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Doubly nonlinear evolution equations and dynamical systems

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1 Introduction

Let V and V^* be a real reflexive Banach space and its dual space, respectively, and let H be a Hilbert space whose dual space H^* is identified with itself such that

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^* \tag{1}$$

with continuous and densely defined canonical injections. Let φ and ψ be proper lower semicontinuous functions from V into $(-\infty, \infty]$, and let $\partial_V\varphi, \partial_V\psi : V \to V^*$ be subdifferential operators of φ, ψ, respectively, defined by

$$\partial_V\varphi(u) := \{\xi \in V^*;\ \varphi(v) - \varphi(u) \geq \langle \xi, v-u\rangle \ \text{ for all } v \in D(\varphi)\}$$

with the domain $D(\partial_V\varphi) := \{u \in D(\varphi);\ \partial_V\varphi(u) \neq \emptyset\}$, where $D(\varphi) := \{u \in V;\ \varphi(u) < \infty\}$, and analogously for $\partial_V\psi(u)$. Moreover, let B be a (possibly) non-monotone and multi-valued operator from V into V^*.

This note provides a brief survey of recent results of the author on the dynamical system generated by the Cauchy problem (CP) for the following doubly nonlinear evolution equation:

$$\partial_V\psi(u'(t)) + \partial_V\varphi(u(t)) + B(u(t)) \ni f \ \text{ in } V^*, \quad 0 < t < \infty, \tag{2}$$

where $f \in V^*$ and $u_0 \in D(\varphi)$ are given data. We first treat the existence of global (in time) strong solutions of (CP) by imposing appropriate conditions such as the coerciveness and the boundedness of $\partial_V\psi$, the precompactness of sub-level sets of φ, and the boundedness and the compactness of B. The main purpose of this note is to discuss the large-time behavior of global solutions for (CP), in particular, the existence of global attractors; however, since the scope of our abstract framework involves the case where (CP) admits multiple solutions, the usual semigroup approach to dynamical systems could be no longer valid. Therefore we exploit the notion of generalized semiflow proposed by J.M. Ball [5] to treat global attractors for (CP). The theory of generalized semiflow was recently started to be applied to various nonlinear PDEs of parabolic type (see [8], [9], [10]) and of hyperbolic type (see [4], [6]) without the uniqueness of solutions.

Furthermore, we apply the preceding abstract theory to a couple of nonlinear PDEs of parabolic type. Gurtin [7] proposed a generalized Allen-Cahn equation, which describes the evolution of an order parameter $u = u(x,t)$, of the form

$$\rho(u, \nabla u, u_t)u_t = \operatorname{div}\left[\partial_{\mathbf{p}}\hat{\psi}(u, \nabla u)\right] - \partial_r\hat{\psi}(u, \nabla u) + f, \tag{3}$$

where $\rho = \rho(r, \mathbf{p}, s) \geq 0$ is a constitutive modulus, $\hat{\psi} = \hat{\psi}(r, \mathbf{p})$ denotes a free energy density and f is an external microforce. A usual Allen-Cahn equation corresponds to the case that

$$\rho \equiv 1 \quad \text{and} \quad \hat{\psi}(r, \mathbf{p}) = \frac{1}{2}|\mathbf{p}|^2 + W(r)$$

with a double-well potential $W(r) = (r^2 - 1)^2$. In Section 5, we treat a generalized Allen-Cahn equation of degenerate type as well as a perturbation problem of a semilinear generalized Allen-Cahn equation.

2 Generalized semiflow

The notion of generalized semiflow is first introduced by J.M. Ball [5]. He also extend the notion of global attractor to generalized semiflows and provide a criterion of the existence of global attractors. We first recall the definition of generalized semiflow.

