Four-term leaping recurrence relations

弘前大学大学院理工学研究科 小松 崇夫（Takao Komatsu）
Graduate School of Science and Technology
Hirosaki University

1 Introduction

Given a three-term linear recurrence relation \( Z_n = T(n)Z_{n-1} + U(n)Z_{n-2} \) \( (n \geq 2) \), where the initial values \( Z_0, Z_1 \) are arbitrary integral values, and \((T(n))_{n \geq 0}, (U(n))_{n \geq 0}\) are integer sequences with \( U(n) \neq 0 \) for all \( n \geq 0 \).

In 2008 Elsener and the author constructed a leaping three-term recurrence relation from the original relation. Namely, for fixed positive integers \( k \) and \( 0 \leq i < k \), they obtained a three-term relation concerning \( z_n = Z_{kn+i} \).

For integers \( a, l \) with \( l \geq 1 \) we define the determinant

\[
K_l(a) = \begin{vmatrix}
T(a) & 1 & 0 \\
-U(a+1) & T(a+1) & 1 \\
0 & -U(a+2) & T(a+2) \\
& & \\
T(a+l-2) & 1 & \\
-U(a+l-1) & T(a+l-1) & \\
\end{vmatrix},
\]

with \( K_0(a) = 1 \). Let

\[
\Omega(M) = U(M-r)U(M-r+1)\ldots U(M-1),
\]
with \( M = (n-1)r+i+2 \). Then we have the following ([3, Theorem 2]).

Theorem 1 Given a three-term recurrence formula

\[
Z_n = T(n)Z_{n-1} + U(n)Z_{n-2} \quad (n \geq 2)
\]

with arbitrary initial values \( Z_0, Z_1 \) and two sequences of integers,

\[
(T(n))_{n \geq 0} = \left(a_0, a_1, a_2, \ldots, a_{\rho}, T_1(k), T_2(k), \ldots, T_w(k)\right)_{k=1}^{\infty},
\]

\[
(U(n))_{n \geq 0} = \left(b_0, b_1, b_2, \ldots, b_{\rho}, U_1(k), U_2(k), \ldots, U_w(k)\right)_{k=1}^{\infty},
\]

where \( U(n) \neq 0 \) for all \( n \geq 0 \), and \( \rho \geq 0, w \geq 1 \) are fixed integers. Then, for any integers \( r \) and \( i \) with \( r \geq 2, 0 \leq \rho \leq i < \rho + r \) and \( n \geq 2 \),

\[
K_{r-1}(M-r) \cdot z_n - \left(K_{r-1}(M)K_r(M-r) + U(M)K_{r-1}(M-r)K_{r-2}(M+1)\right) \cdot z_{n-1} + (-1)^{\rho} \Omega(M)K_{r-1}(M) \cdot z_{n-2} = 0
\]

holds for \( z_n = Z_{rn+i} \). For \( T(a) > 0 \) and \( U(a) > 0 \) for all \( a > \rho \) one has \( K_{r-1}(M) \neq 0 \).

In particular, with \( Z_n = q_n, q_0 = 1, q_1 = a_1 \) or \( Z_n = p_n, p_0 = a_0, p_1 = a_0a_1 + b_1 \), this recurrence formula for \( z_n \) is satisfied by the denominators \( q_{rn+i} \) and numerators \( p_{rn+i} \), respectively, of the convergents of a non-regular continued fraction

\[
\left[a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \cdots + \frac{b_{\rho}}{a_\rho + \frac{U_1(k)}{T_1(k) + \frac{U_2(k)}{T_2(k) + \cdots + \frac{U_w(k)}{T_w(k)}}}}}}\right]_{k=1}^{\infty}.
\]

\(^{1}\)This research was partially supported by the Grant-in-Aid for Scientific Research (C) (No. 18540006), the Japan Society for the Promotion of Science.
In the case of regular continued fractions this result is reduced as follows.

Corollary 1 Given a three-term recurrence formula

\[ Z_n = T(n)Z_{n-1} + Z_{n-2} \quad (n \geq 2) \]

with arbitrary initial values \( Z_0, Z_1 \) and a sequence of integers,

\[ (T(n))_{n \geq 0} = \left( a_0, a_1, a_2, \ldots, a_\rho, \overline{T_1(k), T_2(k), \ldots, T_w(k)} \right)_{k=1}^{\infty}, \]

where \( \rho \geq 0 \) and \( w \geq 1 \) are fixed integers. Then, for any integers \( r \) and \( i \) with \( r \geq 2, 0 \leq \rho \leq i < \rho + r \) and \( n \geq 2 \),

\[ K_{r-1}(M - r) \cdot z_n - \left( K_{r-1}(M)K_r(M - r) + K_{r-1}(M - r)K_{r-2}(M+1) \right) \cdot z_{n-1} \]

\[ + (-1)^r K_{r-1}(M) \cdot z_{n-2} = 0 \]

holds for \( z_n = Z_{rn+i} \). For \( T(a) > 0 \) for all \( a > \rho \) one has \( K_{r-1}(M) \neq 0 \).

