Convex bodies passing through holes

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1. Introduction

For a given convex body, find a “small” wall hole through which the convex body can pass. This type of problems goes back to Zindler [14] in 1920, who considered a convex polytope which can pass through a fairly small circular holes. A related topic known as Prince Rupert’s problem can be found in [2]. Here we concentrate on the case when the convex body is a regular tetrahedron or a regular n-simplex.

For a compact convex body \( K \subset \mathbb{R}^n \), let \( \text{diam}(K) \) and \( \text{width}(K) \) denote the diameter and width of \( K \), respectively. For \( d > 0 \) let \( dK \) denote the convex body with diameter \( d \) and homothetic to \( K \). Let \( S_n, Q_n, \) and \( B_n \) denote the \( n \)-dimensional regular simplex, the \( n \)-dimensional hypercube, and the \( n \)-dimensional ball, respectively. Thus, \( 1S_n \) has side length 1, \( 1Q_n \) has side length \( 1/\sqrt{n} \), and \( 1B_n \) has radius \( 1/2 \).

Let \( H \subset \mathbb{R}^{n-1} \) be a convex body, which we will call a hole. Let \( \Pi \) be the hyperplane containing \( H \), which divides \( \mathbb{R}^n \) into \( \Pi \) and two (open) half spaces \( \Pi^+ \) and \( \Pi^- \). We want to push \( 1S_n \) from \( \Pi^+ \) to \( \Pi^- \) through \( H \). In this situation, we are interested in two types of “small” holes, namely,

\[
\gamma(n, H) := \min\{d : 1S_n \text{ can pass through the hole of } dH \text{ in } \mathbb{R}^n\},
\]

and

\[
\Gamma(n, H) := \min\{d : 1S_n \subset (dH) \times \mathbb{R}\}.
\]

Notice that \( \gamma(n, H) \) and \( \Gamma(n, H) \) do not depend on \( \text{diam}(H) \). For given \( H \), we resize \( H \) so that \( 1S_n \) can pass through the hole \( H \). We will try to find a hole homothetic to \( K \) with minimum diameter, which will give \( \gamma \) or \( \Gamma \). (Recall that \( dH \) is homothetic to \( H \) and \( \text{diam}(dH) = d \).) By definition, \( 1S_n \) can pass through a hole \( H \) by translation perpendicular to the hyperplane containing the hole iff \( \text{diam}(H) \geq \Gamma(n, H) \). Thus we have \( \gamma(n, H) \leq \Gamma(n, H) \).

We have \( \text{width}(1Q_n) = 1/\sqrt{n} \) and \( \text{width}(1B_n) = 1 \). Steinhagen [12] determined the width of \( S_n \) as follows.

\[
\text{width}(1S_n) = \begin{cases} 
\sqrt{\frac{2}{n+1}} & \text{if } n \text{ is odd,} \\
\sqrt{\frac{2n+2}{n(n+2)}} & \text{if } n \text{ is even.}
\end{cases}
\]

(1)

If \( 1S_n \) can pass through a hole \( dH \) by translation, then

\[
\text{width}(dH) \geq \text{width}(1S_n) = (\sqrt{2} - o(1))/\sqrt{n}.
\]

(2)

Let \( n \geq 3 \). If \( 1S_n \) can pass through a hole \( dH \), then \( d \geq \text{width}(1S_2) = \sqrt{3}/2 \). This gives \( \gamma(n, H) \geq \sqrt{3}/2 \).
Brandenberg and Theobald [1] proved the following.

\[
\Gamma(n, B_{n-1}) = \begin{cases} 
\sqrt{\frac{2(n-1)}{n+1}} & \text{if } n \text{ is odd,} \\
\frac{2n-1}{\sqrt{2n(n+1)}} & \text{if } n \text{ is even.}
\end{cases}
\] (3)

2. IN THE 3-SPACE

Itoh, Tanoue, and Zamfirescu [6] proved

\[
\gamma(3, Q_2) = \Gamma(3, Q_2) = 1, \quad \gamma(3, B_2) = 2r = 0.8956..., \] (4)

where \( r \in (0, 1) \) is a unique root of the equation \( 216x^6 - 9x^4 + 38x^2 - 9 = 0 \).

We note that \( \gamma(3, B_2) < \Gamma(3, B_2) = 1 \).

In [9], the following is proved.

\[
\gamma(3, S_2) = \Gamma(3, S_2) = \frac{1 + \sqrt{2}}{\sqrt{6}} = 0.9855... \]

Zamfirescu [13] proved that most convex bodies can be held by a circular frame. Using (4), one can show that a square frame of diagonal length \( d \) can hold \( 1S_3 \) iff \( 1/\sqrt{2} < d < 1 \), and a circular frame of diameter \( d \) can hold \( 1S_3 \) iff \( 1/\sqrt{2} < d < \gamma(3, B_2) \), see [6].

On the other hand, it is shown in [9] that

\[ \text{no triangular frame can hold a convex body.} \] (5)

This is a special property for triangular frames, and in fact, we have the following.

