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Four-term leaping recurrence relations

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1 Introduction

Given a three-term linear recurrence relation $Z_n = T(n)Z_{n-1}+U(n)Z_{n-2}$ $(n \ge 2)$, where the initial values Z_0, Z_1 are arbitrary integral values, and $(T(n))_{n\ge 0}$, $(U(n))_{n\ge 0}$ are integer sequences with $U(n) \ne 0$ for all $n \ge 0$.

In 2008 Elsner and the author constructed a leaping three-term recurrence relation from the original relation. Namely, for fixed positive integers k and $0 \le i < k$, they obtained a three-term relation concerning $z_n = Z_{kn+i}$.

For integers a, l with $l \ge 1$ we define the determinant

$$K_l(a) = \begin{vmatrix} T(a) & 1 & 0 & & & \\ -U(a+1) & T(a+1) & 1 & & & \\ 0 & -U(a+2) & T(a+2) & & & \\ & & & \ddots & & \\ & & & & T(a+l-2) & 1 \\ & & & & -U(a+l-1) & T(a+l-1) \end{vmatrix},$$

with $K_0(a) = 1$. Let

$$\Omega(M) = U(M-r)U(M-r+1)\dots U(M-1)$$

with $M = (n-1)r+i+2$. Then we have the following ([3, Theorem 2]).

Theorem 1 *Given a three-term recurrence formula*

$$Z_n = T(n)Z_{n-1}+U(n)Z_{n-2} \qquad (n \ge 2)$$

with arbitrary initial values Z_0, Z_1 and two sequences of integers,

$$\begin{aligned}(T(n))_{n\ge 0} &= \left(a_0, a_1, a_2, \dots, a_\rho, \overline{T_1(k), T_2(k), \dots, T_w(k)}\right)_{k=1}^{\infty},\\ (U(n))_{n\ge 0} &= \left(b_0, b_1, b_2, \dots, b_\rho, \overline{U_1(k), U_2(k), \dots, U_w(k)}\right)_{k=1}^{\infty},\end{aligned}$$

where $U(n) \ne 0$ for all $n \ge 0$, and $\rho \ge 0$, $w \ge 1$ are fixed integers. Then, for any integers r and i with $r \ge 2$, $0 \le \rho \le i < \rho + r$ and $n \ge 2$,

$$K_{r-1}(M-r)\cdot z_n - \Big(K_{r-1}(M)K_r(M-r) + U(M)K_{r-1}(M-r)K_{r-2}(M+1)\Big)\cdot z_{n-1} + (-1)^r\Omega(M)K_{r-1}(M)\cdot z_{n-2} = 0$$

holds for $z_n = Z_{rn+i}$. For $T(a) > 0$ and $U(a) > 0$ for all $a > \rho$ one has $K_{r-1}(M) \ne 0$.
In particular, with $Z_n = q_n$, $q_0 = 1$, $q_1 = a_1$ or $Z_n = p_n$, $p_0 = a_0$, $p_1 = a_0a_1 + b_1$, this recurrence formula for z_n is satisfied by the denominators q_{rn+i} and numerators p_{rn+i}, respectively, of the convergents of a non-regular continued fraction

$$\left[a_0 + \frac{b_1}{a_1} + \frac{b_2}{a_2} + \dots + \frac{b_\rho}{a_\rho} + \overline{\frac{U_1(k)}{T_1(k)} + \frac{U_2(k)}{T_2(k)} + \dots + \frac{U_w(k)}{T_w(k)}}\right]_{k=1}^{\infty}.$$

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In the case of regular continued fractions this result is reduced as follows.

Corollary 1 *Given a three-term recurrence formula*

$$Z_n = T(n)Z_{n-1} + Z_{n-2} \qquad (n \ge 2)$$

with arbitrary initial values Z_0, Z_1 *and a sequence of integers,*

$$(T(n))_{n\ge0} = \left(a_0, a_1, a_2, \dots, a_\rho, \overline{T_1(k), T_2(k), \dots, T_w(k)}\right)_{k=1}^{\infty},$$

where $\rho \ge 0$ *and* $w \ge 1$ *are fixed integers. Then, for any integers* r *and* i *with* $r \ge 2$, $0 \le \rho \le i < \rho + r$ *and* $n \ge 2$,

$$K_{r-1}(M-r)\cdot z_n - \Big(K_{r-1}(M)K_r(M-r) + K_{r-1}(M-r)K_{r-2}(M+1)\Big)\cdot z_{n-1}$$
$$+(-1)^r K_{r-1}(M)\cdot z_{n-2} = 0$$

holds for $z_n = Z_{rn+i}$. *For* $T(a) > 0$ *for all* $a > \rho$ *one has* $K_{r-1}(M) \ne 0$.
In particular, with $Z_n = q_n$, $q_0 = 1$, $q_1 = a_1$ *or* $Z_n = p_n$, $p_0 = a_0$, $p_1 = a_0a_1 + 1$, *this recurrence formula for* z_n *is satisfied by the denominators* q_{rn+i} *and numerators* p_{rn+i}, *respectively, of the convergents of a regular continued fraction*

$$\left[a_0; a_1, a_2, \dots, a_\rho, \overline{T_1(k), T_2(k), \dots, T_w(k)}\right]_{k=1}^{\infty}.$$

Three-term leaping recurrence relations which are entailed from continued fractions were studied in the case of e by Elsner ([1]). Similar relations were also studied in the case of $e^{1/s}$ ($s \ge 2$) by the author ([7], [8]). Such concepts were extended to the cases of more regular continued fractions and non-regular continued fractions ([2], [3]).