Definition 2.1. *Let X be a metric space with metric $d_X = d_X(\cdot,\cdot)$. A family $\mathcal{G}$ of maps $\varphi : [0,+\infty) \to X$ is said to be a generalized semiflow in X, if the following four conditions are all satisfied:*

(H1) (Existence) *For each $x \in X$ there exists $\varphi \in \mathcal{G}$ such that $\varphi(0) = x$;*

(H2) (Translation invariance) *If $\varphi \in \mathcal{G}$ and $\tau \geq 0$, then the map φ^τ also belongs to $\mathcal{G}$, where $\varphi^\tau(t) := \varphi(t + \tau)$ for $t \in [0, +\infty)$;*

(H3) (Concatenation invariance) *If $\varphi_1, \varphi_2 \in \mathcal{G}$ and $\varphi_2(0) = \varphi_1(\tau)$ for some $\tau \geq 0$, then the map ψ, the concatenation of φ_1 and φ_2 at τ, defined by*

$$\psi(t) := \begin{cases} \varphi_1(t) & \text{if } t \in [0, \tau], \\ \varphi_2(t - \tau) & \text{if } t \in (\tau, +\infty) \end{cases}$$

also belongs to $\mathcal{G}$;

(H4) (Upper semicontinuity) *If $\varphi_n \in \mathcal{G}$, $x \in X$ and $\varphi_n(0) \to x$ in X, then there exist a subsequence $\{n'\}$ of $\{n\}$ and $\varphi \in \mathcal{G}$ such that $\varphi_{n'}(t) \to \varphi(t)$ for each $t \in [0, +\infty)$.*

Let $\mathcal{G}$ be a generalized semiflow in a metric space X. We define a mapping $T(t) : 2^X \to 2^X$ by

$$T(t)E := \{\varphi(t);\ \varphi \in \mathcal{G} \text{ and } \varphi(0) \in E\} \quad \text{for } E \subset X \tag{4}$$

for each $t \geq 0$. One can check from (H1)–(H3) that $\{T(t)\}_{t\geq 0}$ satisfies the semigroup properties, that is, (i) $T(0)$ is the identity mapping in 2^X; (ii) $T(t)T(s) = T(t+s)$ for all $t, s \geq 0$.

Moreover, global attractors for generalized semiflows are defined as follows.

Definition 2.2. *Let $\mathcal{G}$ be a generalized semiflow in a metric space X and let $\{T(t)\}_{t\geq 0}$ be the family of mappings defined as in* (4). *A set $\mathcal{A} \subset X$ is said to be a global attractor for the generalized semiflow $\mathcal{G}$ if the following* (i)–(iii) *hold.*

(i) *$\mathcal{A}$ is compact in X;*

(ii) *$\mathcal{A}$ is invariant under $\{T(t)\}_{t\geq 0}$, i.e., $T(t)\mathcal{A} = \mathcal{A}$, for all $t \geq 0$;*

(iii) *$\mathcal{A}$ attracts any bounded subsets B of X by $\{T(t)\}_{t\geq 0}$, i.e.,*

$$\lim_{t\to+\infty} \operatorname{dist}(T(t)B, \mathcal{A}) = 0,$$

where $\operatorname{dist}(\cdot,\cdot)$ *is defined by*

$$\operatorname{dist}(A, B) := \sup_{a\in A} \inf_{b\in B} d_X(a, b) \quad \textit{for} \ \ A, B \subset X.$$

As in the standard theory of dynamical systems for (single-valued) semigroup operators, we can also introduce the notion of ω-limit set.

Definition 2.3. *Let $\mathcal{G}$ be a generalized semiflow in a metric space X. For $E \subset X$, the ω-limit set of E for $\mathcal{G}$ is given as follows.*

$$\omega(E) := \Big\{ x \in X;\ \textit{there exist sequences } \{\varphi_n\} \textit{ in } \mathcal{G} \textit{ and } \{t_n\} \textit{ on } [0,+\infty) \textit{ such that } \varphi_n(0) \textit{ is bounded and belongs to } E \textit{ for all } n \in \mathbb{N},\quad t_n \to +\infty \textit{ and } \varphi_n(t_n) \to x \Big\}.$$

In order to prove the existence of global attractors for generalized semiflows, we employ the following theorem due to J.M. Ball [5].