In particular, with \( Z_n = q_n, q_0 = 1, q_1 = a_1 \) or \( Z_n = p_n, p_0 = a_0, p_1 = a_0a_1 + 1 \), this recurrence formula for \( z_n \) is satisfied by the denominators \( q_{rn+i} \) and numerators \( p_{rn+i} \), respectively, of the convergents of a regular continued fraction

\[ \left[ a_0; a_1, a_2, \ldots, a_\rho, \overline{T_1(k), T_2(k), \ldots, T_w(k)} \right]_{k=1}^{\infty}. \]

Three-term leaping recurrence relations which are entailed from continued fractions were studied in the case of \( e \) by Elsner ([1]). Similar relations were also studied in the case of \( e^{1/s} (s \geq 2) \) by the author ([7], [8]). Such concepts were extended to the cases of more regular continued fractions and non-regular continued fractions ([2], [3]).

However, it is not easy to find an analogous result for linear four-term recurrence relations

\[ Z_n = U_1(n)Z_{n-1} + U_2(n)Z_{n-2} + U_3(n)Z_{n-3}, \]

where \( U_1(n) \), \( U_2(n) \) and \( U_3(n) \) are general sequences of integers. In this article we shall consider the leaping recurrence relations for four-term recurrence relations where \( U_1(n) = a_1, U_2(n) = a_2 \) and \( U_3(n) = a_3 \) are integer constants. Then we can have the leaping relation

\[ Z_n = b_1Z_{n-k} + b_2Z_{n-2k} + b_3Z_{n-3k} \quad (n = 3k, 3k+1, 3k+2, \ldots) \]

for any leaping step \( k \).

2 Leaping convergents

Let \( \alpha \) be a real number. Continued fraction expansion of \( \alpha \) is denoted by

\[ \alpha = [a_0; a_1, a_2, \ldots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}. \]

The \( n \)-th convergent is given by the irreducible rational number

\[ \frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]. \]

It is well-known that \( p_n \)'s and \( q_n \)'s satisfy the recurrence relations:

\[ p_n = a_np_{n-1} + p_{n-2} \quad (n \geq 0), \quad p_{-1} = 1, \quad p_{-2} = 0, \]
\[ q_n = a_nq_{n-1} + q_{n-2} \quad (n \geq 0), \quad q_{-1} = 1, \quad q_{-2} = 0. \]
Leaping convergents of continued fractions are those of every \( r \)-th convergent of continued fractions:

\[
\frac{P_r}{Q_r}, \quad \frac{P_{r+i}}{Q_{r+i}}, \quad \frac{P_{2r+i}}{Q_{2r+i}}, \quad \ldots, \quad \frac{P_{rn+i}}{Q_{rn+i}}, \quad \ldots
\]

For example, consider

\[
e^{1/s} = [1; \overline{(2k-1)s-1,1,1}]_{k=1}^{\infty} = [1;s-1,1,1,3s-1,1,1,\ldots] \quad (s \geq 2),
\]

then \( p_{3n} = 2s(2n-1)p_{3n-3} + p_{3n-6} \) and \( q_{3n} = 2s(2n-1)q_{3n-3} + q_{3n-6} \) \( (n \geq 2) \) ([7]).

### 3 Three-term relations

Three-term relations have been considered as in the continued fraction expansion of \( e \) ([1]), as that of \( e^{1/s} \ (s \geq 2) \) ([7], [8]), as in that of the type \([1; T_1(k), T_2(k), T_3(k)]_{k=1}^{\infty}\) ([9], [10]). Recently, three-term relations have been developed in the non-regular continued fractions ([2]), and finally as in Theorem 1 here ([3]).

On the other direction, one can simplify the general theorem, entailing some classical results. If \( T(n) = a_1, U(n) = a_2 \ (a_2 \neq 0) \) are integer constants in Theorem 1, we have the following.

**Theorem 2** If the sequence \( \{Z_n\}_n \) satisfies the three-term recurrence relation \( Z_n = a_1 Z_{n-1} + a_2 Z_{n-2} \) \( (a_2 \neq 0) \), then for any positive integer \( k \) we have

\[
Z_n = k \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{(k-i-1)!}{i!(k-2i)!} a_1^{k-2i} a_2^i \cdot Z_{n-k} + (-1)^{k+1} a_2^k \cdot Z_{n-2k} \quad (n \geq 2k).
\]

Moreover, if \( a_1 = a_2 = 1 \), then the sequence \( \{Z_n\}_n \) is called Fibonacci-type sequence. Moreover, if \( Z_0 = 0 \) and \( Z_1 = 1 \), then \( \{Z_n\}_n \) is the Fibonacci sequence \( \{F_n\}_n \).