However, it is not easy to find an anologous result for linear four-term recurrence relations

$$Z_n = U_1(n)Z_{n-1} + U_2(n)Z_{n-2} + U_3(n)Z_{n-3},$$

where $U_1(n)$, $U_2(n)$ and $U_3(n)$ are general sequences of integers. In this article we shall consider the leaping recurrence relations for four-term recurrence relations where $U_1(n) = a_1$, $U_2(n) = a_2$ and $U_3(n) = a_3$ are integer constants. Then we can have the leaping relation

$$Z_n = b_1Z_{n-k} + b_2Z_{n-2k} + b_3Z_{n-3k} \qquad (n = 3k, 3k+1, 3k+2, \dots)$$

for any leaping step k.

2 Leaping convergents

Let α be a real number. Continued fraction expansion of α is denoted by

$$\alpha = [a_0; a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}.$$

The n-th convergent is given by the irreducible rational number

$$\frac{p_n}{q_n} = [a_0; a_1, a_2 \dots, a_n].$$

It is well-known that p_n's and q_n's satisfy the recurrence relations:

$$p_n = a_np_{n-1} + p_{n-2} \qquad (n \ge 0), \qquad p_{-1} = 1, \qquad p_{-2} = 0,$$
$$q_n = a_nq_{n-1} + q_{n-2} \qquad (n \ge 0), \qquad q_{-1} = 1, \qquad q_{-2} = 0.$$

Leaping convergents of continued fractions are those of every r-th convergent of continued fractions;

$$\frac{p_i}{q_i},\quad \frac{p_{r+i}}{q_{r+i}},\quad \frac{p_{2r+i}}{q_{2r+i}},\quad \dots,\quad \frac{p_{rn+i}}{q_{rn+i}},\quad \dots$$

For example, consider

$$e^{1/s} = [1; \overline{(2k-1)s-1, 1, 1}]_{k=1}^{\infty} = [1; s-1, 1, 1, 3s-1, 1, 1, \dots] \quad (s \ge 2),$$

then $p_{3n} = 2s(2n-1)p_{3n-3} + p_{3n-6}$ and $q_{3n} = 2s(2n-1)q_{3n-3} + q_{3n-6}$ $(n \ge 2)$ ([7]).

3 Three-term relations

Three-term relations have been considered as in the continued fraction expansion of e ([1]), as in that of $e^{1/s}$ $(s \ge 2)$ ([7], [8]), as in that of the type $[\,1; \overline{T_1(k), T_2(k), T_3(k)}\,]_{k=1}^{\infty}$ ([9], [10]). Recently , three-term relations have been developed in the non-regular continued fractions ([2]), and finally as in Theorem 1 here ([3]).

On the other direction, one can simplify the general theorem, entailing some classical results. If $T(n) = a_1$, $U(n) = a_2$ $(a_2 \ne 0)$ are integer constants in Theorem 1, we have the following.

Theorem 2 *If the sequence $\{Z_n\}_n$ satisfies the three-term recurrence relation $Z_n = a_1 Z_{n-1} + a_2 Z_{n-2}$ $(a_2 \ne 0)$, then for any positive integer k we have*

$$Z_n = k \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{(k-i-1)!}{i!(k-2i)!} a_1^{k-2i} a_2^i \cdot Z_{n-k} + (-1)^{k+1} a_2^k \cdot Z_{n-2k} \qquad (n \ge 2k).$$

Moreover, if $a_1 = a_2 = 1$, then the sequence $\{Z_n\}_n$ is called Fibonacci-type sequence. Moreover, if $Z_0 = 0$ and $Z_1 = 1$, then $\{Z_n\}_n$ is the Fibonacci sequence $\{F_n\}_n$.

Corollary 2 *For any positive integer k we have*

$$F_n = k \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{(k-i-1)!}{i!(k-2i)!} \cdot F_{n-k} - (-1)^k \cdot F_{n-2k} \qquad (n \ge 2k). \tag{1}$$

If we put $k = 2, 3, \dots, 10$, then we have

$$\begin{aligned}
F_n &= 3F_{n-2} - F_{n-4}\,,\\
F_n &= 4F_{n-3} + F_{n-6}\,,\\
F_n &= 7F_{n-4} - F_{n-8}\,,\\
F_n &= 11F_{n-5} + F_{n-10}\,,\\
F_n &= 18F_{n-6} - F_{n-12}\,,\\
F_n &= 29F_{n-7} + F_{n-14}\,,\\
F_n &= 47F_{n-8} - F_{n-16}\,,\\
F_n &= 76F_{n-9} + F_{n-18}\,,\\
F_n &= 123F_{n-10} - F_{n-20}\,.
\end{aligned}$$