Theorem 2.4 (J.M. Ball [5]). *A generalized semiflow $\mathcal{G}$ in a metric space X has a global attractor $\mathcal{A}$ if and only if the following two conditions are satisfied.*

(i) *$\mathcal{G}$ is point dissipative, that is, one can choose a bounded set B in X such that for all $\varphi \in \mathcal{G}$ there exists $\tau = \tau(\varphi) \geq 0$ satisfying $\varphi(t) \in B$ for all $t \geq \tau$.*

(ii) *$\mathcal{G}$ is asymptotically compact, that is, for any sequences $\{\varphi_n\}$ in $\mathcal{G}$ and $\{t_n\}$ on $[0,+\infty)$, if $\{\varphi_n(0)\}$ is bounded in X and $t_n \to +\infty$, then $\{\varphi_n(t_n)\}$ is precompact in X.*

Moreover, $\mathcal{A}$ is a unique global attractor for $\mathcal{G}$ and given by

$$\mathcal{A} = \bigcup \{\omega(B);\ B \textit{ is a bounded set in } X\} = \omega(X).$$

Furthermore, $\mathcal{A}$ is the maximal compact invariant subset of X under the family of mappings $\{T(t)\}_{t\geq 0}$.

The following proposition gives a sufficient condition for the asymptotic compactness of generalized semiflows.

Proposition 2.5 (J.M. Ball [5]). *Let $\mathcal{G}$ be a generalized semiflow in a metric space X. If $\mathcal{G}$ satisfies the following conditions:*

(i) *$\mathcal{G}$ is eventually bounded, that is, for any bounded set $D \subset X$, there exists $\tau = \tau(D) \geq 0$ such that*

$$\bigcup_{t \geq \tau} T(t)D \text{ is bounded in } X.$$

(ii) *$\mathcal{G}$ is compact, that is, for any sequence $\{u_n\}$ in $\mathcal{G}$, if $\{u_n(0)\}$ is bounded in X, then there exists a subsequence $\{n'\}$ of $\{n\}$ such that $\{u_{n'}(t)\}$ is convergent in X for each $t > 0$,*

then $\mathcal{G}$ is asymptotically compact.

3 Construction of a generalized semiflow

Let us first state our basic assumptions: let $p \in (1, \infty)$, $T > 0$ be fixed.

(A1) There exist positive constants C_i $(i = 1, 2, 3, 4)$ such that

$$C_1 |u|_V^p \leq \psi(u) + C_2 \quad \text{for all } u \in D(\psi),$$
$$|\eta|_{V^*}^{p'} \leq C_3 \psi(u) + C_4 \quad \text{for all } [u, \eta] \in \partial_V \psi.$$

(A2) There exist a reflexive Banach space X_0 and a non-decreasing function ℓ_1 on $[0, \infty)$ such that X_0 is compactly embedded in V and

$$|u|_{X_0} \leq \ell_1(|u|_H + [\varphi(u)]_+) \quad \text{for all } u \in D(\partial_V \varphi),$$

where $[s]_+ := \max\{s, 0\} \geq 0$ for $s \in \mathbb{R}$.

(A3) $D(\partial_V \varphi) \subset D(B)$. For each $\varepsilon > 0$ there exists a constant $c_\varepsilon \geq 0$ such that

$$|g|_{V^*}^{p'} \leq \varepsilon |\xi|_{V^*}^\sigma + c_\varepsilon \left\{ |\varphi(u)| + |u|_V^p + 1 \right\} \quad \text{with } \sigma := \min\{2, p'\}$$

for all $u \in D(\partial_V \varphi)$, $g \in B(u)$ and $\xi \in \partial_V \varphi(u)$.

(A4) Let $S \in (0, T]$ and let (u_n) and (ξ_n) be sequences in $C([0, S]; V)$ and $L^\sigma(0, S; V^*)$ with $\sigma := \min\{2, p'\}$, respectively, such that $u_n \to u$ strongly in $C([0, S]; V)$, $[u_n(t), \xi_n(t)] \in \partial_V \varphi$ for a.e. $t \in (0, S)$, and

$$\sup_{t \in [0, S]} |\varphi(u_n(t))| + \int_0^S |u_n'(t)|_H^p dt + \int_0^S |\xi_n(t)|_{V^*}^\sigma dt$$
$$\text{is bounded for all } n \in \mathbb{N},$$

and let (g_n) be a sequence in $L^{p'}(0, S; V^*)$ such that $g_n(t) \in B(u_n(t))$ for a.e. $t \in (0, S)$ and $g_n \to g$ weakly in $L^{p'}(0, S; V^*)$. Then (g_n) is precompact in $L^{p'}(0, S; V^*)$ and $g(t) \in B(u(t))$ for a.e. $t \in (0, S)$.

