Existence and Multiplicity of Solutions for a Coupled Nonlinear Schrödinger System

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1. Introduction

In the present paper, we consider the multiple existence of non-radial positive solutions of coupled Schrödinger system

\[
\begin{align*}
-\Delta u + \mu_1 u &= u^3 + \beta uv^2 & \text{in } \mathbb{R}^3 \\
-\Delta v + \mu_2 v &= v^3 + \beta u^2 v & \text{in } \mathbb{R}^3
\end{align*}
\]

where \(\mu_1, \mu_2 > 0\) and \(\beta \in \mathbb{R}\).

Coupled nonlinear Schrödinger system (P) models many physical problems. In nonlinear optics, the phenomenon in Kerr-like photorefractive media is described by system (P) (cf. [2]). In this case, the solution \(u\) and \(v\) denote the components of the beam in Kerr-like photorefractive media, and the coupling constant \(\beta\) is the interaction between two components \(u\) and \(v\). In case \(\beta > 0\), the interaction is attractive, while the interaction is repulsive if \(\beta < 0\). The bimodal pulse in optical fibers under birefringent effects is also governed by system (P) (cf. [16]). It is also known that system (P) is a model for a mixture of two Bose-Einstein condensates (cf. [7]).

Motivated by these physical interest, the existence of solutions of (P) has been investigated by several authors. In the case that \(\beta > 0\), problem (P) was studied by Ambrosetti & Colorado[3], Maia, Montefusco & Pellacci[15] and Lin & Wei[12]. They proved the existence of least energy solutions \((u, v)\) of (P) with \(u, v > 0\). By the result of Troy (cf. [22]), we know that in this case, all positive solutions \((u, v)\) of (P) satisfying

\[(1.1) \quad u(x) \to 0, v(x) \to 0, \quad \text{as } |x| \to \infty\]

are radially symmetric functions. On the other hand, in case that \(\beta < 0\), it is known that there is no least energy solution of (P) (cf. [12]). Moreover, positive solutions of (P) satisfying (1.1) is not always radial. In case that \(\beta < 0\) and \(|\beta|\) is small, there are positive solutions with one component concentrating on the origin and the other component concentrating around a regular polygon (cf. [14]). The existence of non-radial positive solutions was also considered in [24] for the case that \(\beta < 0\) and \(\mu_1 = \mu_2\). In this case, there are infinitely many nonradial solutions if \(\beta < -1\). Recently, Sirakov[21]
established the existence of ground state solutions of (P) in the case that coefficients of nonlinear terms $u^3$ and $v^3$ in (P) are different.

On the other hand, the existence of sign changing solutions of nonlinear scalar elliptic problem

\[(1.2) \quad -\Delta u + u = |u|^{p-1} u, \quad u \in H^1_0(\Omega)\]

has been investigated by many authors in the last decade. Here $\Omega$ is a domain in $\mathbb{R}^N (N \geq 3)$, and $p \in (1, (N + 2)/(N - 2)]$. We refer to [5], [6] and [17] for related results of sign changing solutions of (1.2). The existence of sign changing solutions of (P) with $\beta > 0$ was considered in [10]. For the problem (P) with $\mathbb{R}^3$ replaced by a bounded domain, we refer to [13] and [19].

In the present paper, we first see the multiple existence of solutions of (P) in the case that $\beta > 0$. Next we consider the case that $\beta < 0$ and $|\beta|$ is small, i.e. the case that the interaction of two solutions are small and repulsive. We will show the multiple existence of nonradial solutions of (P) in this case with $\mu_1 \neq \mu_2$. Our results improve the results in [14]. (See Remark 2 and Remark 3).