**Corollary 2** For any positive integer \( k \) we have

\[
F_n = k \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{(k-i-1)!}{i!(k-2i)!} \cdot F_{n-k} - (-1)^k \cdot F_{n-2k} \quad (n \geq 2k).
\]

If we put \( k = 2, 3, \ldots, 10 \), then we have

\[
\begin{align*}
F_n &= 3F_{n-2} - F_{n-4}, \\
F_n &= 4F_{n-3} + F_{n-6}, \\
F_n &= 7F_{n-4} - F_{n-8}, \\
F_n &= 11F_{n-5} + F_{n-10}, \\
F_n &= 18F_{n-6} - F_{n-12}, \\
F_n &= 29F_{n-7} + F_{n-14}, \\
F_n &= 47F_{n-8} - F_{n-16}, \\
F_n &= 76F_{n-9} + F_{n-18}, \\
F_n &= 123F_{n-10} - F_{n-20}.
\end{align*}
\]

There is a classical result corresponding to this corollary (Ruggles, 1963 [11, identity 105, p.92]):

\[
F_n = L_k F_{n-k} + (-1)^{k+1} F_{n-2k},
\]

(2)
where $F_n$ and $L_n$ are Fibonacci number and Lucas numbers, respectively. Namely, they satisfy the three-term relations

\[
F_n = F_{n-1} + F_{n-2} \quad (n \geq 2), \quad F_0 = 0, \quad F_1 = 1; \\
L_n = L_{n-1} + L_{n-2} \quad (n \geq 2), \quad L_0 = 2, \quad L_1 = 1.
\]

Comparing (2) with (1), we have

Corollary 3

\[
L_k = k \sum_{i=0}^{[k/2]} \frac{(k-i-1)!}{i!(k-2i)!} \quad (k \geq 1).
\]

Proof of Theorem 2. Set $K_l = K_l(c)$. Then, $\{K_l\}_{l \geq 0}$ satisfies the recurrence relation:

\[
K_l = a_1 K_{l-1} + a_2 K_{l-2} \quad (l \geq 2), \quad K_0 = 1, \quad K_1 = a_1.
\]

Hence, for $l \geq 0$ we have

\[
K_l = \sum_{i=0}^{[l/2]} \frac{(l-i)!}{i!(l-2i)!} a_1^{l-2i} a_2^i.
\]

Applying Theorem 1 with $\Omega(M) = a_2^r$, we have

\[
Z_n = (K_r + a_2 K_{r-2}) \cdot Z_{n-1} - (-1)^r a_2^r \cdot Z_{n-2}.
\]

Since

\[
K_r + a_2 K_{r-2} = \sum_{i=0}^{[r/2]} \frac{(r-i)!}{i!(r-2i)!} a_1^{r-2i} a_2^i + \sum_{i=0}^{[r/2]-1} \frac{(r-i-2)!}{i!(r-2i-2)!} a_1^{r-2i-2} a_2^{i+1}
\]

\[
= r \sum_{i=0}^{[r/2]} \frac{(r-i-1)!}{i!(r-2i)!} a_1^{r-2i} a_2^i,
\]

we obtain the desired result.

4 Four-term relations

Consider the four-term recurrence relation

\[
Z_n = U_1(n) Z_{n-1} + U_2(n) Z_{n-2} + U_3(n) Z_{n-3}.
\]

For the moment, the corresponding result to Theorem 1 has not been known. However, one can relax the conditions, in order to get some typical results. If $U_1(n) = a_1, U_2(n) = a_2, U_3(n) = a_3$ are constants for all $n$, we have the following four-term leaping recurrence relation.

Theorem 3 If the sequence $\{Z_n\}_n$ satisfies the four-term recurrence relation $Z_n = a_1 Z_{n-1} + a_2 Z_{n-2} + a_3 Z_{n-3}$ for any positive integer $k$

\[
Z_n = k \sum_{j=0}^{[k/3]} \sum_{i=0}^{[(k-3j)/2]} \frac{(k-i-2j-1)!}{i!j!(k-2i-3j)!} a_1^{k-2i-3j} a_2^i a_3^{j} \cdot Z_{n-k}
\]

\[- k \sum_{j=0}^{[k/3]} \sum_{i=0}^{[(k-3j)/2]} (-1)^{k+i+j} \frac{(k-i-2j-1)!}{i!j!(k-2i-3j)!} a_1^{k-2i-3j} a_2^{i+2j} a_3^j \cdot Z_{n-2k} + a_3^k \cdot Z_{n-3k}
\]

$(n \geq 3k)$.  

In 2001 F. T. Howard obtained a similar result ([5]):

$$Z_{n} = J_{k}Z_{n-k} - a_{3}^{k}J_{-k}Z_{n-2k} + a_{3}^{k}Z_{n-3k},$$  \hspace{1cm} (3) 

where $J_{n}$ satisfies

$$J_{n} = a_{1}J_{n-1} + a_{2}J_{n-2} + a_{3}J_{n-3} \quad (n \geq 3),$$

$$J_{0} = 3, \quad J_{1} = a_{1}, \quad J_{2} = a_{1}^{2} + 2a_{2}.$$  \hspace{1cm} (4) 

$J_{-n} (n = 1, 2, \ldots)$ are determined by

$$J_{-n} = \frac{1}{a_{3}}(J_{-n+3} - a_{1}J_{-n+2} - a_{2}J_{-n+1}) \quad (n \geq 1).$$  \hspace{1cm} (5) 