There is a classical result corresponding to this corollary (Ruggles, 1963 [11, identity 105, p.92]):

$$F_n = L_k F_{n-k} + (-1)^{k+1} F_{n-2k}\,, \tag{2}$$

where F_n and L_n are Fibonacci number and Lucas numbers, respectively. Namely, they satisfy the three-term relations

$$\begin{array}{llll} F_n = F_{n-1} + F_{n-2} & (n \ge 2), & F_0 = 0, & F_1 = 1; \\ L_n = L_{n-1} + L_{n-2} & (n \ge 2), & L_0 = 2, & L_1 = 1. \end{array}$$

Comparing (2) with (1), we have

Corollary 3

$$L_k = k \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{(k-i-1)!}{i!(k-2i)!} \quad (k \ge 1).$$

Proof of Theorem 2. Set $K_l = K_l(c)$. Then, $\{K_l\}_{l \ge 0}$ satisfies the recurrence relation:

$$K_l = a_1 K_{l-1} + a_2 K_{l-2} \quad (l \ge 2), \qquad K_0 = 1, \quad K_1 = a_1.$$

Hence, for $l \ge 0$ we have

$$K_l = \sum_{i=0}^{\lfloor l/2 \rfloor} \frac{(l-i)!}{i!(l-2i)!} a_1^{l-2i} a_2^i.$$

Applying Theorem 1 with $\Omega(M) = a_2^r$, we have

$$Z_n = (K_r + a_2 K_{r-2}) \cdot Z_{n-1} - (-1)^r a_2^r \cdot Z_{n-2}.$$

Since

$$\begin{aligned} K_r + a_2 K_{r-2} &= \sum_{i=0}^{\lfloor r/2 \rfloor} \frac{(r-i)!}{i!(r-2i)!} a_1^{r-2i} a_2^i + \sum_{i=0}^{\lfloor r/2 \rfloor - 1} \frac{(r-i-2)!}{i!(r-2i-2)!} a_1^{r-2i-2} a_2^{i+1} \\ &= r \sum_{i=0}^{\lfloor r/2 \rfloor} \frac{(r-i-1)!}{i!(r-2i)!} a_1^{r-2i} a_2^i, \end{aligned}$$

we obtain the desired result. ∎

4 Four-term relations

Consider the four-term recurrence relation

$$Z_n = U_1(n) Z_{n-1} + U_2(n) Z_{n-2} + U_3(n) Z_{n-3}.$$

For the moment, the corresponding result to Theorem 1 has not been known. However, one can relax the conditions, in order to get some typical results. If $U_1(n) = a_1$, $U_2(n) = a_2$, $U_3(n) = a_3$ are constants for all n, we have the following four-term leaping recurrence relation.

Theorem 3 *If the sequence $\{Z_n\}_n$ satisfies the four-term recurrence relation $Z_n = a_1 Z_{n-1} + a_2 Z_{n-2} + a_3 Z_{n-3}$ ($a_3 \ne 0$), then for any positive integer k*

$$\begin{aligned} Z_n = & \, k \sum_{j=0}^{\lfloor k/3 \rfloor} \sum_{i=0}^{\lfloor (k-3j)/2 \rfloor} \frac{(k-i-2j-1)!}{i!j!(k-2i-3j)!} a_1^{k-2i-3j} a_2^i a_3^j \cdot Z_{n-k} \\ & - k \sum_{j=0}^{\lfloor k/3 \rfloor} \sum_{i=0}^{\lfloor (k-3j)/2 \rfloor} (-1)^{k+i+j} \frac{(k-i-2j-1)!}{i!j!(k-2i-3j)!} a_1^i a_2^{k-2i-3j} a_3^{i+2j} \cdot Z_{n-2k} + a_3^k \cdot Z_{n-3k} \\ & (n \ge 3k). \end{aligned}$$

In 2001 F. T. Howard obtained a similar result ([5]):

$$Z_n = J_k Z_{n-k} - a_3^k J_{-k} Z_{n-2k} + a_3^k Z_{n-3k}\,, \tag{3}$$

where J_n satisfies

$$J_n = a_1 J_{n-1} + a_2 J_{n-2} + a_3 J_{n-3} \quad (n \ge 3),$$
$$J_0 = 3, \quad J_1 = a_1, \quad J_2 = a_1^2 + 2a_2\,.$$

J_{-n} $(n = 1, 2, \dots)$ are determined by

$$J_{-n} = \frac{1}{a_3}(J_{-n+3} - a_1 J_{-n+2} - a_2 J_{-n+1}) \quad (n \ge 1)\,.$$

Comparing (3) with Theorem 3, we obtain

Corollary 4

$$J_k = k \sum_{j=0}^{\lfloor k/3 \rfloor} \sum_{i=0}^{\lfloor (k-3j)/2 \rfloor} \frac{(k-i-2j-1)!}{i!j!(k-2i-3j)!} a_1^{k-2i-3j} a_2^i a_3^j,$$

$$J_{-k} = k \sum_{j=0}^{\lfloor k/3 \rfloor} \sum_{i=0}^{\lfloor (k-3j)/2 \rfloor} (-1)^{k+i+j} \frac{(k-i-2j-1)!}{i!j!(k-2i-3j)!} a_1^i a_2^{k-2i-3j} a_3^{-k+i+2j}$$

