Title
Existence and Multiplicity of Solutions for a Coupled Nonlinear Schrödinger System (Nonlinear Analysis and Convex Analysis)

Author(s)
Hirano, Norimichi

Citation
数理解析研究所講究録 2009, 1643: 232-241

Issue Date
2009-04

URL
http://hdl.handle.net/2433/140613

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
Existence and Multiplicity of Solutions for a Coupled Nonlinear Shrödinger System

Norimichi Hirano

Graduate School of Environment and Information Sciences, Yokohama National University

1. Introduction

In the present paper, we consider the multiple existence of non-radial positive solutions of coupled Schrödinger system

\[
(P) \begin{cases}
-\Delta u + \mu_1 u = u^3 + \beta uv^2 & \text{in } \mathbb{R}^3 \\
-\Delta v + \mu_2 v = v^3 + \beta u^2 v & \text{in } \mathbb{R}^3
\end{cases}
\]

where \( \mu_1, \mu_2 > 0 \) and \( \beta \in \mathbb{R} \).

Coupled nonlinear Schrödinger system \((P)\) models many physical problems. In nonlinear optics, the phenomenon in Kerr-like photorefractive media is described by system \((P)\) (cf. [2]). In this case, the solution \( u \) and \( v \) denote the components of the beam in Kerr-like photorefractive media, and the coupling constant \( \beta \) is the interaction between two components \( u \) and \( v \). In case \( \beta > 0 \), the interaction is attractive, while the interaction is repulsive if \( \beta < 0 \). The bimodal pulse in optical fibers under birefringent effects is also governed by system \((P)\) (cf. [16]). It is also known that system \((P)\) is a model for a mixture of two Bose-Einstein condensates (cf. [7]).

Motivated by these physical interest, the existence of solutions of \((P)\) has been investigated by several authors. In the case that \( \beta > 0 \), problem \((P)\) was studied by Ambrosetti & Colorado [3], Maia, Montefusco & Pellacci [15] and Lin & Wei [12]. They proved the existence of least energy solutions \((u, v)\) of \((P)\) with \( u, v > 0 \). By the result of Troy (cf. [22]), we know that in this case, all positive solutions \((u, v)\) of \((P)\) satisfying

\[
(1.1) \quad u(x) \to 0, v(x) \to 0, \quad \text{as } |x| \to \infty
\]

are radially symmetric functions. On the other hand, in case that \( \beta < 0 \), it is known that there is no least energy solution of \((P)\) (cf. [12]). Moreover, positive solutions of \((P)\) satisfying (1.1) is not always radial. In case that \( \beta < 0 \) and \( |\beta| \) is small, there are positive solutions with one component concentrating on the origin and the other component concentrating around a regular polygon (cf. [14]). The existence of non radial positive solutions was also considered in [24] for the case that \( \beta < 0 \) and \( \mu_1 = \mu_2 \). In this case, there are infinitely many nonradial solutions if \( \beta < -1 \). Recently, Sirakov [21]
established the existence of ground state solutions of (P) in the case that coefficients of nonlinear terms $u^3$ and $v^3$ in (P) are different.

On the other hand, the existence of sign changing solutions of nonlinear scalar elliptic problem

\begin{equation}
-\Delta u + u = |u|^{p-1} u, \quad u \in H_0^1(\Omega)
\end{equation}

has been investigated by many authors in the last decade. Here $\Omega$ is a domain in $\mathbb{R}^N (N \geq 3)$ and $p \in (1, (N + 2)/(N - 2))$. We refer to [5], [6] and [17] for related results of sign changing solutions of (1.2). The existence of sign changing solutions of (P) with $\beta > 0$ was considered in [10]. For the problem (P) with $\mathbb{R}^3$ replaced by a bounded domain, we refer to [13] and [19].

In the present paper, we first see the multiple existence of solutions of (P) in the case that $\beta > 0$. Next we consider the case that $\beta < 0$ and $|\beta|$ is small, i.e., the case that the interaction of two solutions are small and repulsive. We will show the multiple existence of nonradial solutions of (P) in this case with $\mu_1 \neq \mu_2$. Our results improve the results in [14]. (See Remark 2 and Remark 3).