Comparing (3) with Theorem 3, we obtain Corollary 4

$$J_{k} = k \sum_{j=0}^{\lfloor k/3 \rfloor} \sum_{i=0}^{\lfloor (k-3j)/2 \rfloor} \frac{(k-i-2j-1)!}{i!j!(k-2i-3j)!}a_{1}^{i}a_{2}^{k-2i-3j}a_{3}^{-k+i+2j},$$

$$J_{-k} = k \sum_{j=0}^{\lfloor k/3 \rfloor} \sum_{i=0}^{\lfloor (k-3j)/2 \rfloor} (-1)^{k+i+j}(k-i-2j-1)! \frac{(k-i-2j-1)!}{i!j!(k-2i-3j)!}a_{1}^{i}a_{2}^{k-2i-3j}a_{3}^{-k+i+2j}.$$  \hspace{1cm} (6) 

In the case of $k = 12,$

$$Z_{n} - (a_{1}^{12} + 12a_{1}^{10}a_{2} + 54a_{1}^{8}a_{2}^{2} + 112a_{1}^{6}a_{2}^{3} + 105a_{1}^{4}a_{2}^{4} + 36a_{1}^{2}a_{2}^{5} + 2a_{2}^{6} + 12a_{1}^{9}a_{3} + 96a_{1}^{7}a_{2}a_{3} + 252a_{1}^{5}a_{2}^{2}a_{3} + 240a_{1}^{3}a_{2}^{3}a_{3} + 60a_{1}a_{2}^{4}a_{3} + 42a_{1}^{6}a_{3}^{2} + 180a_{1}^{5}a_{2}a_{3}^{2} + 180a_{2}^{7}a_{3}^{2} + 24a_{2}^{5}a_{3}^{3} + 40a_{2}^{3}a_{3}^{4} + 48a_{1}a_{2}a_{3}^{3} + 3a_{3}^{5})Z_{n-12}$$

$$+ 180a_{1}a_{2}a_{3}^{2} + 252a_{2}^{5}a_{3}^{2} + 180a_{2}^{7}a_{3}^{2} + 24a_{2}^{5}a_{3}^{3} + 40a_{2}^{3}a_{3}^{4} + 48a_{1}a_{2}a_{3}^{3} + 3a_{3}^{5})Z_{n-12}$$

$$+ 180a_{1}a_{2}a_{3}^{2} + 252a_{2}^{5}a_{3}^{2} + 180a_{2}^{7}a_{3}^{2} + 24a_{2}^{5}a_{3}^{3} + 40a_{2}^{3}a_{3}^{4} + 48a_{1}a_{2}a_{3}^{3} + 3a_{3}^{5})Z_{n-12}$$

$$+ 180a_{1}a_{2}a_{3}^{2} + 252a_{2}^{5}a_{3}^{2} + 180a_{2}^{7}a_{3}^{2} + 24a_{2}^{5}a_{3}^{3} + 40a_{2}^{3}a_{3}^{4} + 48a_{1}a_{2}a_{3}^{3} + 3a_{3}^{5})Z_{n-12} - a_{3}^{12}Z_{n-36} = 0.$$  \hspace{1cm} (7) 

Put $a_{1} = a_{2} = a_{3} = 1.$ Then the four-term recurrence relation $T_{n} = T_{n-1} + T_{n-2} + T_{n-3}$ yields Tribonacci numbers $\{T_{n}\}_{n \geq 0}.$ If $T_{0} = 0,$ $T_{1} = T_{2} = 1$ and $T_{3} = 2,$ then the Tribonacci sequence is given by

$$1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 5768, 10609, 19513, 35890, 66012, 121415, \ldots$$

([11, p.527], [12, A000073]). Putting $k = 2, 3, \ldots, 10$ in Theorem 3, we have

$$T_{n} = 3T_{n-2} + T_{n-4} + T_{n-6},$$

$$T_{n} = 7T_{n-3} - 5T_{n-5} + T_{n-9},$$

$$T_{n} = 11T_{n-4} + 5T_{n-8} + T_{n-12},$$

$$T_{n} = 21T_{n-5} + T_{n-10} + T_{n-15},$$

$$T_{n} = 39T_{n-6} - 11T_{n-12} + T_{n-18},$$

$$T_{n} = 71T_{n-7} + 15T_{n-14} + T_{n-21},$$

$$T_{n} = 131T_{n-8} - 3T_{n-16} + T_{n-24},$$

$$T_{n} = 241T_{n-9} - 23T_{n-18} + T_{n-27},$$

$$T_{n} = 443T_{n-10} + 41T_{n-20} + T_{n-30}.$$
5 Five-term relations

Consider the sequence \( \{Z_n\}_n \) satisfying the five-term recurrence relation
\[
Z_n = a_1 Z_{n-1} + a_2 Z_{n-2} + a_3 Z_{n-3} + a_4 Z_{n-4} (a_4 \neq 0)
\]
Then how can we determine the integer constants \( b_1, b_2, b_3, b_4 \), satisfying
\[
Z_n = b_1 Z_{n-k} + b_2 Z_{n-2k} + b_3 Z_{n-3k} + b_4 Z_{n-4k}
\]
for any positive integer \( k (1 < k < n/4) \)?

