots
\]

That gives \( \Lambda \times (A(z)) \) where

\[
A(z) = (A_0(z), A_1(z) \ldots, A_q(z))^t
\]

• The exponents at \( z = \infty \) are:

\[
\sigma_\infty, -n, \ldots, -n.
\]

• At \( z = 0 \) one finds

\[
\sigma_0, 0, \ldots, 0.
\]
\( \sigma_0 \) is the exponent given by analytic continuation at 0 of \( R_\infty(z) \). 

- At \( z = 1 \):

\[ 0, 1, \cdots, q - 1, \sigma_1 \]

Fuchs relation for Fuchsian differential equations of order \((q+1)\) gives:

\[
\sigma_0 + \sigma_\infty + \sigma_1 - qn + \frac{q(q-1)}{2} = \frac{q(q+1)}{2}
\]

\[
\sigma_0 + \sigma_\infty + \sigma_1 = q(n + 1)
\]

(There does not exist apparent singularities and we find exactly the number of coefficient of the polynomials \( A_k(z) \).)

- Let \( \sigma_0 \), and at \( z = 1 \) (\( \sigma_1 \)), be the exponents related to the analytic continuations of \( R_\infty(z) \) (which depend on additional assumptions on the polynomials \( A_k(z) \)).

- The Riemann scheme related to this equation gives the main theorem:

**Theorem 7 (Main Riemann scheme)**

\[
P \begin{pmatrix}
0 & \infty & 1 \\
\sigma_0 & \sigma_\infty & \sigma_1 \\
0 & -n & 0 \\
0 & -n & 1 \\
\vdots & \vdots & \vdots \\
0 & -n & q - 1
\end{pmatrix} \left| \frac{1}{z} \right|
\]

which can be written:

\[
(1/z)^{\sigma_\infty} P \begin{pmatrix}
0 & \infty & 1 \\
\sigma_0 & \sigma_\infty + \sigma_0 & \sigma_1 \\
-\sigma_\infty - n & \sigma_\infty & 1 \\
-\sigma_\infty - n & \sigma_\infty & 2 \\
\vdots & \vdots & \vdots \\
-\sigma_\infty - n & \sigma_\infty & 0
\end{pmatrix} \left| \frac{1}{z} \right|
\]

- The elements of this local system are solutions of the following differential equation:

\[
(\theta^q(\theta + 1 - \sigma_0) - z(\theta + \sigma_\infty)(\theta - n)^q)(y) = 0.
\]

Within a multiplicative constant this give the formulae of the main theorem for the remainder as well as the Fuchsian differential equation.

- If one puts \( A_q(z) = \sum_{j=0}^{n} c(j) z^j \), the other polynomials are obtained by
the use of Frobenius method for solving Fuchsian linear differential equations, i.e., for $1 \leq k \leq q - 1$,

$$A_{q-k}(z) = \frac{d^k}{dt^k} \left[ \sum_{j=0}^{n} c(j + t)z^j \right]_{t=0}$$

Let us recall that the logarithmic solutions of the Fuchsian differential equation are given by

$$R_k(z) = \frac{d^k}{dt^k} \left[ \sum_{j=0}^{n} c(j + t)z^{j+t} \right]_{t=0}.$$

- The Padé case is related to $\sigma_0 = \sigma_1 = 0$.

### 3.3 D-modules

- But if $\sigma_0 \geq 1 + n$, i.e. if there exist relations between the analytic continuation of the power series $R_\infty(z)$ at $z = \infty$ and at $z = 0$, we find that the rank of the $D$-module $\frac{Q(z)(\theta)}{L(\theta)} \cong \mathbb{Q}(z)[Li_1(1/z), \cdots, Li_q(1/z)]$.

- The previous fact has been verified by Rivoal himself and has been generalized by Nesterenko.

There exists also an elementary proof using a decomposition in partial fraction of $R_\infty(z)$ [Ba,Ri].