(A5) Let $S \in (0,T]$ and $u \in C([0,S];V) \cap W^{1,p}(0,S;H)$ be such that $\sup_{t\in[0,S]} |\varphi(u(t))| < \infty$ and suppose that there exists $\xi \in L^{p'}(0,S;V^*)$ such that $\xi(t) \in \partial_V \varphi(u(t))$ for a.e. $t \in (0,S)$. Then there exists a V^*-valued strongly measurable function g such that $g(t) \in B(u(t))$ for a.e. $t \in (0,S)$. Moreover, the set $B(u)$ is convex for all $u \in D(B)$.

Remark 3.1. In case B is single-valued, (A4) and (A5) are satisfied under the following simple condition:

(A4)$'$ The map $u \mapsto B(u(\cdot))$ is continuous from $C([0,T];V)$ into $L^{p'}(0,T;V^*)$.

The following theorem is concerned with the existence of global (in time) strong solutions.

Theorem 3.2 (Global existence, [1]). *Let $p \in (1,\infty)$ and $T > 0$ be fixed. Suppose that (A1)–(A5) are all satisfied. Then, for all $f \in V^*$ and $u_0 \in D(\varphi)$, there exists at least one strong solution $u \in W^{1,p}(0,T;V)$ on $[0,T]$.*

Remarks and an outline of a proof. To prove this theorem, we are facing the following hurdles on our way to the goal.

- Defect of useful properties for maximal monotone operators in $V \times V^*$: The Yosida approximations and resolvents of maximal monotone operators are Lipschitz continuous in Hilbert space settings. However, they are not so in the V-V^* setting.
- Strong nonlinearity arising from the nonlinear operator $\partial_V \psi$: It prevents us from establishing energy estimates. For example, it could be somewhat difficult to extract valuable informations from the multiplication of (CP) and $u(t)$.
- Non-uniqueness of solutions even for unperturbed problems: Let us define the solution operator $\mathcal{S} : g \mapsto u$ for

$$(\mathrm{CP})_g \quad \partial_V \psi(u'(t)) + \partial_V \varphi(u(t)) + g(t) \ni f \quad \text{in } V^*, \quad 0 < t < T,$$

  and find a fixed point g_* of $\mathcal{F} : g \mapsto B(u(\cdot))$, i.e., $g_* \in B(u_*(\cdot))$ with $u_* := \mathcal{S}(g_*)$. Then

$$\partial_V \psi(u_*'(t)) + \partial_V \varphi(u_*(t)) + g_*(t) \ni f \quad \text{and} \quad g_*(t) \in B(u_*(t)).$$

  Here, $\mathcal{S}$ may be multi-valued, since solutions for $(\mathrm{CP})_g$ are not unique. However, usual fixed point theorems require the convexity of $\mathcal{F}(g)$.

To cope with such difficulties, in [1] this theorem is proved as follows:

Phase 1. Let us introduce approximate problems for (CP) given by

$$(\mathrm{CP})_\lambda \quad \begin{cases} \lambda u'(t) + \partial_V \psi(u'(t)) + \partial_H \tilde{\varphi}_\lambda(u(t)) + B(J_\lambda u(t)) \ni f \quad \text{in } V^*, \\ u(0) = u_0, \end{cases}$$

where $\tilde{\varphi}$ is an extension of φ onto H, J_λ and $\partial_H \tilde{\varphi}_\lambda$ denote the resolvent and the Yosida approximation of $\partial_H \tilde{\varphi}$ respectively. Solutions of $(\mathrm{CP})_\lambda$ will be constructed in Phase 2 below.