To state our main results, we need some notations. We denote by $B_r(x)$ the open ball in $\mathbb{R}^3$ centered at $x \in \mathbb{R}^3$ with radius $r > 0$. The inner product in $\mathbb{R}^3$ is denoted by $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$. We put $H = H^1(\mathbb{R}^3)$ and $H = B \times H$. We set $\mu_0 = 1$. We denote by $\| \cdot \|_{\mu_i}$ the norm of $H$ defined by $\|u\|_{\mu_i}^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + \mu_i |u|^2)dx$ for $u \in H$ and $i \in \{0, 1, 2\}$. For simplicity of notations, we put $|u(x)|_{\mu_i}^2 = |\nabla u(x)|^2 + \mu_i |u(x)|^2$ for $u \in H$ and $x \in \mathbb{R}^3$. For each function $u \in H$, we set $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \max\{-u(x), 0\}$. For each $p \geq 1$, we denote by $|u|_p$ the norm of the space $L^p(\mathbb{R}^3)$. The Hilbert space $H$ is equipped with the norm defined by $\|u\|^2 = \|u\|_{\mu_1}^2 + \|v\|_{\mu_2}^2$ for $U = (u, v) \in H$. We recall that for each $i \in \{0, 1, 2\}$, problem

\begin{equation}
(P_i) \quad \begin{cases}
-\Delta u + \mu_i u &= u^3 & \text{in } \mathbb{R}^3 \\
u(x) &= 0 & \text{in } \mathbb{R}^3 \\
u(x) &\to 0 & \text{as } |x| \to \infty
\end{cases}
\end{equation}

has a radial solution, denoted by $U_i$ (cf. [8], [11]). The function $U_i$ is the unique smooth solution of $(P_i)$ up to translation. Moreover we know that $U_i$ satisfies $I_i(U_i) = c_i = \min \{ I(v) : v \in S_i \}$, where $I_i$ is the functional associated with problem $(P_i)$ defined by

\begin{equation}
I_i(v) = \frac{1}{2} \|v\|_{\mu_i}^2 - \frac{1}{4} |v^+|_4^4 & \quad \text{for } v \in H
\end{equation}

and $S_i$ is the set defined by

$S_i = \{ v \in H : \|v\|_{\mu_i}^2 = |v^+|_4^4 \}$ for $i = 1, 2$. 

For each $x \in \mathbb{R}^3$ and $i \in \{0, 1, 2\}$, we put $U_{i,x}(\cdot) = U_i(\cdot - x)$. It is also known that

(1.4) $|U_i(x)|_{\mu_i} |x| \exp(\sqrt{\mu_i} |x|) \to c > 0$, as $|x| \to \infty$ for $i \in \{0, 1, 2\}$.

(cf. [11]). For each $u \in L^4(\mathbb{R}^3)$, we put $\hat{u}(x) = \int_{B_1(x)} |u(x)|^4 \, dx$ for $x \in \mathbb{R}^3$. Then from (1.4), we can choose $R_0 > 0$ such that

(1.5) $\hat{U}_i(z) < \frac{|\hat{U}|_{\infty}}{3}$ for all $z \in \mathbb{R}^3 \setminus B_{R_0}(0)$ and $i \in \{1, 2\}$.

Since we consider the case that $\mu_1 \neq \mu_2$, we may assume without any loss of generality that $\mu_2 < \mu_1$. We can now state our main results.

**Theorem 1.** Suppose that the following condition holds:

$$0 < 2\sqrt{\mu_2} < \sqrt{\mu_1}.$$  

Then there exists $\beta_0 > 0$ such that for each $\beta \in (0, \beta_0)$, problem $(P)$ possesses at least one ground state solution $U_0 \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ and one nonradial sign changing solution $U \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$.