For a proof, we can use the following relations:

$$(\theta + a_0)_p/(a_0)_{p+1} F_q \left( \begin{array}{c} a_0, a_1, \cdots, a_q \\ b_1, b_2, \cdots, b_q \end{array} \right) \left| z \right) = q+1F_q \left( \begin{array}{c} a_0 + p, a_1, \cdots, a_q \\ b_1, b_2, \cdots, b_q \end{array} \right)$$

If, for instance $a_0 + p = b_1$, we obtain

$$qF_{q-1} \left( \begin{array}{c} a_1, \cdots, a_q \\ b_2, \cdots, b_q \end{array} \right)$$

### 3.4 The well-poised case

- We consider the differential operator

$$H(\theta) = (\theta + a_0)_p/(a_0)$$

for $a_0 = \sigma_\infty; p = \sigma_\infty - n - 1$.

In this case we can write,
\[ R_\infty(z) = H(\theta)(\tilde{R}(z)) \]

\( \tilde{R}(z) \) being a solution of a Fuchsian differential equation of order \( q \) and \( H(\theta) \) commutes with the monodromy.

We obtain a shift for the linear combination of polylogarithmic functions \( Li_k(1/z) \) with, for \( 1 \leq k \leq q - 1 \) : the same polynomials.

The new polynomial \( A_{q-1} \) replaces the previous polynomial \( A_q(z) \) i.e \( A_q(z) = 0 \), etc.

- The new linear form becomes:

\[
R_\infty(z) = \sum_{j=1}^{q-1} A_j(s) Li_j(1/z) + \tilde{A}_0(z)
\]

\[
\tilde{A}_0(z) = -[A_q(z) Li_{q-1}(1/z) + \cdots A_2(z) Li_1(1/z)]_{n-1}
\]

(polynomial part at the order \( n - 1 \)). This gives the "well-poised-case".

- In the literature concerning special functions, [AAR]: if the parameters of the hypergeometric power series satisfy

**Definition 5** \( a_0 + 1 = a_1 + b_1 = \cdots = a_q + b_q \) the power-series is said well-poised.

- It is said very-well-poised if it is well-poised and \( a_1 = \frac{1}{2} a_0 + 1 \)

**Remark 5** In the present problem, in the very-well poised case, one finds that the first polynomial \( A_{q-1}(z) \) satisfies \( A_{q-1}(1) = 0 \).

The differential equation satisfied by this polynomial is of order \( q + 1 \) but the local system is of rank \( r_L = q - 2 \) over \( \mathbb{C}[z] \).

- Let us consider the relation

\[ \sigma_\infty - \sigma_0 + 1 = 1 - n \]  \hspace{1cm} (11)

This relation means that in the above differential equation \( y(z) \) is a solution if and only if \( z^n y(1/z) \) is also a solution. In this case, the remainder \( R_\infty(z) \) can be written [Well-poised remainder]

\[
R_\infty(z) = (1/z)^\sigma_\infty \cdot q+1 F_q \left( \begin{array}{c} 2\sigma_\infty, \cdots, \sigma_\infty, \sigma_\infty \\ \sigma_\infty + n + 1, \cdots, \sigma_\infty + n + 1 \end{array} \right | 1/z \)
\]

- The polynomial \( A_q(z) \) satisfies the relation

**Theorem 8 (A reciprocal polynomial)** \( A_q(z) = (-1)^{(q+1)n} z^n A_q(1/z) \).
Let us write $A_q(z) = \sum_{j=0}^{n} c_j z^j$.

For $0 \leq j \leq n$, we find, $c_j = c_{n-j}$.

Since these polynomials are solutions of a Fuchsian differential equation, the other polynomials are computed by Frobenius method.