Phase 2, Step 1. To prove the existence of solutions for the approximation problems described above, we first prove the existence and uniqueness of solutions for unperturbed problems (i.e., g is given):

$$(\mathrm{CP})_{\lambda,g} \quad \begin{cases} \lambda u'(t) + \partial_V \psi(u'(t)) + \partial_H \tilde{\varphi}_\lambda(u(t)) + g(t) \ni f \ \text{ in } V^*, \\ u(0) = u_0. \end{cases}$$

Here we note that the existence and uniqueness of solutions for $(\mathrm{CP})_{\lambda,g}$ follows immediately in a Hilbert space setting, i.e., $V = V^* = H$, since $(\mathrm{CP})_{\lambda,g}$ is rewritten as

$$\begin{cases} u'(t) = (\lambda I + \partial_H \psi)^{-1}\Big(f - g(t) - \partial_H \varphi_\lambda(u(t))\Big) \ \text{ in } H, \\ u(0) = u_0, \end{cases}$$

and the resolvents and Yosida approximations for maximal monotone operators are Lipschitz continuous in H. However, in the case of Banach space settings, we have to prepare additional arguments.

Phase 2, Step 2. Since the solution of $(\mathrm{CP})_{\lambda,g}$ is unique, by applying a Kakutani-Schauder-type fixed point theorem, we can find a fixed point u_λ of the mapping

$$\mathcal{F}_\lambda : g \mapsto B(J_\lambda u(\cdot)),$$

where u is a solution of $(\mathrm{CP})_{\lambda,g}$ on $[0, T_*]$ with some $T_* > 0$ independent of λ. Then u_λ solves $(\mathrm{CP})_{\lambda,g}$ on $[0, T_*]$.

Phase 3. Establishing a priori estimates (particularly, from the multiplication of $(\mathrm{CP})_\lambda$ and $u'_\lambda(t)$) and deriving the convergences for u_λ, we can obtain local (in time) solutions of (CP).

Phase 4. Finally, we globally (in time) extend the local solutions. □

Now let us construct a generalized semiflow from all global (in time) strong solutions for (CP). We set $X := D(\varphi)$ with the distance $d_X(u,v) := |u - v|_V + |\varphi(u) - \varphi(v)|$ for $u, v \in X$, and moreover, we define

$$\mathcal{G} := \{u \in C([0,\infty); X);\ u \text{ is a strong solution of (2) on } [0,\infty)\}.$$

Then $\mathcal{G}$ becomes a generalized semiflow on X. More precisely, we have:

Theorem 3.3 (Formation of generalized semiflow, [2]). *Let $p \in (1,\infty)$ be given. Suppose that* (A1)–(A5) *are all satisfied for any $T > 0$. Then, for all $f \in V^*$, the set $\mathcal{G}$ is a generalized semiflow on X.*

An outline of a proof. From Theorem 3.2, we can obtain (H1) immediately. Moreover, (H2) and (H3) follow from the definition of strong solutions. In [2], the author proved (H4) as follows: Let $u_n \in \mathcal{G}$ and $u_0 \in X$ be such that $u_n(0) \to u_0$ in X. As in Theorem 3.2, we establish energy estimates for u_n. Then for every $T > 0$, we can take a subsequence (n_k^T) of (n) such that

$$u_{n_k^T} \to u \quad \text{strongly in } C([0,T]; V),$$

$$\int_0^T \varphi(u_{n_k^T}(t))dt \to \int_0^T \varphi(u(t))dt,$$

which implies

$$u_{n_k}(t) \to u(t) \qquad \text{strongly in } V \quad \text{for all } t \geq 0 \tag{5}$$

with some subsequence (n_k) of (n_k^T). Moreover, u solves (CP) on $[0,\infty)$. Hence it remains to show $\varphi(u_{n_k}(t)) \to \varphi(u(t))$ for all $t \geq 0$.

From the definition of $\partial_V \varphi$ and (5),

$$\liminf_{n_k \to \infty} \varphi(u_{n_k}(t)) = \varphi(u(t)) \quad \text{for a.e. } t > 0. \tag{6}$$

Here we used the fact that

$$p(\cdot) := \liminf_{n\to\infty} |\xi_n(\cdot)|_{V^*} \in L^1_{loc}([0,\infty))$$

with a section $\xi_n(t) \in \partial_V \varphi(u_n(t))$ by Fatou's lemma.