**Theorem 1.** [9] Every non-triangular frame holds some tetrahedron in \( \mathbb{R}^3 \).

Debrunner and Mani-Levitska [3] proved that any section of a right cylinder by a plane contains a congruent copy of the base, see also [7]. This together with (5) implies the following: if a convex body, not necessarily smooth, can pass through a triangular hole, then the convex body can pass through the hole by translation perpendicular to the wall, see [9].

Itoh and Zamfirescu [5] found a hole \( H \subset \mathbb{R}^2 \) with \( \text{diam}(H) = \text{width}(1S_2) = \sqrt{3}/2 \) and \( \text{width}(H) = \text{width}(1S_3) = \sqrt{2}/2 \), such that \( 1S_3 \) can pass through \( H \).

3. HIGHER DIMENSIONS

3.1. **The hole** \( S_{n-1} \). Recall that any plane section of a right triangular prism contains a congruent copy of a base of the prism [3, 7]. The situation in higher dimension is different. In [3], it is proved that if \( n > 3 \), then for any right cylinder with convex polytope base, one can find a hyperplane section which does not contain a congruent copy of the base. Nevertheless, we have the following.
Theorem 2. [9] Let $K \subset \mathbb{R}^n$ be a compact convex body, and let $\Delta_{n-1}$ be a general $(n-1)$-simplex. If $K$ can pass through the hole $\Delta_{n-1}$, then this can be done by translation only.

Problem 1. Is it possible to take the translation in Theorem 2 perpendicular to the wall? Or equivalently, do $\gamma(n, S_{n-1})$ and $\Gamma(n, S_{n-1})$ coincide?

Theorem 3.

$$\gamma(n, S_{n-1}) \geq \begin{cases} \sqrt{1 - \frac{1}{n}} & \text{if } n \text{ is odd,} \\ \sqrt{1 - \frac{1}{n+2}} & \text{if } n \text{ is even.} \end{cases}$$

Proof. Suppose that $1S_n$ can pass through the hole of $dS_{n-1}$. By Theorem 2, this can be done by translation only. Thus we can apply (2) with (1), which implies the desired inequality.

The above result together with $\gamma(n, S_{n-1}) \leq \Gamma(n, S_{n-1}) \leq 1$ gives

$$\lim_{n \to \infty} \gamma(n, S_{n-1}) = \lim_{n \to \infty} \Gamma(n, S_{n-1}) = 1.$$ If the simplex does pass through a hole, then in particular the volume of some central hyperplane section of that simplex is no bigger than the volume of the hole. After the RIMS workshop, Jiří Matoušek suggested showing $\gamma(n, S_{n-1}) \to 1$ by using this simple observation. He also told us the information from Keith Ball: it is conjectured that the smallest central hyperplane section of $S_n$ is obtained by a hyperplane parallel to a facet of the simplex. According to Keith Ball's suggestion, we asked Matthieu Fradelizi about the volume of central slices of a simplex. Then, Fradelizi told us that a result in [4] implies that the volume of the smallest central hyperplane section of $S_n$ is more than $\text{vol}(S_{n-1})/(2\sqrt{3})$, and this is enough for proving $\gamma(n, S_{n-1}) \to 1$.

Since the diameter of circumsphere of $1S_n$ is $\sqrt{2(n-1)/n}$, we have

$$\Gamma(n, S_{n-1}) \sqrt{\frac{2(n-1)}{n}} \geq \Gamma(n, B_{n-1}).$$

This together with (3) implies

$$\Gamma(n, S_{n-1}) \geq \sqrt{1 - \frac{1}{n+1}}$$

for $n$ odd. (For $n$ even, Theorem 3 gives a better lower bound for $\Gamma(n, S_{n-1})$.) Actually $S_n$ can pass through a hole smaller than its facet.

Theorem 4. $\Gamma(n, S_{n-1}) < 1$ for all $n \geq 2$.

Let us try the case $n = 3$ to get a feel. Let $S_2 = A_0A_1A_2, A_0 = (0, 1/2), A_1 = (0, -1/2), A_2 = (\sqrt{3}/2, 0)$, and let $\mathcal{P}$ be the right triangular prism with base
$A_0A_1A_2$. We put the unit regular tetrahedron $S_3 = B_0B_1B_2B_3$ in the prism, namely, we set

$$B_0 = (0, 1/2, 0), B_1 = (0, -1/2, 0), B_2 = (1/\sqrt{2}, 0, 1/2), B_3 = (1/\sqrt{2}, 0, -1/2).$$

Now we move the tetrahedron very slightly keeping it inside $\mathcal{P}$ so that all vertices are off the faces of $\mathcal{P}$. This can be done by rotating the tetrahedron along the $x$-axis, and push it in the direction of $x$-axis. This gives $\Gamma(3, S_2) < 1$.