In the case of \( k = 5 \)
\[
Z_n = (a_1^5 + 5a_1^3a_2 + 5a_1a_2^3 + 5a_1^2a_3 + 5a_2a_3)Z_{n-5} + (a_1^2 - 5a_1a_2^2 + 5a_1^2a_3 + 5a_1^3a_4 - 5a_2^2a_4)Z_{n-10} + (a_5 - 5a_2a_3a_4 + 5a_1a_2a_4 + 5a_1a_3^2 - 5a_2a_4^2)Z_{n-15} + a_4^2 Z_{n-20}.
\]

In the case of \( k = 6 \)
\[
Z_n = (a_1^6 + 6a_1^4a_2 + 9a_1^2a_2^2 + 2a_2^3 + 6a_1^3a_3 + 12a_1a_2a_3 + 3a_3^2 + 6a_1^2a_4 + 6a_2a_4)Z_{n-6} + (-a_2^6 + 6a_1a_2^4a_3 - 9a_1^2a_2^2a_3^2 + 6a_2^3a_3^2 + 2a_1^3a_3^3 - 12a_1a_2a_3^3 - 3a_3^4 - 6a_1^2a_2^3a_4 - 6a_2^4a_4 + 12a_1^3a_2a_3a_4 + 18a_1^2a_3^2a_4 - 3a_1^4a_4^2 - 9a_2a_4^2 + 18a_1^3a_4^2 + 2a_4^3 - a_4^6)Z_{n-12} + (a_3^6 - 6a_2a_3^4a_4 + 9a_2^2a_3^2a_4^2 + 6a_1a_3^3a_4^2 - 2a_2^3a_4^3 - 12a_1a_2a_3a_4^3 + 6a_3^2a_4^3 + 3a_1^2a_4^4 - 6a_2a_4^4)Z_{n-18} - a_4^6 Z_{n-24}.
\]

Tetranacci Numbers \( \{F_k^{(4)}\}_{k \geq 1} \) are the \( n = 4 \) case of the Fibonacci \( n \)-step Numbers, defined by
\[
F_k^{(4)} = F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)} + F_{k-4}^{(4)} (k \geq 5) \text{ with } F_1^{(4)} = F_2^{(4)} = 1, F_3^{(4)} = 2 \text{ and } F_4^{(4)} = 4.
\]
The first terms are
1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, 5536, 10671, 20569, 39648, 76424, 147312, 283953, 547337, 1055026, 2033628, 3919944, 7555935, 14564533, 28074040, 54114452, 104308960, ...

([12, A000078]). They satisfy the recurrence relations:
\[
\begin{align*}
F_k^{(4)} &= 3F_{k-2}^{(4)} + 3F_{k-4}^{(4)} - F_{k-6}^{(4)} - F_{k-8}^{(4)}, \\
F_k^{(4)} &= 7F_{k-3}^{(4)} + F_{k-6}^{(4)} + F_{k-9}^{(4)} + F_{k-12}^{(4)}, \\
F_k^{(4)} &= 15F_{k-4}^{(4)} - 17F_{k-6}^{(4)} + 1F_{k-8}^{(4)} - F_{k-16}^{(4)}, \\
F_k^{(4)} &= 26F_{k-5}^{(4)} + 16F_{k-10}^{(4)} + 6F_{k-15}^{(4)} + F_{k-20}^{(4)}, \\
F_k^{(4)} &= 51F_{k-6}^{(4)} + 15F_{k-12}^{(4)} - F_{k-18}^{(4)} - F_{k-24}^{(4)}, \\
F_k^{(4)} &= 99F_{k-7}^{(4)} - 13F_{k-14}^{(4)} + F_{k-21}^{(4)} + F_{k-28}^{(4)}, \\
F_k^{(4)} &= 191F_{k-8}^{(4)} - 81F_{k-16}^{(4)} + 15F_{k-24}^{(4)} - F_{k-32}^{(4)}, \\
F_k^{(4)} &= 367F_{k-9}^{(4)} + 127F_{k-18}^{(4)} + 19F_{k-27}^{(4)} + F_{k-36}^{(4)}, \\
F_k^{(4)} &= 708F_{k-10}^{(4)} + 58F_{k-20}^{(4)} + F_{k-30}^{(4)} - F_{k-40}^{(4)}.
\end{align*}
\]

\( b_1, b_3 \) and \( b_4 \) are calculated as follows.
Theorem 4
\[
b_1 = k \sum_{\kappa=0}^{\lfloor k/4 \rfloor} \sum_{i=0}^{\lfloor (k-i-2j-3\kappa-1)/2 \rfloor} \frac{(k-i-2j-3\kappa-1)!}{i!j!\kappa!(k-2i-3j-4\kappa)!} a_1^{k-2i-3j-4\kappa} a_2^i a_3^j a_4^\kappa,
\]
\[
b_2 = k \sum_{\kappa=0}^{\lfloor k/4 \rfloor} \sum_{i=0}^{\lfloor (k-i-2j-3\kappa-1)/2 \rfloor} (-1)^i \frac{(k-i-2j-3\kappa-1)!}{i!j!\kappa!(k-2i-3j-4\kappa)!} a_1^i a_2^j a_3^{k-2i-3j-4\kappa} a_4^{j+2i+3\kappa},
\]
\[
b_4 = (-1)^{k-1} a_4^k.
\]

However, it is not easy to find an explicit form of \(b_2\). This shall be discussed in the next section.