In the case of $k = 12$,

$$\begin{aligned} Z_n - (&a_1^{12} + 12a_1^{10}a_2 + 54a_1^8a_2^2 + 112a_1^6a_2^3 + 105a_1^4a_2^4 + 36a_1^2a_2^5 + 2a_2^6 \\ &+ 12a_1^9a_3 + 96a_1^7a_2a_3 + 252a_1^5a_2^2a_3 + 240a_1^3a_2^3a_3 + 60a_1a_2^4a_3 + 42a_1^6a_3^2 \\ &+ 180a_1^4a_2a_3^2 + 180a_1^2a_2^2a_3^2 + 24a_2^3a_3^2 + 40a_1^3a_3^3 + 48a_1a_2a_3^3 + 3a_3^4)Z_{n-12} \\ &+ (a_2^{12} - 12a_1a_2^{10}a_3 + 54a_1^2a_2^8a_3^2 - 12a_2^9a_3^2 - 112a_1^3a_2^6a_3^3 + 96a_1a_2^7a_3^3 \\ &+ 105a_1^4a_2^4a_3^4 - 252a_1^2a_2^5a_3^4 + 42a_2^6a_3^4 - 36a_1^5a_2^2a_3^5 + 240a_1^3a_2^3a_3^5 - 180a_1a_2^4a_3^5 \\ &+ 2a_1^6a_3^6 - 60a_1^4a_2a_3^6 + 180a_1^2a_2^2a_3^6 - 40a_2^3a_3^6 - 24a_1^3a_3^7 \\ &+ 48a_1a_2a_3^7 + 3a_3^8)Z_{n-24} - a_3^{12}Z_{n-36} = 0\,. \end{aligned}$$

Put $a_1 = a_2 = a_3 = 1$. Then the four-term recurrence relation $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ yields Tribonacci numbers $\{T_n\}_{n \ge 0}$. If $(T_0 = 0,)$ $T_1 = T_2 = 1$ and $T_3 = 2$, then the Tribonacci sequence is given by

$$1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 5768, 10609, 19513, 35890, 66012, 121415, \dots$$

([11, p.527], [12, A000073]). Putting $k = 2, 3, \dots, 10$ in Theorem 3, we have

$$\begin{aligned} T_n &= 3T_{n-2} + T_{n-4} + T_{n-6}\,, \\ T_n &= 7T_{n-3} - 5T_{n-6} + T_{n-9}\,, \\ T_n &= 11T_{n-4} + 5T_{n-8} + T_{n-12}\,, \\ T_n &= 21T_{n-5} + T_{n-10} + T_{n-15}\,, \\ T_n &= 39T_{n-6} - 11T_{n-12} + T_{n-18}\,, \\ T_n &= 71T_{n-7} + 15T_{n-14} + T_{n-21}\,, \\ T_n &= 131T_{n-8} - 3T_{n-16} + T_{n-24}\,, \\ T_n &= 241T_{n-9} - 23T_{n-18} + T_{n-27}\,, \\ T_n &= 443T_{n-10} + 41T_{n-20} + T_{n-30}\,. \end{aligned}$$

5 Five-term relations

Consider the sequence $\{Z_n\}_n$ satisfying the five-term recurrence relation $Z_n = a_1 Z_{n-1} + a_2 Z_{n-2} + a_3 Z_{n-3} + a_4 Z_{n-4}$ ($a_4 \neq 0$). Then how can we determine the integer constants b_1, b_2, b_3, b_4, satisfying

$$Z_n = b_1 Z_{n-k} + b_2 Z_{n-2k} + b_3 Z_{n-3k} + b_4 Z_{n-4k}$$

for any positive integer k ($1 < k < n/4$)?

In the case of $k = 5$

$$\begin{aligned} Z_n = &(a_1^5 + 5a_1^3 a_2 + 5a_1 a_2^2 + 5a_1^2 a_3 + 5a_2 a_3 + 5a_1 a_4) Z_{n-5} \\ &+ (a_2^5 - 5a_1 a_2^3 a_3 + 5a_1^2 a_2 a_3^2 - 5a_2^2 a_3^2 + 5a_1 a_3^3 + 5a_1^2 a_2^2 a_4 + 5a_2^3 a_4 \\ &- 5a_1^3 a_3 a_4 + 5a_1 a_2 a_3 a_4 + 5a_3^2 a_4 - 5a_1^2 a_4^2 + 5a_2 a_4^2) Z_{n-10} \\ &+ (a_3^5 - 5a_2 a_3^3 a_4 + 5a_2^2 a_3 a_4^2 + 5a_1 a_3^2 a_4^2 - 5a_1 a_2 a_4^3 + 5a_3 a_4^3) Z_{n-15} + a_4^5 Z_{n-20}. \end{aligned}$$