**Theorem 2.** Suppose that $\sqrt{\mu_1/\mu_2}$ is irrational. Then for each $i \in \{2, 4, 6, 8, 12, 20\}$, there exists $\beta_i \in (-1, 0)$ such that for each $\beta \in (\beta_i, 0)$, there exists a positive solution $U_i \in H^1 \times H^1$ of $(P)$ such that $ic_1 + c_2 < \Phi(U_i) < ic_1 + 2c_2$ and $U_i$ has the form

(1.6) $\Phi(U_i) = (U, V) = (\sum_{j=1}^i U_{1,x_j} + u, U_2 + v)$

where $\{x_1, x_2, \ldots, x_i\}$ forms a regular $i$-polyhedra in $\mathbb{R}^3$ in case $i \neq 2$ and $x_1 = -x_2$ in case that $i = 2$, and $u, v \in H$ such that $\|u, v\|$ is so small that

(1.7) $\hat{U}(z) < \frac{1}{2} |\hat{U}|_{\infty}$ for $z \in \mathbb{R}^3 \setminus (\bigcup_{j=1}^i B_{R_0}(x_j))$ and

$\hat{V}(z) < \frac{1}{2} |\hat{V}|_{\infty}$ for $z \in \mathbb{R}^3 \setminus B_{R_0}(0)$.

**Remark 1.** The expression (1.6) of $U_i = (U, V)$ is unique when $\|u, v\|$ is so small that condition (1.7) holds, i.e., for each $U_i, (x_1, x_2, \ldots, x_i)$ is uniquely determined.

**Remark 2.** The assertion of Theorem 1 implies that for $\beta < 0$ with $|\beta|$ sufficiently small, problem $(P)$ possesses at least 6 nonradial positive solutions. In [14], the existence of positive solutions of $(P)$ of the form (1.6) was established in the case that $\{x_1, x_2, \ldots, x_i\}$ forms regular cube or tetrahedra under the assumption

$$\sqrt{\frac{\mu_1}{\mu_2}} < \begin{cases} \frac{\sqrt{3}}{3} & \text{for the cube} \\ \frac{\sqrt{3}}{\sqrt{2}} & \text{for the tetrahedra.}\end{cases}$$
Our argument employed in this paper does not require the ratio of $\sqrt{\mu_1}$ and $\sqrt{\mu_2}$.

Theorem 3. Suppose that $\sqrt{\mu_1/\mu_2}$ is irrational. Then for each $k \in \mathbb{N}$, there exists $\beta_k \in (-1,0)$ such that for each $\beta \in (\beta_k,0)$, the problem (P) has a positive solution $\tilde{U}_k$ such that $kc_1 + c_2 < \Phi(\tilde{U}_k) < kc_1 + 2c_2$ and $\tilde{U}_k$ has the form

\begin{equation}
\tilde{U}_k = (U, V) = (\sum_{j=1}^{k} U_{1j} + u, U_2 + v)
\end{equation}

where $\{x_1, x_2, \ldots, x_k\} \subset \mathbb{R}^3$ form a regular $k$-polygon in a two-dimensional subspace of $\mathbb{R}^3$, and $u, v \in H$ such that $\| (u, v) \|$ is so small that

$\tilde{U}(z) < \frac{1}{2} \| \tilde{U} \|_\infty$ for $z \in \mathbb{R}^3 \setminus (\bigcup_{j=1}^{k} B_{R_0}(x_j))$ and $\tilde{V}(z) < \frac{1}{2} \| \tilde{V} \|_\infty$ for $z \in \mathbb{R}^3 \setminus B_{R_0}(0)$.

Remark 3. The existence of positive solutions of (P) of the form (1.8) was proved in [14] in the case that the spacial dimension is 2 and $\mu_1, \mu_2$ satisfy

$\sqrt{\frac{\mu_1}{\mu_2}} < \sin \frac{\pi}{k}$.

We give a sketch of the proof of Theorem 2 for the case $i = 2$. The proofs of Theorem 2 for $i \neq 2$ and the proof of Theorem 3 are slight modifications of that of the case $i = 2$ of Theorem 2. The detail of the proofs can be found in [9].

2. Preliminaries

Throughout the rest of this paper, we assume that $\sqrt{\mu_1/\mu_2}$ is irrational.