To state our main results, we need some notations. We denote by $B_r(x)$ the open ball in $\mathbb{R}^3$ centered at $x \in \mathbb{R}^3$ with radius $r > 0$. The inner product in $\mathbb{R}^3$ is denoted by $\langle \cdot , \cdot \rangle_{\mathbb{R}^3}$. We put $H = H^1(\mathbb{R}^3)$ and $H = H \times H$. We set $\mu_0 = 1$. We denote by $\|\cdot\|_{\mu_i}$ the norm of $H$ defined by $\|u\|_{\mu_i}^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + \mu_i |u|^2) \, dx$ for $u \in H$ and $i \in \{0, 1, 2\}$. For simplicity of notations, we put $\|u(x)\|_{\mu_i}^2 = |\nabla u(x)|^2 + \mu_i |u(x)|^2$ for $u \in H$ and $x \in \mathbb{R}^3$. For each function $u \in H$, we set $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \max\{-u(x), 0\}$. For each $p \geq 1$, we denote by $|\cdot|_p$ the norm of the space $L^p(\mathbb{R}^3)$. The Hilbert space $H$ is equipped with the norm defined by $\|U\|^2 = \|u\|_{\mu_1}^2 + \|v\|_{\mu_2}^2$ for $U = (u, v) \in H$. We recall that for each $i \in \{0, 1, 2\}$, problem

\[
(P_i) \quad \begin{cases} 
-\Delta u + \mu_i u = u^3 & \text{in } \mathbb{R}^3 \\
u(x) > 0 & \text{in } \mathbb{R}^3 \\
u(x) \to 0 & \text{as } |x| \to \infty 
\end{cases}
\]

has a radial solution, denoted by $U_i$ (cf. [8], [11]). The function $U_i$ is the unique smooth solution of $(P_i)$ up to translation. Moreover we know that $U_i$ satisfies $I_i(U_i) = c_i = \min\{I(v) : v \in S_i\}$, where $I_i$ is the functional associated with problem $(P_i)$ defined by

\[(1.3) \quad I_i(v) = \frac{1}{2} \|v\|_{\mu_i}^2 - \frac{1}{4} |v^+|^4 \quad \text{for } v \in H\]

and $S_i$ is the set defined by

\[S_i = \{ v \in H : \|v\|_{\mu_i}^2 = |v^+|^4 \} \quad \text{for } i = 1, 2.\]
For each $x \in \mathbb{R}^3$ and $i \in \{0, 1, 2\}$, we put $U_{i, x} (\cdot) = U_{i} (\cdot - x)$. It is also known that
\begin{equation}
|U_{i} (x)|_{\mu_{i}} |x| \exp (\sqrt{\mu_{i}} |x|) \rightarrow c > 0, \text{ as } |x| \rightarrow \infty \quad \text{for } i \in \{0, 1, 2\}.
\end{equation}
(cf. [11]). For each $u \in L^{4}(R^3)$, we put $\hat{u}(x) = \int_{B_{1}(x)} |u(x)|^4 \, dx$ for $x \in \mathbb{R}^3$. Then from (1.4), we can choose $R_{0} > 0$ such that
\begin{equation}
\hat{U}_{i}(z) < \frac{1}{3} |\hat{U}_{i}|_{\infty} \quad \text{for all } z \in \mathbb{R}^3 \setminus B_{R_{0}}(0) \text{ and } i \in \{1, 2\}.
\end{equation}
Since we consider the case that $\mu_{1} \neq \mu_{2}$, we may assume without any loss of generality that $\mu_{2} < \mu_{1}$. We can now state our main results.

**Theorem 1.** Suppose that the following condition holds:

\[ 0 < 2\sqrt{\mu_{2}} < \sqrt{\mu_{1}}. \]

Then there exists $\beta_{0} > 0$ such that for each $\beta \in (0, \beta_{0})$, problem $(P)$ possesses at least one ground state solution $U_{0} \in H^{1}(\mathbb{R}^3) \times H^{1}(\mathbb{R}^3)$ and one nonradial sign changing solution $U \in H^{1}(\mathbb{R}^3) \times H^{1}(\mathbb{R}^3)$.