In the case of $k = 6$,

$$\begin{aligned} Z_n = &(a_1^6 + 6a_1^4 a_2 + 9a_1^2 a_2^2 + 2a_2^3 + 6a_1^3 a_3 + 12a_1 a_2 a_3 + 3a_3^2 + 6a_1^2 a_4 + 6a_2 a_4) Z_{n-6} \\ &+ (-a_2^6 + 6a_1 a_2^4 a_3 - 9a_1^2 a_2^2 a_3^2 + 6a_2^3 a_3^2 + 2a_1^3 a_3^3 - 12a_1 a_2 a_3^3 - 3a_3^4 - 6a_1^2 a_2^3 a_4 \\ &- 6a_2^4 a_4 + 12a_1^3 a_2 a_3 a_4 + 18a_1^2 a_3^2 a_4 - 3a_1^4 a_4^2 - 9a_2^2 a_4^2 + 18a_1 a_3 a_4^2 + 2a_4^3) Z_{n-12} \\ &+ (a_3^6 - 6a_2 a_3^4 a_4 + 9a_2^2 a_3^2 a_4^2 + 6a_1 a_3^3 a_4^2 - 2a_2^3 a_4^3 - 12a_1 a_2 a_3 a_4^3 + 6a_3^2 a_4^3 + 3a_1^2 a_4^4 - 6a_2 a_4^4) Z_{n-18} \\ &- a_4^6 Z_{n-24}. \end{aligned}$$

Tetranacci Numbers $\{F_k^{(4)}\}_{k \ge 1}$ are the $n = 4$ case of the Fibonacci n-step Numbers, defined by $F_k^{(4)} = F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)} + F_{k-4}^{(4)}$ ($k \ge 5$) with $F_1^{(4)} = F_2^{(4)} = 1$, $F_3^{(4)} = 2$ and $F_4^{(4)} = 4$. The first terms are

$$\begin{aligned} &1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, 5536, 10671, 20569, 39648, 76424, 147312, \\ &283953, 547337, 1055026, 2033628, 3919944, 7555935, 14564533, 28074040, 54114452, 104308960, \dots \end{aligned}$$

([12, A000078]). They satisfy the recurrence relations:

$$\begin{aligned} F_k^{(4)} &= 3F_{k-2}^{(4)} + 3F_{k-4}^{(4)} - F_{k-6}^{(4)} - F_{k-8}^{(4)}, \\ F_k^{(4)} &= 7F_{k-3}^{(4)} + F_{k-6}^{(4)} + F_{k-9}^{(4)} + F_{k-12}^{(4)}, \\ F_k^{(4)} &= 15F_{k-4}^{(4)} - 17F_{k-8}^{(4)} + 7F_{k-12}^{(4)} - F_{k-16}^{(4)}, \\ F_k^{(4)} &= 26F_{k-5}^{(4)} + 16F_{k-10}^{(4)} + 6F_{k-15}^{(4)} + F_{k-20}^{(4)}, \\ F_k^{(4)} &= 51F_{k-6}^{(4)} + 15F_{k-12}^{(4)} - F_{k-18}^{(4)} - F_{k-24}^{(4)}, \\ F_k^{(4)} &= 99F_{k-7}^{(4)} - 13F_{k-14}^{(4)} + F_{k-21}^{(4)} + F_{k-28}^{(4)}, \\ F_k^{(4)} &= 191F_{k-8}^{(4)} - 81F_{k-16}^{(4)} + 15F_{k-24}^{(4)} - F_{k-32}^{(4)}, \\ F_k^{(4)} &= 367F_{k-9}^{(4)} + 127F_{k-18}^{(4)} + 19F_{k-27}^{(4)} + F_{k-36}^{(4)}, \\ F_k^{(4)} &= 708F_{k-10}^{(4)} + 58F_{k-20}^{(4)} + 4F_{k-30}^{(4)} - F_{k-40}^{(4)}. \end{aligned}$$

b_1, b_3 and b_4 are calculated as follows.

Theorem 4

$$b_1 = k\sum_{\kappa=0}^{\lfloor k/4\rfloor}\sum_{j=0}^{\lfloor (k-4\kappa)/3\rfloor}\sum_{i=0}^{\lfloor (k-3j-4\kappa)/2\rfloor}\frac{(k-i-2j-3\kappa-1)!}{i!j!\kappa!(k-2i-3j-4\kappa)!}a_1^{k-2i-3j-4\kappa}a_2^i a_3^j a_4^\kappa\,,$$

$$b_3 = k\sum_{\kappa=0}^{\lfloor k/4\rfloor}\sum_{j=0}^{\lfloor (k-4\kappa)/3\rfloor}\sum_{i=0}^{\lfloor (k-3j-4\kappa)/2\rfloor}(-1)^i\frac{(k-i-2j-3\kappa-1)!}{i!j!\kappa!(k-2i-3j-4\kappa)!}a_1^j a_2^i a_3^{k-2i-3j-4\kappa}a_4^{i+2j+3\kappa}\,,$$

$$b_4 = (-1)^{k-1}a_4^k\,.$$

However, it is not easy to find an explicit form of b_2. This shall be discussed in the next section.

In 2005 Latushkin and Ushakov ([6]) obtained a different form of five-term leaping relations.

$$Z_n = H_k Z_{n-k} + \frac{H_{2k}-H_k^2}{2}Z_{n-2k} + (-a_4)^k H_{-k}Z_{n-3k} - (-a_4)^k Z_{n-4k}\,, \tag{4}$$

where

$$H_n = x_1^n + x_2^n + x_3^n + x_4^n \quad (n\in\mathbb{Z})$$

and x_1, x_2, x_3 and x_4 are the complex roots (including multiple roots) of the equation $x^4 - a_1x^3 - a_2x^2 - a_3x - a_4 = 0$. On the other hand, the sequence $\{H_n\}_n$ satisfies the recurrence relation:

$$H_n = a_1H_{n-1} + a_2H_{n-2} + a_3H_{n-3} + a_4H_{n-4} \quad (n\in\mathbb{Z})\,.$$

The initial values are determined by

$$\begin{aligned}
H_0 &= 4,\\
H_1 &= a_1,\\
H_2 &= a_1H_1 + 2a_2 = a_1^2 + 2a_2,\\
H_3 &= a_1H_2 + a_2H_1 + 3a_3 = a_1^3 + 3a_1a_2 + 3a_3,\\
H_4 &= a_1H_3 + a_2H_2 + a_3H_1 + 4a_4 = a_1^4 + 4a_1a_2 + 4a_1a_3 + 2a_2^2 + 4a_4.
\end{aligned}$$

Comparing their results (4) with ours in Theorem 4, we get the following.