For each $u, v \in H = H^1(\mathbb{R}^3)$, we put $\langle u, v \rangle = \int_{\mathbb{R}^3} uv$. We denote by $\langle \cdot, \cdot \rangle_{\mu_i}$ the inner product of $H$ defined by $\langle u, v \rangle_{\mu_i} = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + \mu_i uv)$ for $u, v \in H$ and $i \in \{0, 1, 2\}$. The inner product of $H$ is defined by $\langle U_1, U_2 \rangle_H = \langle U_1, U_2 \rangle_{\mu_1} + \langle V_1, V_2 \rangle_{\mu_2}$ for $U_1 = (U_1, V_1), U_2 = (U_2, V_2) \in H$. For each $u \in L^4(\mathbb{R}^3)$, we put $\Omega(u) = \{ x \in \mathbb{R}^3 : \hat{u}(x) \geq \frac{1}{2} \| \hat{u} \|_{L^\infty} \}$ and

$B(u) = \frac{\int_{\Omega(u)} \hat{u}(x) \hat{u}(x) - \| \hat{u} \|_{L^\infty}^2)}{\int_{\Omega(u)} (\hat{u}(x) - \| \hat{u} \|_{L^\infty})^2} dx.

The mapping $B$ is called generalized barycenter, which is introduced in [18] (cf. also [4]). By Sobolev's embedding theorem([1]), for $p \in [2, 6]$ there exists $m_p > 0$ such that

$|z|_{L^p(B_r(0))} \leq m_p \left( |\nabla z|_{L^2(B_r(0))}^2 + \mu_i |z|_{L^2(B_r(0))}^2 \right)^{1/2}$.
for $r > 1, z \in H^1(B_r(0))$, and $i \in \{0, 1, 2\}$. For each $a \in R$, and a functional $F : H \rightarrow R$, we denote by $F^a$ the level set defined by $F^a = \{ v \in H : F(v) \leq a \}$. The same notation is used for functionals defined on $H$.

It is easy to see that for each $u \in H \setminus \{0\}$ with $u^+ \neq 0$, there exists a unique positive number $s$ such that $su \in S_i$ (cf. [25]). It follows from the definitions of $U_i$ that $U_i(x) = \sqrt{\mu_i}U_0(\sqrt{\mu_i}x)$ on $R^3$. Then one can see that

$$c_1 = \sqrt{\mu_1}c_0 > c_2 = \sqrt{\mu_2}c_0.$$  

Let $i \in \{0, 1, 2\}$. It is known that $\{U_i,x : x \in R^3\}$ is a nondegenerate critical set of $I_i$ (cf. [23]). More precisely, we have there exists $\lambda > 0$ such that

$$\|u\|_{\mu_i}^2 - 3\langle u^2u, u \rangle \geq \lambda \|u\|_{\mu}^2$$  

for all $u \in \{U_i, \frac{\partial U_i}{\partial x_1}, \frac{\partial U_i}{\partial x_2}, \frac{\partial U_i}{\partial x_3}\}$.

We define a functional $\Phi : H \rightarrow R$ associated with problem (P) by

$$\Phi(U) = \frac{1}{2}(\|U\|_{\mu_1}^2 + \|V\|_{\mu_2}^2) - \frac{1}{4}(\|U^+\|_4^4 + \|V^+\|_4^4) - \frac{\beta}{4} \int_{R^3} (U^+)^2(V^+)^2$$

$$= \Phi_1(U) + \Phi_2(U)$$  

for $U = (U, V) \in H$,

where

$$\Phi_1(U) = \frac{1}{2} \|U\|_{\mu_1}^2 - \frac{1}{4} \|U^+\|_4^4 - \frac{\beta}{4} \int_{R^3} (U^+)^2(V^+)^2$$

and

$$\Phi_2(U) = \frac{1}{2} \|V\|_{\mu_2}^2 - \frac{1}{4} \|V^+\|_4^4 - \frac{\beta}{4} \int_{R^3} (U^+)^2(V^+)^2.$$  