On the other hand, we can verify that

$$\zeta_n(t) := \varphi(u_n(t)) - Ct\left(|f|_{V^*}^{p'} + 1\right) - C\int_0^t \{\varphi(u_n(\tau)) + |u_n(\tau)|_V^p\}\, d\tau$$

is non-increasing for all $t \geq 0$. By Helly's lemma (see, e.g., [3]),

$$\zeta_n(t) \to \phi(t) \quad \text{for all } t \geq 0$$

with some function $\phi : [0,\infty) \to [-\infty,\infty]$. Moreover, by (6),

$$\begin{aligned} \phi(t) &= \liminf_{n_k\to\infty} \zeta_{n_k}(t) \\ &= \varphi(u(t)) - Ct\left(|f|_{V^*}^{p'} + 1\right) - C\int_0^t \{\varphi(u(\tau)) + |u(\tau)|_V^p\}\, d\tau =: \zeta(t) \end{aligned}$$

for a.e. $t > 0$. From the continuity of $\zeta(\cdot)$ and the assumption that $\varphi(u_n(0)) \to \varphi(u_0)$, we have $\phi(t) = \zeta(t)$ for all $t \geq 0$. Thus

$$\varphi(u_{n_k}(t)) \to \varphi(u(t)) \quad \text{for all } t \geq 0.$$

Consequently, $u_{n_k}(t) \to u(t)$ in X for all $t \geq 0$. □

4 Existence of global attractors

The existence of global attractors for the generalized semiflow $\mathcal{G}$ is ensured under some structure condition for $\partial_V \varphi$ and B, which yields a dissipative estimate.

Theorem 4.1 (Existence of global attractors, [2]). *Suppose that*

(A6) *There exist constants $\alpha > 0$ and $C_5 \geq 0$ such that*

$$\alpha\{\varphi(u) + |u|_V^p\} \leq \langle \xi + g, u\rangle + C_5$$

for all $u \in D(\partial_V \varphi)$, $\xi \in \partial_V \varphi(u)$ and $g \in B(u)$.

In addition, assume $f \in V^*$ and (A1)–(A5) *for any $T > 0$. Then the generalized semiflow $\mathcal{G}$ has a global attractor $\mathcal{A}$, and $\mathcal{A}$ is a unique maximal compact invariant subset of X.*

An outline of a proof. We first establish a dissipative estimate by using (A6).

Lemma 4.2 ([2]). *Under the same assumptions as in Theorem 4.1, there exist a constant $R \geq 0$ and an increasing function $T_0(\cdot)$ on $[0, \infty)$ such that*

$$\varphi(u(t)) + |u(t)|_V^p \leq R \qquad \text{for all } u_0 \in X,\ u \in \mathcal{G} \text{ satisfying } u(0) = u_0 \text{ and } t \geq T_0(\varphi(u_0) + |u_0|_V^p). \tag{7}$$

Proof. For simplicity, we assume that $\psi(0) = 0$ and all operators are single-valued, and we also denote by C a non-negative constant, which does not depend on the elements of the corresponding space or set and may vary from line to line. Multiply $u'(t)$ to get

$$\psi(u'(t)) + \frac{d}{dt}\varphi(u(t)) = \langle f - B(u(t)), u'(t)\rangle,$$

which together with (A1) and (A3) gives

$$\begin{aligned} &\frac{1}{2}\psi(u'(t)) + \frac{d}{dt}\{\varphi(u(t)) + |u(t)|_V^p\} \\ &\leq\ C\left(|f|_{V^*}^{p'} + 1\right) + C\{\varphi(u(t)) + |u(t)|_V^p\}. \end{aligned} \tag{8}$$

On the other hand, by (A1), (A6) and (CP),

$$\begin{aligned} \alpha\{\varphi(u(t)) + |u(t)|_V^p\} &\leq \langle \partial_V\varphi(u(t)) + B(u(t)), u(t)\rangle + C \\ &= \langle f - \partial_V\psi(u'(t)), u(t)\rangle \\ &\leq C\left(|f|_{V^*}^{p'} + 1\right) + \frac{\alpha}{2}|u(t)|_V^p + C\psi(u'(t)). \end{aligned} \tag{9}$$