Corollary 5

$$H_k = k\sum_{\kappa=0}^{\lfloor k/4\rfloor}\sum_{j=0}^{\lfloor (k-4\kappa)/3\rfloor}\sum_{i=0}^{\lfloor (k-3j-4\kappa)/2\rfloor}\frac{(k-i-2j-3\kappa-1)!}{i!j!\kappa!(k-2i-3j-4\kappa)!}a_1^{k-2i-3j-4\kappa}a_2^i a_3^j a_4^\kappa\,,$$

$$H_{-k} = k\sum_{\kappa=0}^{\lfloor k/4\rfloor}\sum_{j=0}^{\lfloor (k-4\kappa)/3\rfloor}\sum_{i=0}^{\lfloor (k-3j-4\kappa)/2\rfloor}(-1)^{i+k}\frac{(k-i-2j-3\kappa-1)!}{i!j!\kappa!(k-2i-3j-4\kappa)!}a_1^j a_2^i a_3^{k-2i-3j-4\kappa}a_4^{-k+i+2j+3\kappa}\,.$$

Pentanacci Numbers $\{F_k^{(5)}\}_{k\ge1}$ are the $n=5$ case of the Fibonacci n-step Numbers, defined by $F_k^{(5)} = F_{k-1}^{(5)} + F_{k-2}^{(5)} + F_{k-3}^{(5)} + F_{k-4}^{(5)} + F_{k-5}^{(5)}$ $(k\ge6)$ with $F_1^{(5)} = F_2^{(5)} = 1$, $F_3^{(5)} = 2$, $F_4^{(5)} = 4$ and $F_5^{(5)} = 8$. The first terms are

1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3525, 6930, 13624, 26784, 52656, 103519, 203513, 400096, 786568, 1546352, 3040048, 5976577, 11749641, 23099186, 45411804, 89277256, 175514464, …

([12, A001591]). They satisfy the recurrence relations:

$$
\begin{aligned}
F_k^{(5)} &= 3F_{k-2}^{(5)} + 3F_{k-4}^{(5)} + F_{k-6}^{(5)} + F_{k-8}^{(5)} + F_{k-10}^{(5)}\,,\\
F_k^{(5)} &= 7F_{k-3}^{(5)} + 4F_{k-6}^{(5)} + 4F_{k-9}^{(5)} + F_{k-12}^{(5)} + F_{k-15}^{(5)}\,,\\
F_k^{(5)} &= 15F_{k-4}^{(5)} - F_{k-8}^{(5)} + F_{k-12}^{(5)} + F_{k-16}^{(5)} + F_{k-20}^{(5)}\,,\\
F_k^{(5)} &= 31F_{k-5}^{(5)} - 49F_{k-10}^{(5)} + 31F_{k-15}^{(5)} - 9F_{k-20}^{(5)} + F_{k-25}^{(5)}\,,\\
F_k^{(5)} &= 57F_{k-6}^{(5)} + 42F_{k-12}^{(5)} + 22F_{k-18}^{(5)} + 7F_{k-24}^{(5)} + F_{k-30}^{(5)}\,,\\
F_k^{(5)} &= 113F_{k-7}^{(5)} + 57F_{k-14}^{(5)} + F_{k-21}^{(5)} + F_{k-28}^{(5)} + F_{k-35}^{(5)}\,,\\
F_k^{(5)} &= 223F_{k-8}^{(5)} + 31F_{k-16}^{(5)} + 33F_{k-24}^{(5)} + F_{k-32}^{(5)} + F_{k-40}^{(5)}\,,\\
F_k^{(5)} &= 439F_{k-9}^{(5)} - 140F_{k-18}^{(5)} + 4F_{k-27}^{(5)} + F_{k-36}^{(5)} + F_{k-45}^{(5)}\,,\\
F_k^{(5)} &= 863F_{k-10}^{(5)} - 497F_{k-20}^{(5)} + 141F_{k-30}^{(5)} - 19F_{k-40}^{(5)} + F_{k-50}^{(5)}\,.
\end{aligned}
$$

6 A form of b_2 in five-term leaping relations

An explicit form of b_2 has not been known yet. Instead, there is a way to express b_2 by matrices.