Then a direct computation shows

$$\langle \nabla \Phi(U), V \rangle_H = \left\langle \begin{pmatrix} \nabla_u \Phi(U) \\ \nabla_v \Phi(U) \end{pmatrix}, \begin{pmatrix} W \\ Z \end{pmatrix} \right\rangle_H$$

$$= \langle -\Delta U + \mu_1 U - (U^+)^3 - \beta U^+(V^+)^2, W \rangle$$

$$+ \langle -\Delta V + \mu_2 V - (V^+)^3 - \beta (U^+)^2V^+, Z \rangle$$

for $U = (U, V), V = (W, Z) \in H$. We put

$$M_+ = \{(U, V) \in H \setminus \{0\} : \|U\|_{\mu_1}^2 = \|U^+\|_4^4 + \beta \int_{R^3} (U^+)^2(V^+)^2, \|V\|_{\mu_2}^2 = \|V^+\|_4^4 + \beta \int_{R^3} (U^+)^2(V^+)^2 \}.$$  

Then one can see that $U = (U, V) \in M_+$ is a critical point of $\Phi$ if and only if $U$ is a positive solution of problem (P). From the definition, we have

$$\Phi_1(U) = \frac{1}{4} \|U\|_{\mu_1}^2, \Phi_2(U) = \frac{1}{4} \|V\|_{\mu_2}^2$$  

and

$$\Phi(U) = \frac{1}{4}(\|U\|_{\mu_1}^2 + \|V\|_{\mu_2}^2)$$  

for $U = (U, V) \in M_+$. We also have that for each $U = (U, V) \in M_+$ there exists $(s, t) \in R^+ \times R^+$ such that $(sU, tV) \in M_+$. In fact, for each $U = (U, V), U \neq$
0, V \not\equiv 0, (sU, tV) \in M_+ \text{ if and only if }

\begin{align*}
\begin{pmatrix}
   s^2 \\
   t^2 
\end{pmatrix} = A^{-1} \begin{pmatrix}
   \|U\|_{\mu_1}^2 \\
   \|V\|_{\mu_2}^2
\end{pmatrix}
\end{align*}

where

\[ A = \begin{pmatrix}
   |U^+|^4_4 \\
   \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 |V^+|^4_4
\end{pmatrix} \begin{pmatrix}
   (s^2 t^2) \\
   (s^2 t^2)
\end{pmatrix} \] \quad \because \beta \in (-1, 0).

Since \beta \in (-1, 0), we have by the Schwartz's inequality that \(A^{-1}\) exists and then there exists a unique solution \((s, t) \in R^+ \times R^+\). For given \(U = (U, V) \in H\) with \(U \not\equiv 0, V \not\equiv 0\), we put \(NU = N(U, V) = (N_1 U, N_2 V) = (sU, tV) \in M_+\).

Now to prove Theorem 1 for the case that \(i = 2\), we define \(H_2 \subset H, H_2 \subset H\) and \(M_2 \subset M_+\) by

\[ H_2 = \{ u \in H : u(x) = u(-x) \text{ for } x \in \mathbb{R}^3 \}, \quad \mathbb{H}_2 = H_2 \times H_2, \]

and

\[ M_2 = M_+ \cap \mathbb{H}_2. \]

Since \(\sqrt{\mu_1/\mu_2}\) is irrational, we can choose \(\delta_2 \in (0, c_2)\) so small that

\begin{equation}
\begin{align*}
   c_1 + k c_2 &\notin [2c_1 + c_2 - \delta_2, 2c_1 + c_2 + \delta_2] \quad \text{for all } k \in \mathbb{N}.
\end{align*}
\end{equation}

The following Lemmata are crucial for our argument.

**Lemma 1.** (1) There exists \(\beta_1 \in (-1, 0)\) such that for each \(\beta \in (\beta_1, 0)\) and each critical point \(U \in M_2 \cap \Phi^{2c_1 + 2c_2} \cdot \Phi\),
\[ \Phi(U) \in \bigcup_{i \geq 1, j \geq 1} [ic_1 + jc_2 - \delta_2 / 2, ic_1 + jc_2 + \delta_2 / 2]. \]

(2) Let \(\beta \in (\beta_1, 0)\) and \(\{U_n\} \subset M_2\) such that \(\lim_{n \to \infty} \nabla \Phi(U_n) = 0\) and \(\lim_{n \to \infty} \Phi(U_n) = 2c_1 + c_2 + \epsilon\) with \(\epsilon \in (0, \delta_2 / 2)\). Then there exists a convergence subsequence \(\{U_{n_k}\} \subset \{U_n\}\).