Then by (8) + ε(9) with $\varepsilon > 0$ small enough, we have

$$\phi'(t) + C\phi(t) \leq C \qquad \text{for a.e. } t > 0$$

with $\phi(t) := \varphi(u(t)) + |u(t)|_V^p$. This yields that

$$\phi(t) \leq C(1 + e^{-\beta t}) \qquad \text{with some } \beta > 0,$$

which implies that there exists $R > 0$ such that $\phi(t) \leq R$ for all $t \geq T_0$ with some constant $T_0 = T_0(\phi(0)) > 0$ depending on $\phi(0) = \varphi(u_0) + |u_0|_V^p$. □

Hence we can prove that $u(t)$ will eventually enter a ball in X with an estimate from above for the arrival time and eternally stay there. This lemma further implies that $\mathcal{G}$ is eventually bounded and point dissipative in X.

Moreover, the compactness of $\mathcal{G}$ also follows as in the proof of (H4) (see Theorem 3.3). Consequently, the general theory due to J.M.Ball (see Theorem 2.4) ensures the existence of a unique global attractor for the generalized semiflow X. □

5 Applications to generalized Allen-Cahn equations

Finally, we briefly discuss the applications of the preceding abstract theory to a couple of nonlinear PDE problems of parabolic type arising from Gurtin's generalized Allen-Cahn equations. Let Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary $\partial\Omega$. For given functions $u_0, f : \Omega \to \mathbb{R}$, we first deal with

$$\left.\begin{array}{ll} \alpha(u_t(x,t)) - \Delta_m u(x,t) + \partial_r W(x, u(x,t)) \ni f(x), & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,t) = u_0(x), & x \in \Omega, \end{array}\right\} \tag{10}$$

where $\alpha(r) = |r|^{p-2}r$ with $p \geq 2$ and Δ_m stands for the so-called m-Laplace operator given by

$$\Delta_m u(x) = \nabla \cdot \left(|\nabla u(x)|^{m-2} \nabla u(x) \right), \quad 1 < m < \infty.$$

Moreover, $\partial_r W$ stands for the derivative in r of a potential $W = W(x,r) : \Omega \times \mathbb{R} \to (-\infty, +\infty]$ given by

$$W(x,r) := j(r) + \int_0^r g(x,\rho) d\rho \quad \text{for } x \in \Omega,\ r \in \mathbb{R} \tag{11}$$

with a lower semicontinuous convex function $j : \mathbb{R} \to (-\infty, +\infty]$ and a (possibly non-monotone) Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$. Hence $\partial_r W(x,r) = \partial j(r) + g(x,r)$. Then (10) can be regarded as a special case of (3); more precisely, ρ and $\hat{\psi}$ are given such that

$$\alpha(s) = \rho(s)s \quad \text{and} \quad \hat{\psi}(r, \mathbf{p}) = \frac{1}{m}|\mathbf{p}|^m + W(r).$$

In order to reduce (10) to an abstract Cauchy problem such as (CP), we set $V = L^p(\Omega)$, $H = L^2(\Omega)$, $V^* = L^{p'}(\Omega)$ and define $\varphi : V \to [0,\infty]$, $\psi : V \to [0,\infty)$ by

$$\varphi(u) := \begin{cases} \dfrac{1}{m}\displaystyle\int_\Omega |\nabla u(x)|^m dx + \int_\Omega j(u(x)) dx & \text{if } u \in W_0^{1,m}(\Omega),\ j(u(\cdot)) \in L^1(\Omega), \\ \infty & \text{otherwise} \end{cases}$$

and

$$\psi(u) := \frac{1}{p}\int_\Omega |u(x)|^p dx.$$

Then $\partial_V \varphi(u)$ and $\partial_V \psi(u)$ coincide with $-\Delta_m u + \partial j(u(\cdot))$ equipped with the boundary condition $u|_{\partial\Omega} = 0$ and $\alpha(u) = |u|^{p-2}u$ in V^*. Furthermore, let us set a mapping $B : V \to V^*$ by

$$B(u) := g(\cdot, u(\cdot))$$

with the domain $D(B) = \{u \in V;\ g(\cdot, u(\cdot)) \in V^*\}$. Then (10) is reduced to (CP).