$$
\begin{aligned}
b_2 = {} & a_2\Lambda_{k-1} - 2a_3\Phi_{k-1} + 3a_4\Psi_{k-1}\\
& - a_3\Phi_{k-1} + 2a_4\Lambda_{k-2} - a_1a_3\Lambda_{k-2} + 2a_2a_4\Lambda_{k-3} - 3a_4^2\Lambda_{k-4}\,,
\end{aligned}
$$

where

$$
\Lambda_n = \begin{vmatrix}
-a_2 & -a_1 & 1 & 0 & & & & \\
-a_3 & -a_2 & -a_1 & 1 & & & & \\
-a_4 & -a_3 & -a_2 & -a_1 & & & & \\
0 & & & \ddots & & & & \\
 & & & & -a_2 & -a_1 & 1 & 0 \\
 & & & & -a_3 & -a_2 & -a_1 & 1 \\
 & & & & -a_4 & -a_3 & -a_2 & -a_1 \\
 & & & & 0 & -a_4 & -a_3 & -a_2
\end{vmatrix},
$$

$$
\Phi_n = \begin{vmatrix}
-a_1 & 1 & 0 & & & & & \\
-a_3 & -a_2 & -a_1 & 1 & 0 & & & \\
-a_4 & -a_3 & -a_2 & -a_1 & 1 & & & \\
0 & & & \ddots & & & & \\
 & & & & -a_2 & -a_1 & 1 & 0 \\
 & & & & -a_3 & -a_2 & -a_1 & 1 \\
 & & & & -a_4 & -a_3 & -a_2 & -a_1 \\
 & & & & 0 & -a_4 & -a_3 & -a_2
\end{vmatrix},
$$

$$
\Psi_n = \begin{vmatrix}
-a_1 & 1 & 0 & & & & & \\
-a_2 & -a_1 & 1 & 0 & & & & \\
-a_4 & -a_3 & -a_2 & -a_1 & 1 & & & \\
0 & & & \ddots & & & & \\
 & & & & -a_2 & -a_1 & 1 & 0 \\
 & & & & -a_3 & -a_2 & -a_1 & 1 \\
 & & & & -a_4 & -a_3 & -a_2 & -a_1 \\
 & & & & 0 & -a_4 & -a_3 & -a_2
\end{vmatrix}.
$$

Notice that

$$\begin{aligned}\Lambda_n &= -a_2\Lambda_{n-1} + a_3\Phi_{n-1} - a_4\Psi_{n-1}\,,\\ \Phi_n &= -a_1\Lambda_{n-1} + a_3\Lambda_{n-2} - a_4\Phi_{n-2}\,,\\ \Psi_n &= -a_1\Phi_{n-1} + a_2\Lambda_{n-2} - a_4\Lambda_{n-3}\,.\end{aligned}$$

b_2 may be expanded as follows.

$$\begin{aligned}b_2 = (-1)^{k-1}k\Big(&a_2^k - (a_1a_3 - a_4)a_2^{k-2} - (a_3^2 - a_1^2a_4)a_2^{k-3}\\
&+\left(\frac{(k-3)!}{2!(k-4)!}a_1^2a_3^2 - \frac{(k-4)!}{(k-4)!}(k-6)a_1a_3a_4 + \frac{k-3}{2}a_4^2\right)a_2^{k-4}\\
&+(a_3^2 - a_1^2a_4)\left(\frac{(k-4)!}{(k-5)!}a_1a_3 - (k-6)a_4\right)a_2^{k-5}\\
&-\left(\frac{(k-4)!}{3!(k-6)!}a_1^3a_3^3 - \frac{(k-5)!}{2!(k-6)!}(k-12)a_1^2a_3^2a_4 + \frac{k^2-15k+60}{2}a_1a_3a_4^2\right.\\
&\left.-\frac{(k-4)!}{3!(k-6)!}a_4^3 - \frac{k-5}{2}(a_1^4a_4^2 + a_3^4)\right)a_2^{k-6}\\
&-(a_3^2 - a_1^2a_4)\left(\frac{(k-5)!}{2!(k-7)!}a_1^2a_3^2 - \frac{(k-6)!}{(k-7)!}(k-10)a_1a_3a_4 + \frac{k^2-15k+60}{2}a_4^2\right)a_2^{k-7}\\
&+\left(\frac{(k-5)!}{4!(k-8)!}a_1^4a_3^4 - \frac{(k-6)!}{3!(k-8)!}(k-20)a_1^3a_3^3a_4\right.\\
&+\frac{(k-7)!}{2!2!(k-8)!}(k-12)(k-15)a_1^2a_3^2a_4^2 - \frac{(k-10)(k^2-17k+84)}{3!}a_1a_3a_4^3\\
&\left.+\frac{(k-5)!}{4!(k-8)!}a_4^4 - \frac{k-7}{2}(a_1^4a_4^2 + a_3^4)((k-6)a_1a_3 - (k-10)a_4)\right)a_2^{k-8}\\
&+\Big((a_3^2 - a_1^2a_4)\Big(\frac{(k-6)!}{3!(k-9)!}a_1^3a_3^3 - \frac{(k-7)!}{2!(k-9)!}(k-15)a_1^2a_3^2a_4\\
&\quad+\frac{(k-8)!}{2!(k-9)!}(k^2-23k+140)a_1a_3a_4^2 - \frac{(k-9)!}{3!(k-9)!}(k-10)(k^2-17k+84)a_4^3\Big)\\
&\quad-(a_3^6 - a_1^6a_4^3)\frac{(k-7)!}{3!(k-9)!}\Big)a_2^{k-9}\\
&-\cdots\Big)\,.\end{aligned}$$

However, its simplified form has not been known.