**Lemma 2.** For given \(\epsilon > 0\), there exists \(\beta_\epsilon \in (\beta_1, 0)\) such that for each \(\beta \in (\beta_\epsilon, 0)\) and for each \(x \in \mathbb{R}^3 \setminus \{0\}\),
\[ \Phi(N(U_{1,x} + U_{1,-x}, U_2)) < 2c_1 + c_2 + \epsilon \]
3. Sketch of the Proof of Theorem 2 for $i = 2$.

Throughout this section we assume that $\beta \in (\beta_1, 0)$. We put

$$b_R(U) = \int_{\mathbb{R}^3 \setminus B_R(0)} |U|_{\mu_1}^2$$

for $U \in H$ and $R > 0$

and

$$\Lambda_{2,\epsilon}(R) = \left\{ \mathcal{U} = (u, v) \in \Phi^{2c_1 + c_2 + \epsilon} \cap M_2 : b_R(U) \geq 8c_1 - \min \left\{ \frac{1}{2m_4}, c_1 \right\} \right\}$$

for each $\epsilon > 0$ and $R > 0$.

**Proposition 1.** For $\epsilon > 0$ sufficiently small, there exists $(R_\epsilon, \delta_\epsilon, \alpha_\epsilon, \gamma_\epsilon) \in (\mathbb{R}^+)^4$ such that

$$\lim_{\epsilon \to 0} \delta_\epsilon = \lim_{\epsilon \to 0} \alpha_\epsilon = \lim_{\epsilon \to 0} \gamma_\epsilon = 0$$

and each $U = (U, V) \in \Lambda_{2,\epsilon}(R_\epsilon)$ has the form

$$U = (\alpha(U_{1,x} + U_{1,-x}) + u, \gamma U_2 + v)$$

where $\alpha \in (1 - \alpha_\epsilon, 1 + \alpha_\epsilon), \gamma \in (1 - \gamma_\epsilon, 1 + \gamma_\epsilon),

$$|x| \geq R_\epsilon, \quad x = B(U|_{B_{R_\epsilon}(x)}), \quad \hat{U}(z) < \frac{1}{2} |\hat{U}|_{\infty} \quad \text{for} \ z \in \mathbb{R}^3 \setminus \bigcup_{i=\pm 1} B_{R_\epsilon}(ix),

$$\hat{V}(z) < \frac{1}{2} |\hat{V}|_{\infty} \quad \text{for} \ z \in \mathbb{R}^3 \setminus B_{R_\epsilon}(0),

and

$$|u|_{\mu_1}^2 + |v|_{\mu_2}^2 \leq \delta_\epsilon.$$

**Remark 4.** By (3.2) and the definition of $B$, one can see that for each $U \in \Lambda_{2,\epsilon}(R_\epsilon), (x, -x) \in R^3 \times R^3$ in (3.1) is uniquely determined, and the mapping $U \in \Lambda_{2,\epsilon}(R_\epsilon) \to (x, -x) \in R^3 \times R^3$ is continuous. We define a continuous mapping $\eta : \Lambda_{2,\epsilon}(R_\epsilon) \to \mathbb{R}^+$ by

$$\eta(U) = |x| \quad \text{for} \ U \in \Lambda_{2,\epsilon}(R_\epsilon).$$

We also need the following Proposition.