• For $1 \leq k \leq q-2$ the polynomial coefficients of $A_k(z)$ satisfy the relations:

$$\frac{d^k}{dt^k}(c_{j+t})|_{t=0} = (-1)^k \frac{d^k}{dt^k}(c_{n-(j+t)})|_{t=0}.$$

We find:

$$A_{q-k}(z) = (-1)^{(q+1)n+k} z^n A_{q-k}(1/z).$$

### 3.5 Arithmetic applications

• For $k = 2 \cdots q-1$, the polynomials $A_k(z)$ are such that

$$A_{q-2}(1) = A_{q-4}(1) \cdots A_2(1) = A_1(1) = 0.$$

In particular if $q-1 = 2a+1$ is odd, we obtain the famous Rivoal’s relation on linear form of $\zeta(2k+1)$ [Ba,Ri]. The remainder can thus be written:

**Theorem 9**

$$R_{\infty}(1) = A_{2a+1}(1)\zeta(2a+1) + \cdots + A_3(1)\zeta(3) + \bar{A}_0(1).$$

We have multiplied the remainder by a normalized constant related to various integral values which represent $R_{\infty}(z)$ and also by a common denominator $D_n$ such that:

$A_q(z) \in \mathbb{Z}[z]$ and $d_n^k A_{q-k}(z) \in \mathbb{Z}[z]

• We put

$$\sigma_\infty = rn+1, \sigma_0 = \sigma_\infty + n$$

with the parameter $r$ satisfying: $1 \leq r \leq \frac{q-1}{2}$. ($\sigma_1 \geq 1$.

These assumptions permits us to compute the remainder $R_{\infty}(z)$ at $z = 1$.

In this case,

$$A_q(z) = \phantom{q+1} F_q \left( \begin{array}{c} rn+1, -n, \ldots, -n \\ -(r+1)n, 1, \ldots, 1 \end{array} | z \right).$$

The remainder is given by

$$R_{\infty}(1) = C(n, r, q). q+1 F_q \left( \begin{array}{c} (2r+1)n+2, rn+1, \ldots, rn+1 \\ (r+1)n + 2, \ldots, (r+1)n + 2 \end{array} | 1 \right).$$
\[ C(n, q, r) = \frac{n!^{q-1-2r}((2r+1)n+1)!}{((r+1)n+1)!^q} \]

- The remainder can also be written using Euler’s integral

\[ R_\infty(z) = \frac{(2r+1)n!}{n!^{2r+1}} \int_{[0,1]^q} \left[ \frac{\prod_{l=1}^{q}(t_l^r(1-t_l))}{(1-t_1\cdots t_q)^{2r+1}} \right]^n dt_1 \cdots dt_q \]

4 Apéry, Gutnik, Nesterenko, \( \zeta(2) \) and \( \zeta(3) \)

- Let us recall Beukers’s and Gutnik’s method, [Be], [Gu] concerning simultaneous approximations of \( \zeta(2) \) and \( \zeta(3) \).
- Linear Algebra shows that there exists four polynomials

\[ A_3(z), A_2(z), A_1(z), A_0(z) \]

of degree \( n \) such that:

\[ R_1(z) = A_3(z)Li_2(1/z) + A_2(z)Li_1(1/z) + A_1(z) \]
\[ R_2(z) = 2A_3(1/z)Li_3(z) + A_2(Li_2(1/z) + A_0(z) \]

satisfying \( \text{Ord}_\infty R_1(z) \geq n+1, \text{Ord}_\infty R_2(\infty) \geq n+1 \) and \( A_2(1) = 0. \)

Remark 6 The main idea to motivate the introduction of \( R_2(z) \) comes from ”Frobenius method of perturbing the power series”.

In this aim we introduce the function:

\[ \text{Li}_k(z, s) = \sum_{n=1}^{\infty} \frac{z^{n+s}}{(n+s)^k} \]

where \( s \) denotes a ’formal’ variable.