7 $(s+1)$-term recurrence relations

We may extend terms to five, six, seven, and so on. In 1999 Howard got a general term leaping relation ([4]). Young also found a different form ([13]). This result holds for more-term recurrence relations, but it is not so useful practically in order to obtain an explicit form for any given s.

If the sequence $\{Z_n\}_{n\ge 0}$ satisfies the relation

$$Z_n = a_1Z_{n-1} + a_2Z_{n-2} + \cdots + a_sZ_{n-s} \quad (a_s \ne 0)\,,$$

where Z_0, Z_1, …, Z_{s-1} are arbitrary initial values, then we have

$$Z_{rn+i} = c_{r,r}Z_{r(n-1)+i} - c_{r,2r}Z_{r(n-2)+i} + \cdots + (-1)^{s-1}c_{r,sr}Z_{r(n-s)+i}\,,$$

where $c_{r,r}$, $c_{r,2r}$, ..., $c_{r,sr}$ are determined by

$$\prod_{\nu=0}^{r-1}(1-a_1(\zeta^\nu x)-a_2(\zeta^\nu x)^2-\cdots-a_s(\zeta^\nu x)^s)=1-c_{r,r}x^r+c_{r,2r}x^{2r}-\cdots+(-1)^s c_{r,sr}x^{sr},$$

where ζ is a primitive r-th root of unity.

As straight generalization of our theorems 2, 3, 4, we obtain the following .

Theorem 5 *If*

$$Z_n=b_1Z_{n-k}+b_2Z_{n-2k}+\cdots+b_{s-1}Z_{n-(s-1)k}+b_sZ_{n-sk},$$

then

$$b_1=k\sum_{\substack{2i_1+3i_2+\cdots+si_{s-1}\le k\\ i_1,i_2,\ldots,i_{s-1}\ge 0}}\frac{(k-i_1-2i_2-\cdots-(s-1)i_{s-1}-1)!}{i_1!i_2!\ldots i_{s-1}!(k-2i_1-3i_2-\cdots-si_{s-1})!}\times a_1^{k-2i_1-3i_2-\cdots-si_{s-1}}a_2^{i_1}a_3^{i_2}\ldots a_s^{i_{s-1}},$$

$$b_{s-1}=k\sum_{\substack{2i_1+3i_2+\cdots+si_{s-1}\le k\\ i_1,i_2,\ldots,i_{s-1}\ge 0}}(-1)^{I-1}\frac{(k-i_1-2i_2-\cdots-(s-1)i_{s-1}-1)!}{i_1!i_2!\ldots i_{s-1}!(k-2i_1-3i_2-\cdots-si_{s-1})!}\times a_1^{i_{s-2}}a_2^{i_{s-3}}\ldots a_{s-2}^{i_1}a_{s-1}^{n-2i_1-3i_2-\cdots-si_{s-1}}a_s^{i_1+2i_2+\cdots+(s-1)i_{s-1}},$$

where

$$I=\begin{cases}i_1+i_3+\cdots+i_{s-2}+i_{s-1} & \text{if } s \text{ is odd;}\\ i_1+i_3+\cdots+i_{s-3} & \text{if } s \text{ is even,}\end{cases}$$

and

$$b_s=\begin{cases}a_s^k & \text{if } s \text{ is odd;}\\ (-1)^{k-1}a_s^k & \text{if } s \text{ is even.}\end{cases}$$

8 Periodicity

In [2, Theorem 3] a result about periodicity of three-term leaping relations is obtained.

Theorem 6 *Given a three-term recurrence formula*

$$Z_n = T(n)Z_{n-1}+U(n)Z_{n-2} \qquad (n\ge 2)$$

with arbitrary initial values Z_0, Z_1 and two sequences of integers $(T(n))_{n\ge0}$ and $(U(n))_{n\ge0}$, which both are (ultimately) periodic modulo m with periods of length r, say

$$\begin{aligned}(T(n) \bmod m)_{n\ge0} &= \left(a_0,a_1,a_2,\ldots,a_\rho,\overline{T_1,T_2,\ldots,T_r}\right),\\ (U(n) \bmod m)_{n\ge0} &= \left(b_0,b_1,b_2,\ldots,b_\rho,\overline{U_1,U_2,\ldots,U_r}\right).\end{aligned}$$

Then, the sequence $(Z(n))_{n\ge0}$ is (ultimately) periodic modulo m. If $\rho\in\{0,1\}$ and $U(n)=1$ for all $n\ge\rho$, then the sequence $(Z(n))_{n\ge0}$ is periodic modulo m.

This result can be extended to the case of any term leaping relations.

Theorem 7 *Given a $(s+1)$-term recurrence formula $Z_n=T_1(n)Z_{n-1}+T_2(n)Z_{n-2}+\cdots+T_s(n)Z_{n-s}$ $(n\ge 2)$ with arbitrary initial values Z_0, Z_1, ..., Z_s and s sequences of integers $(T_j(n))_{n\ge0}$ $(j=1,2,\ldots,s)$, which all are (ultimately) periodic modulo m with periods of length r, then, the sequence $(Z(n))_{n\ge0}$ is (ultimately) periodic modulo m.*

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