**Proposition 2.** There exists $M_0 > 0$ satisfying that for $\epsilon > 0$ sufficiently small,

$$\Phi(U) \geq 2c_1 + c_2 - \beta M_0 e^{-2\mu_4 |x|}$$

for each $U \in \Lambda_{2,\epsilon}(R_\epsilon),

where $x \in R^3$ such that $U$ has the form (3.1).
Now for $x \in R^3 \setminus \{0\}$, we define a class $\Gamma_2(x) \subset C([0,1], M_2)$ by

$$\Gamma_2(x) = \{ p \in C([0,1], M_2) : p(0) = N(U_1, U_2), p(1) = N(U_{1,x} + U_{1,-x}, U_2) \}$$

and put

$$c_2(x) = \inf_{p\in \Gamma_2(x)} \sup_{t\in [0,1]} \Phi(p(t)).$$

We also note that from the definitions of $N$ and $\Phi$, we have that $N(U_{1,x} + U_{1,-x}, U_2) \to 0$ in $H$ as $|x| \to \infty$ and then

$$\lim_{|x| \to \infty} \Phi(N(U_{1,x} + U_{1,-x}, U_2)) = 2I_1(U_1) + I_2(U_2) = 2c_1 + c_2.$$

Based on the preliminary results above, we can prove Theorem 2 for $i = 2$.

**Proof of Theorem 2.** Let $\epsilon \in (0, \delta_2/2)$ sufficiently small. Let $\beta \in (\beta_\epsilon, 0)$. To complete the proof, it is sufficient to show that there exists $\delta > 0$ and $R > 0$ such that

$$2c_1 + c_2 + \delta < c_2(x) < 2c_1 + c_2 + \delta_2/2 \quad \text{for } |x| > R.$$

In fact, if the inequalities above hold, we have by (3.6) that we can choose $x \in R^3$ such that $|x| > R$ and

$$\Phi(N(U_{1,x} + U_{1,-x}, U_2)) < 2c_1 + c_2 + \delta.$$

That is $\Phi(p(1)) < c_2(x)$ for all $p \in \Gamma_2(x).$ We also have $\Phi(p(0)) < \frac{7}{4}c_1 + c_2.$ Then since the Palais-Smale condition holds by (2) of Lemma 1 on $\Phi(2c_1 + c_2 + c_2 + \delta_2/2)$, we have by a standard mountain pass argument that there exists a critical point $U$ of $\Phi$ with $\Phi(U) = c_2(x).$

From the definition of $\epsilon$ and Lemma 2, one can see the pass $p \in \Gamma_2(x)$ defined by

$$p(s) = N(U_{1,sx} + U_{1,-sx}, U_2), \quad s \in [0,1]$$

satisfies $\max_{s \in [0,1]} \Phi(p(s)) \leq 2c_1 + c_2 + \epsilon$. Then the second inequality of (3.7) holds. We now show that the first inequality of (3.7) holds. We first see that there exists $\bar{R} > 2R_e$ such that

$$b_{R_e}(U) \geq 8c_1 - \frac{1}{2} \min \left\{ \frac{1}{2m_4^2}, c_1 \right\} \quad \text{for } U = (U, V) \in \Lambda_{2,\epsilon}(R_e) \text{ with } \eta(U) \geq \bar{R},$$

where $\eta$ is the function defined by (3.5). By Proposition 1, each $U = (U, V) \in \Lambda_{2,\epsilon}(R_e)$ has the form

$$\mathcal{U} = (\alpha(U_{1,x} + U_{1,-x}) + u, \gamma U_2 + v)$$
with $\alpha \in (1-\alpha_{\epsilon}, 1+\alpha_{\epsilon}), \gamma \in (1-\gamma_{\epsilon}, 1+\gamma_{\epsilon})$ and $(u, v) \in \{u_{1,x}, u_{1,-x}\}^\perp \times \{u_{2}\}^\perp$ with $\Vert u \Vert_{\mu_{1}}^{2} \leq \delta_{\epsilon}$ and $\Vert v \Vert_{\mu_{2}}^{2} \leq \delta_{\epsilon}$. Since $\lim_{\epsilon \to 0} \delta_{\epsilon} = \lim_{\epsilon \to 0} \alpha_{\epsilon} = 0$, we may assume that $\epsilon > 0$ is sufficiently small that

\begin{equation}
8\alpha_{\epsilon}^{2}c_{1} - \delta_{\epsilon} > 8c_{1} - \frac{1}{2} \min \left\{ \frac{1}{2m_{4}^{4}}, c_{1} \right\}.
\end{equation}