Since,

\[ \frac{\partial \text{Li}_k(z, s)}{\partial s} |_{s=0} = \text{Li}_k(z) \log z - k\text{Li}_{k+1}(z) \]

Using the following function:

\[ R_1(1/z, s) = A_3(z)Li_2(1/z, s) + A_2(z)Li_1(1/z, s) + A_1(z, s)(1/z)^s \]

- An easy computation shows that:

\[ \frac{\partial R_1(1/z, s)}{\partial s} |_{s=0} = \]
with $A_1(z) = A_1(z,s)|_{s=0}$ and

$$A_0(z) = \frac{\partial A_1(z,s)}{\partial s}|_{s=0}.$$ 

We put now

$$\tilde{R}_2(z) = \log(1/z).R_1(z) - R_2(z) \quad (12)$$

We can construct a linear differential operator $L$ of order at least 4 such that at $z = \infty$.

Since

$$\tilde{R}_2(z) = \log(1/z).R_1(z) - R_2(z)$$

is a (logarithmic) solution of $L = 0$.

Monodromy around 0 shows that $L(R_1(z)) = 0$.

Now if we put:

$$R_3(z) = A_3(z) \log(1/z) + A_2(z),$$

monodromy around 1 shows that $L(R_3(z)) = 0$.

Monodromy around 0 for $R_4(z) = A_3(z)$ yields $L(R_4(z)) = 0$.

**Theorem 10** *The 'Levelt basis' of solutions of $L$ at 0 is*

$$\tilde{R}_2(z), R_1(z), R_3(z), R_4(z)$$

They are linearly independent solutions at $z = \infty$ of a Fuchsian differential equation of order 4.

- The Riemann scheme of $L$ is:

$$P = \begin{pmatrix} 0 & \infty & 1 \\ 0 & -n & 0 \\ 0 & -n & 1 \\ 0 & n+1 & 2 \\ 0 & n+1 & 1 \end{pmatrix}.$$ 

The unique differential hypergeometric equation related to this Riemann scheme is

$$\theta^4 - z(\theta - n)^2(\theta + n + 1)^2 = 0.$$ 

- This Riemann scheme gives the famous Apéry's polynomial $[Ap]$

$$A_3(z) = 4F_3\left( -n, -n, n+1, n+1 \mid z \right)$$
\[
= \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 z^n
\]

- We also see that \( R_3(1) = 0 \) i.e \( A_2(1) = 0 \) (see the Riemann scheme)!
- We find the form of the remainder, only by studying this Riemann scheme!

One finds that \( R_1(z) \) is equal (with the choice of a multiplicative normalisation 's constant) to

\[
\frac{n!^4}{(2n)!^2} (1/z)^{n+1} \sum_{n=0}^{\infty} \binom{n+1}{2n+1, 2n+1, 1} |1/z|
\]

If one puts,

\[
R_1(z) = \frac{n!^4}{(2n+1)!^2} \frac{1}{z^{n+1}} r_1(z)
\]

and \( r_1(z) = \sum_{n=0}^{\infty} c_n (1/z)^n \),

- The 'logarithmic' solution belonging to the exponent \( n+1 \) is given by

\[
r_2(z) = \frac{\partial}{\partial t} (\sum_{k=0}^{\infty} c_{n+t} (1/z)^{n+t})|_{t=0}
\]

4.1 \( \zeta(3) \) is irrational!

- Since \( \log 1 = 0 \), we find:

\[
\frac{\partial}{\partial t} (\sum_{k=0}^{\infty} c_{n+t} (1/z)^{n+t})|_{t=0}
\]

(which gives Beukers or Nesterenko's integral for the remainder.) Since, \( d_n^3 A_0(1) \in \mathbb{Z} \), we obtain

\[
(2A_3(1)\zeta(3) + A_0(1))d_n^3 = r_2(1).d_n^3
\]

Since,

\[
\lim_{n \to \infty} d_n^3 r_2(1) = 0
\]

the irrationality of \( \zeta(3) \) is proved!

We can conclude that in many cases, the study of the Riemann scheme gives a complete answer for the determination of simultaneous rational approximation of polylogarithmic functions.
References


[Ap] Apéry, R. *Irrationalité de $\zeta(2)$ et $\zeta(3)$*, Asterisque 61, 11-13 (197).


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