Let us introduce the following assumptions.

(a1) $g = g(x,r)$ is a Carathéodory function, i.e., measurable in x and continuous in r. Moreover, there exist constants $q \geq 2$, $C_6 \geq 0$ and a function $a_1 \in L^1(\Omega)$ such that

$$|g(x,r)|^{p'} \leq C_6 |r|^{p'(q-1)} + a_1(x)$$

for a.e. $x \in \Omega$ and all $r \in \mathbb{R}$.

(a2) there exist constants $\sigma > 1$ and $C_7 \geq 0$ such that

$$|r|^\sigma \leq C_7\Big(j(r) + 1\Big) \quad \text{for all} \quad r \in \mathbb{R}.$$

Our result reads,

Theorem 5.1 ([2]). *In addition to* (a1) *and* (a2), *assume that*

$$2 \leq p < \max\{m^*, \sigma\} \quad \textit{and} \quad p'(q-1) < \max\{m, p, \sigma\},$$

where m^* is the Sobolev critical exponent, i.e., $m^* := Nm/(N-m)_+$. Then, for $f \in L^{p'}(\Omega)$ and $u_0 \in W_0^{1,m}(\Omega)$ satisfying $j(u_0(\cdot)) \in L^1(\Omega)$, the initial-boundary value problem (10) *admits at least one L^p-solution on $(0, \infty)$. Moreover, the set of solutions for* (10) *forms a generalized semiflow $\mathcal{G}$ in a phase space $X := \{v \in W_0^{1,m}(\Omega);\ j(v(\cdot)) \in L^1(\Omega)\}$. Furthermore, if $p \leq \max\{m, \sigma\}$, then $\mathcal{G}$ possesses a global attractor in X.*

The following generalized problem also falls within our abstract theory.

$$\left.\begin{array}{ll} \alpha(u_t(x,t)) - \Delta u(x,t) + N(x, u(x,t), \nabla u(x,t)) \ni f(x), & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,t) = u_0(x), & x \in \Omega, \end{array}\right\} \quad (12)$$

where $N = N(x, r, \mathbf{p})$ is written as follows

$$N(x, r, \mathbf{p}) = \partial j(r) + h(x, r, \mathbf{p}) \quad \text{for} \quad x \in \Omega,\ r \in \mathbb{R},\ \mathbf{p} \in \mathbb{R}^N.$$

It could be emphasized that this problem may not be written as a (generalized) gradient system such as (3), since the nonlinear term N depends on the gradient of u. We discuss the existence of global (in time) solutions and their long-time behavior for (10) and (12).

(a3) $h = h(x, r, \mathbf{p})$ is a Carathéodory function, i.e., measurable in x and continuous in r and $\mathbf{p}$. There exist constants $q_1, q_2 \geq 2$, $C_3 \geq 0$ and a function $a_2 \in L^1(\Omega)$ such that

$$|h(x, r, \mathbf{p})|^{p'} \leq C_3\left(|r|^{p'(q_1-1)} + |\mathbf{p}|^{p'(q_2-1)}\right) + a_2(x)$$

for a.e. $x \in \Omega$ and all $r \in \mathbb{R}$ and $\mathbf{p} \in \mathbb{R}^N$.

Then we have:

Theorem 5.2 ([2]). *In addition to* (a2) *and* (a3), *assume that*

$$2 \leq p < \max\{2^*, \sigma\}, \quad p'(q_1 - 1) < \max\{p, \sigma\} \quad \textit{and} \quad p'(q_2 - 1) < 2.$$

Then, for $f \in L^{p'}(\Omega)$ and $u_0 \in H_0^1(\Omega)$ satisfying $j(u_0(\cdot)) \in L^1(\Omega)$, the initial-boundary value problem (12) *admits at least one L^p-solution on $(0, \infty)$. Moreover, the set of solutions for* (12) *forms a generalized semiflow $\mathcal{G}$ in a phase space $X := \{v \in H_0^1(\Omega);\ j(v(\cdot)) \in L^1(\Omega)\}$. Furthermore, if $p = 2$ or $p \leq \sigma$, then $\mathcal{G}$ possesses a global attractor in X.*

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