Then noting that

$$b_{R_{\epsilon}}(U) \geq \alpha^{2} \Vert U_{1,x} + U_{1,-x} \Vert_{\mu_{1}}^{2} - \Vert u \Vert_{\mu_{1}}^{2} - 2 \int_{B_{R_{\epsilon}}(0)} \vert U_{1_{x}} + U_{1,-x} \vert_{\mu_{1}}^{2}$$

and

$$\Vert U_{1,x} + U_{1,-x} \Vert_{\mu_{1}}^{2} \to 8c_{1} \text{ and } \int_{B_{R_{\epsilon}}(0)} \vert U_{1,x} + U_{1,-x} \vert_{\mu_{1}}^{2} \to 0, \text{ as } |x| \to \infty,$$

we find by (3.10) that there exists $\overline{R}$ such that for each $U = (U, V) \in \Lambda_{2,\epsilon}(R_{\epsilon})$ with $\eta(U) \geq \overline{R}, (3.8)$ holds. Now we choose $x \in \mathbb{R}^{3}$ so large that $|x| > \overline{R}$. Then

$$b_{R_{\epsilon}}(N_{1}(U_{1,x} + U_{1,-x})) \geq 8c_{1} - \frac{1}{2} \min \left\{ \frac{1}{2m_{4}^{4}}, c_{1} \right\}.$$ \(\square\)

Let $p = (p_{1}, p_{2}) \in \Gamma_{2}(x)$ such that $\sup_{t \in [0,1]} \Phi(p(t)) \leq 2c_{1} + c_{2} + \epsilon.$ From the definition,

$$\eta(p_{1}(1)) = \eta(N_{1}(U_{1,x} + U_{1,-x})) > \overline{R} \text{ and } b_{R_{\epsilon}}(p_{1}(1)) \geq 8c_{1} - \frac{1}{2} \min \left\{ \frac{1}{2m_{4}^{4}}, c_{1} \right\}.$$ \(\square\)

On the other hand, recalling that $\Phi_{2}(U) \geq c_{2}$, we have that $\Phi_{1}(U) \leq \frac{7}{4}c_{1}$. Then by the definition of $\epsilon$, $b_{R_{\epsilon}}(p_{1}(0)) < 7c_{1} - \frac{1}{2} \min \left\{ \frac{1}{2m_{4}^{4}}, c_{1} \right\}$, there exists $t \in (0,1)$ such that $b_{R_{\epsilon}}(p_{1}(t)) = 8c_{1} - \frac{1}{2} \min \left\{ \frac{1}{2m_{4}^{4}}, c_{1} \right\}.$ Then by (3.8), $\eta(p_{1}(t)) < \overline{R}.$ Therefore by the continuity of $\eta$, we have that there exists $t_{0} \in (0,t)$ such that $\eta(p_{1}(t_{0})) = \overline{R}.$ By Proposition 2, we have

$$\Phi(p(t_{0})) \geq 2c_{1} + c_{2} + \beta M_{0}e^{-2\sqrt{\mu_{2}}\overline{R}}.$$ \(\square\)

Therefore we obtain that $\sup_{t \in [0,1]} \Phi(p(t)) > 2c_{1} + c_{2} + \beta M_{0}e^{-2\sqrt{\mu_{2}}\overline{R}}.$ Thus by the mountain pass theorem (cf. [20]), we find that there exists a critical point $U$ of $\Phi$ with $\Phi(U) = c_{2}(x).$

\section*{References}

12. T. C. Lin and J. Wei, \textit{Ground state of n coupled nonlinear schrodinger equations in} $\mathbb{R}^n$, Communications in Mathematical Physics \textbf{255} (2005), 629–653.
21. B. Sirakov, \textit{Least energy solitary waves for a system of nonlinear schrodinger equations in} $\mathbb{R}^n$, Communications in Mathematical Physics \textbf{271} (2007), 199–221.