In 2005 Latushkin and Ushakov ([6]) obtained a different form of five-term leaping relations.
\[
Z_n = H_k Z_{n-k} + \frac{H_{2k} - H_k^2}{2} Z_{n-2k} + (-a_4)^k H_{-k} Z_{n-3k} - (-a_4)^k Z_{n-4k}, \tag{4}
\]
where
\[
H_n = x_1^n + x_2^n + x_3^n + x_4^n \quad (n \in \mathbb{Z})
\]
and \(x_1, x_2, x_3\) and \(x_4\) are the complex roots (including multiple roots) of the equation \(x^4 - a_1 x^3 - a_2 x^2 - a_3 x - a_4 = 0\). On the other hand, the sequence \(\{H_n\}_n\) satisfies the recurrence relation:
\[
H_n = a_1 H_{n-1} + a_2 H_{n-2} + a_3 H_{n-3} + a_4 H_{n-4} \quad (n \in \mathbb{Z}).
\]
The initial values are determined by
\[
H_0 = 4,
\]
\[
H_1 = a_1,
\]
\[
H_2 = a_1 H_1 + 2a_2 = a_1^2 + 2a_2,
\]
\[
H_3 = a_1 H_2 + a_2 H_1 + 3a_3 = a_1^3 + 3a_1 a_2 + 3a_3,
\]
\[
H_4 = a_1 H_3 + a_2 H_2 + a_3 H_1 + 4a_4 = a_1^4 + 4a_1 a_2 + 4a_1 a_3 + 2a_2^2 + 4a_4.
\]
Comparing their results (4) with ours in Theorem 4, we get the following.

Corollary 5
\[
H_k = k \sum_{\kappa=0}^{\lfloor k/4 \rfloor} \sum_{i=0}^{\lfloor (k-i-2j-3\kappa-1)/2 \rfloor} \frac{(k-i-2j-3\kappa-1)!}{i!j!\kappa!(k-2i-3j-4\kappa)!} a_1^{k-2i-3j-4\kappa} a_2^i a_3^j a_4^\kappa,
\]
\[
H_{-k} = k \sum_{\kappa=0}^{\lfloor k/4 \rfloor} \sum_{i=0}^{\lfloor (k-i-2j-3\kappa-1)/2 \rfloor} (-1)^i k^{i+k} \frac{(k-i-2j-3\kappa-1)!}{i!j!\kappa!(k-2i-3j-4\kappa)!} a_1^i a_2^j a_3^{k-2i-3j-4\kappa} a_4^{j+2i+3\kappa}\]

Pentanacci Numbers \(\{F_k^{(5)}\}_{k \geq 1}\) are the \(n = 5\) case of the Fibonacci \(n\)-step Numbers, defined by
\[
F_k^{(5)} = F_{k-1}^{(5)} + F_{k-2}^{(5)} + F_{k-3}^{(5)} + F_{k-4}^{(5)} + F_{k-5}^{(5)} \quad (k \geq 6)
\]
with \(F_1^{(5)} = F_2^{(5)} = 1, F_3^{(5)} = 2, F_4^{(5)} = 4\) and \(F_5^{(5)} = 8\). The first terms are
\[
1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3525, 6930, 13624, 26784, 52656, 103519, 203513,
400096, 786568, 1546352, 3080048, 6076577, 11749641, 23099186, 45411804, 89277256, 175514464, \ldots
\]
They satisfy the recurrence relations:

\[
F_k^{(5)} = 3F_{k-2}^{(5)} + 3F_{k-4}^{(5)} + F_{k-6}^{(5)} + F_{k-8}^{(5)} + F_{k-10}^{(5)}, \\
F_k^{(6)} = 7F_{k-3}^{(6)} + 4F_{k-6}^{(6)} + 4F_{k-9}^{(6)} + F_{k-12}^{(5)} + F_{k-15}^{(5)}, \\
F_k^{(6)} = 15F_{k-4}^{(5)} - F_{k-6}^{(5)} + F_{k-12}^{(6)} + F_{k-16}^{(6)} + F_{k-20}^{(6)}, \\
F_k^{(6)} = 31F_{k-5}^{(5)} - 49F_{k-10}^{(6)} + 31F_{k-15}^{(5)} - 9F_{k-20}^{(5)} + F_{k-25}^{(5)}, \\
F_k^{(5)} = 57F_{k-6}^{(6)} + 42F_{k-12}^{(6)} + 22F_{k-18}^{(6)} + 7F_{k-24}^{(5)} + F_{k-30}^{(6)}, \\
F_k^{(6)} = 113F_{k-7}^{(6)} + 7F_{k-14}^{(5)} + F_{k-21}^{(5)} + F_{k-28}^{(5)} + F_{k-35}^{(5)}, \\
F_k^{(5)} = 223F_{k-8}^{(6)} + 31F_{k-16}^{(6)} + 33F_{k-24}^{(5)} + F_{k-32}^{(6)} + F_{k-40}^{(6)}, \\
F_k^{(6)} = 439F_{k-9}^{(5)} - 140F_{k-18}^{(5)} + 4F_{k-27}^{(6)} + F_{k-36}^{(5)} + F_{k-45}^{(5)}, \\
F_k^{(5)} = 863F_{k-10}^{(5)} - 497F_{k-20}^{(6)} + 141F_{k-30}^{(5)} - 19F_{k-40}^{(6)} + F_{k-50}^{(6)}.
\]

6 A form of $b_2$ in five-term leaping relations

An explicit form of $b_2$ has not been known yet. Instead, there is a way to express $b_2$ by matrices.

\[ b_2 = a_2 \Lambda_{k-1} - 2a_3 \Phi_{k-1} + 3a_4 \Psi_{k-1} \]
\[ - a_3 \Phi_{k-1} + 2a_4 \Phi_{k-1} + a_4 \Psi_{k-1} - a_4 \Psi_{k-1} - a_1 \Psi_{k-1} \]

where

\[
\Lambda_n = \begin{bmatrix}
-a_2 & -a_1 & 1 & 0 \\
-a_3 & -a_2 & -a_1 & 1 \\
-a_4 & -a_3 & -a_2 & -a_1 \\
0 & -a_4 & -a_3 & -a_2 \\
\end{bmatrix},
\]

\[
\Phi_n = \begin{bmatrix}
-a_1 & 1 & 0 \\
-a_3 & -a_2 & -a_1 & 1 \\
-a_4 & -a_3 & -a_2 & -a_1 \\
0 & -a_4 & -a_3 & -a_2 \\
\end{bmatrix},
\]

\[
\Psi_n = \begin{bmatrix}
-a_1 & 1 & 0 \\
-a_2 & -a_1 & 1 & 0 \\
-a_4 & -a_3 & -a_2 & -a_1 \\
0 & -a_4 & -a_3 & -a_2 \\
\end{bmatrix}.
\]
Notice that

\[ \Lambda_n = -a_2 \Lambda_{n-1} + a_3 \Phi_{n-1} - a_4 \Psi_{n-1}, \]
\[ \Phi_n = -a_1 \Lambda_{n-1} + a_3 \Lambda_{n-2} - a_4 \Phi_{n-2}, \]
\[ \Psi_n = -a_1 \Phi_{n-1} + a_2 \Lambda_{n-2} - a_4 \Lambda_{n-3}. \]

\( b_2 \) may be expanded as follows.

\[
b_2 = (-1)^{k-1} k \left( a_2^2 - (a_1 a_3 - a_4) a_2^{k-2} - (a_3^2 - a_1^2 a_4) a_2^{k-3} \right) 
+ \left( \frac{(k-3)!}{2!(k-4)!} a_2^2 a_3^2 - \frac{(k-4)!}{(k-6)!} a_1 a_3 a_4 + \frac{k-3}{2} a_4^2 \right) a_2^{k-4} 
+ \left( \frac{(k-4)!}{(k-5)!} (a_1 a_3 - (k-6) a_4) a_2^{k-5} \right) 
- \left( \frac{(k-4)!}{3!(k-6)!} a_3^3 - \frac{(k-5)!}{2!(k-6)!} a_1 a_3 a_4 + \frac{k^2 - 15k + 60}{2} a_1 a_3 a_4 \right) a_2^{k-6} 
- \left( \frac{(k-5)!}{2!(k-7)!} a_2^2 a_3^2 - \frac{(k-6)!}{(k-7)!} a_1 a_3 a_4 + \frac{k^2 - 15k + 60}{2} a_2^2 a_4 \right) a_2^{k-7} 
+ \left( \frac{(k-5)!}{4!(k-8)!} a_1 a_3^4 - \frac{(k-6)!}{3!(k-8)!} a_2 a_3^3 - \frac{(k-7)!}{2!(k-8)!} a_1 a_3 a_4 + \frac{k^2 - 15k + 60}{2} a_3^3 \right) a_2^{k-8} 
+ \left( \frac{(k-6)!}{3!(k-9)!} a_2 a_3^3 - \frac{(k-7)!}{2!(k-9)!} a_1 a_3 a_4 + \frac{k^2 - 15k + 60}{2} a_2 a_4 \right) a_2^{k-9} 
- \left( a_3^3 - a_1 a_3 \right) (a_1 a_3 - (k-6) a_4) a_2^{k-10} 
- \cdots . \]

However, its simplified form has not been known.