3.2. The hole $Q_{n-1}$. In [8] the following is proved: for every $\epsilon > 0$ there is an $N$ such that for every $n > N$ one has

$$1S_n \subset (2 + \epsilon)Q_n.$$

This gives

$$\lim_{n \to \infty} \Gamma(n, Q_{n-1}) \leq 2.$$

Clearly we have $\Gamma(n, Q_{n-1}) \geq \Gamma(n, B_{n-1})$, and we get a lower bound for $\Gamma(n, Q_{n-1})$ from (3). Here we include a simple proof of the following slightly weaker bound.

**Theorem 5.** We have

$$\Gamma(n, Q_{n-1}) \geq \sqrt{\frac{2(n-1)}{n+1}}, \quad \text{(6)}$$

with equality holding iff there exists an Hadamard matrix of order $n + 1$.

**Proof.** Let $d = \Gamma(n, Q_{n-1})$. Then $1S_n$ can pass through a hole of $dQ_{n-1}$ by translation. So (2) and (1) imply

$$\text{width}(dQ_{n-1}) = \frac{d}{\sqrt{n-1}} \geq \text{width}(1S_n) \geq \sqrt{\frac{2}{n+1}},$$

which gives (6). Moreover, if $1S_n \subset \ell Q_n$, then we have

$$\ell \geq \frac{\sqrt{n}}{\sqrt{n-1}} \Gamma(n, Q_{n-1}) \geq \sqrt{\frac{2n}{n+1}}.$$

It is known that $\ell = \sqrt{(2n)/(n+1)}$ iff there exists an Hadamard matrix of order $n + 1$, see e.g., [11].

**Problem 2.**

$$\gamma(n, Q_{n-1}) = \Gamma(n, Q_{n-1}) = \sqrt{2} - o(1)$$
3.3. **The hole $B_{n-1}$.** We have $\Gamma(n, B_{n-1}) \rightarrow \sqrt{2}$ by (3). On the other hand, the following result shows $\gamma(n, B_{n-1}) \rightarrow 3/(2\sqrt{2})$. Namely, “rotation” does help for escaping from the ball hole.

**Theorem 6.** [10]

(i) For $n$ even,

$$\gamma(n, B_{n-1}) = \frac{3}{2\sqrt{2}} \left(1 + \frac{1}{n}\right)^{-1/2} = \frac{3}{2\sqrt{2}} \left(1 - \frac{1}{2n} + \frac{3}{8n^2} - \frac{5}{16n^3} + O(n^{-4})\right).$$

(ii) Let $r^2$ be a unique real root of the cubic equation

$$8(n+1)n^3X^3 + a_2X^2 + a_1X + a_0 = 0,$$

where

$$a_0 = -\left(\frac{9}{256}\right)(n^2 - 1)^2(n^4 - 4n^3 + 2n^2 + 4n + 13),$$

$$a_1 = \frac{1}{16}(n^2 - 1)(2n^6 - 6n^5 - 15n^4 + 38n^3 + 42n^2 + 48n - 29),$$

$$a_2 = \frac{1}{4}(8n^6 - 8n^5 - 41n^4 - 28n^3 - 10n^2 + 36n + 27).$$

Then, for $n$ odd,

$$\gamma(n, B_{n-1}) = 2r = \frac{3}{2\sqrt{2}} \left(1 - \frac{1}{2n} + \frac{3}{8n^2} - \frac{13}{16n^3} + O(n^{-4})\right).$$

3.4. **Hole having minimum volume.** In [5], the following problem is posed.

**Problem 3.** Find the minimum $(n - 1)$-dimensional volume of a compact hole in a hyperplane of $\mathbb{R}^n$ such that $1S_n$ can pass through it.

The following variation seems to be easier.

**Problem 4.** Find the minimum $(n - 1)$-dimensional volume of a compact hole in a hyperplane of $\mathbb{R}^n$ such that $1S_n$ can pass through it by translation perpendicular to the hyperplane.

We list possible candidates. Put $\sqrt{2}S_n$ in $\mathbb{R}^{n+1}$ so that the vertices are $e_1, \ldots, e_{n+1}$, where $e_i$ is the $i$-th standard base of $\mathbb{R}^{n+1}$.

Project the $\sqrt{2}S_n$ in the direction of

$$(1, -1, 0, \ldots, 0).$$

Then the hole created by the shadow has volume

$$\frac{1}{(n-1)!} \sqrt{\frac{n+1}{2}}.$$  \(7\)

Next suppose that $n$ is odd and write $n = 2k + 1$. Project the $\sqrt{2}S_n$ in the direction of

$$(1, \ldots, 1, -1, \ldots, -1).$$
Then the corresponding hole has volume

\[ \frac{2}{(n-1)!}. \]  

Finally suppose that \( n \) is even and write \( n = 2k \). Project the \( \sqrt{2}S_n \) in the direction of

\[ \frac{k}{k+1, \ldots, k+1, -k, \ldots, -k}. \]

In this case, the volume of the hole is

\[ \frac{2}{(n-1)!} \sqrt{\frac{n}{n+2}}. \]  

(9)

Among the above examples, the smallest one is (7) for \( n \leq 5 \). For \( n = 7 \), (7) and (8) coincide. For the other cases, (8) and (9) give the smallest one.

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**REFERENCES**