7 (s+1)-term recurrence relations

We may extend terms to five, six, seven, and so on. In 1999 Howard got a general term leaping relation \([4]\). Young also found a different form \([13]\). This result holds for more-term recurrence relations, but it is not so useful practically in order to obtain an explicit form for any given \( s \).

If the sequence \( \{Z_n\}_{n \geq 0} \) satisfies the relation

\[ Z_n = a_1 Z_{n-1} + a_2 Z_{n-2} + \cdots + a_s Z_{n-s} \ (a_s \neq 0), \]

where \( Z_0, Z_1, \ldots, Z_{s-1} \) are arbitrary initial values, then we have

\[ Z_{n+i} = c_{r,1} Z_{r(n-1)+i} - c_{r,2} Z_{r(n-2)+i} + \cdots + (-1)^{s-1} c_{r,s} Z_{r(n-s)+i}, \]
where $c_{r,r}$, $c_{r,2}$ are determined by
\[
\prod_{\nu=0}^{r-1} (1-a_1(\zeta^\nu x)-a_2(\zeta^\nu x)^2-\cdots-a_s(\zeta^\nu x)^s) = 1-c_{r,r}x^r+c_{r,2r}x^{2r}-\cdots+(-1)^sc_{r,sr}x^{sr},
\]
where $\zeta$ is a primitive $r$-th root of unity.

As straight generalization of our theorems 2, 3, 4, we obtain the following.

**Theorem 5** If
\[
Z_n = b_1 Z_{n-k} + b_2 Z_{n-2k} + \cdots + b_{s-1} Z_{n-(s-1)k} + b_s Z_{n-sk},
\]
then
\[
b_1 = k \sum_{i_1+i_2+\cdots+i_{s-1}=0}^{2i_1+3i_2+\cdots+s-1} \frac{(k-i_1-2i_2-\cdots-(s-1)i_{s-1}-1)!}{i_1!i_2!\cdots i_{s-1}!(k-2i_1-3i_2-\cdots-si_{s-1})!} \times a_1^{i_1}a_2^{i_2}\cdots a_{s-1}^{i_{s-1}},
\]

\[
b_{s-1} = k \sum_{i_1+i_2+\cdots+i_{s-1}+i_s=0}^{2i_1+3i_2+\cdots+s-1} \frac{(-1)^{i_1-i_2-\cdots-(s-1)i_{s-1}}}{i_1!i_2!\cdots i_{s-1}!(k-2i_1-3i_2-\cdots-si_{s-1})!} \times a_1^{i_1}a_2^{i_2}\cdots a_{s-1}^{i_{s-1}}a_s^{i_s},
\]
where
\[
I = \begin{cases} 
  i_1 + i_3 + \cdots + i_{s-2} + i_{s-1} & \text{if } s \text{ is odd;} \\
  i_1 + i_3 + \cdots + i_{s-3} & \text{if } s \text{ is even},
\end{cases}
\]
and
\[
b_s = \begin{cases} 
  a_s^k & \text{if } s \text{ is odd;} \\
  (-1)^{k-1}a_s^k & \text{if } s \text{ is even.}
\end{cases}
\]

8 Periodicity

In [2, Theorem 3] a result about periodicity of three-term leaping relations is obtained.

**Theorem 6** Given a three-term recurrence formula
\[
Z_n = T(n)Z_{n-1} + U(n)Z_{n-2} \quad (n \geq 2)
\]
with arbitrary initial values $Z_0$, $Z_1$ and two sequences of integers $(T(n))_{n \geq 0}$ and $(U(n))_{n \geq 0}$, which both are (ultimately) periodic modulo $m$ with periods of length $r$, say
\[
(T(n) \mod m)_{n \geq 0} = (a_0, a_1, a_2, \ldots, a_p, T_1, T_2, \ldots, T_r),
\]
\[
(U(n) \mod m)_{n \geq 0} = (b_0, b_1, b_2, \ldots, b_p, U_1, U_2, \ldots, U_r).
\]
Then, the sequence $(Z(n))_{n \geq 0}$ is (ultimately) periodic modulo $m$. If $\rho \in \{0,1\}$ and $U(n) = 1$ for all $n \geq \rho$, then the sequence $(Z(n))_{n \geq 0}$ is periodic modulo $m$.

This result can be extended to the case of any term leaping relations.

**Theorem 7** Given a $(s+1)$-term recurrence formula $Z_n = T_1(n)Z_{n-1} + T_2(n)Z_{n-2} + \cdots + T_s(n)Z_{n-s} \quad (n \geq 2)$ with arbitrary initial values $Z_0$, $Z_1$, \ldots, $Z_s$ and $s$ sequences of integers $(T_j(n))_{n \geq 0}$ ($j = 1, 2, \ldots, s$), which all are (ultimately) periodic modulo $m$ with periods of length $r$, then, the sequence $(Z(n))_{n \geq 0}$ is (ultimately) periodic modulo $m$. 
References


Graduate School of Science and Technology
Hirosaki University, Hirosaki, 036-8561, Japan
komatsu@cc.hirosaki-u.ac.jp