**Theorem 2.** Suppose that $\sqrt{\mu_{1}/\mu_{2}}$ is irrational. Then for each $i \in \{2, 4, 6, 8, 12, 20\}$, there exists $\beta_{i} \in (-1, 0)$ such that for each $\beta \in (\beta_{i}, 0)$, there exists a positive solution $U_{i} \in H^{1}(\mathbb{R}^3) \times H^{1}(\mathbb{R}^3)$ such that $i c_{1} + c_{2} < \Phi(U_{i}) < i c_{1} + 2c_{2}$ and $U_{i}$ has the form

\begin{equation}
U_{i} = (U, V) = \left( \sum_{j=1}^{i} U_{1, x_{j}} + u, U_{2} + v \right)
\end{equation}

where $\{x_{1}, x_{2}, \ldots, x_{i}\}$ forms a regular $i$-polyhedra in $\mathbb{R}^3$ in case $i \neq 2$ and $x_{1} = -x_{2}$ in case that $i = 2$, and $u, v \in H$ such that $
\Vert(u, v)\Vert$ is so small that
\begin{align}
\hat{U}(z) < \frac{1}{2} |\hat{U}|_{\infty} \quad &\text{for } z \in \mathbb{R}^3 \setminus (\bigcup_{j=1}^{i} B_{R_{0}}(x_{j})) \\
\hat{V}(z) < \frac{1}{2} |\hat{V}|_{\infty} \quad &\text{for } z \in \mathbb{R}^3 \setminus B_{R_{0}}(0).
\end{align}

**Remark 1.** The expression (1.6) of $U_{i} = (U, V)$ is unique when $
\Vert(u, v)\Vert$ is so small that condition (1.7) holds, i.e., for each $U_{i}, (x_{1}, x_{2}, \ldots, x_{i})$ is uniquely determined.

**Remark 2.** The assertion of Theorem 1 implies that for $\beta < 0$ with $|\beta|$ sufficiently small, problem $(P)$ possesses at least 6 nonradial positive solutions. In [14], the existence of positive solutions of $(P)$ of the form (1.6) was established in the case that $\{x_{1}, x_{2}, \ldots, x_{i}\}$ forms regular cube or tetrahedra under the assumption

\[ \sqrt{\frac{\mu_{1}}{\mu_{2}}} \leq \begin{cases} \frac{\sqrt{3}}{3} & \text{for the cube} \\
\frac{\sqrt{8}}{3} & \text{for the tetrahedra}. \end{cases} \]
Our argument employed in this paper does not require the ratio of \( \sqrt{\mu_1} \) and \( \sqrt{\mu_2} \).

**Theorem 3.** Suppose that \( \sqrt{\mu_1/\mu_2} \) is irrational. Then for each \( k \in \mathbb{N} \), there exists \( \beta_k \in (-1,0) \) such that for each \( \beta \in (\beta_k,0) \), the problem (P) has a positive solution \( \tilde{U}_k \) such that \( k \beta_1 + c_2 < \Phi(\tilde{U}_k) < k \beta_1 + 2c_2 \) and \( \tilde{U}_k \) has the form

\[
\tilde{U}_k = (U,V) = \left( \sum_{j=1}^{k} U_{1j} + u, U_2 + v \right) 
\]

where \( \{x_1,x_2, \ldots, x_k\} \subset \mathbb{R}^3 \) form a regular \( k \)-polygon in a two-dimensional subspace of \( \mathbb{R}^3 \) and \( u, v \in H \) such that \( \|(u,v)\| \) is so small that

\[
\tilde{U}(z) < \frac{1}{2} \left\| \tilde{U} \right\|_{\infty} \text{ for } z \in \mathbb{R}^3 \backslash (\bigcup_{j=1}^{k} B_{R_0}(x_j)) \text{ and } \tilde{V}(z) < \frac{1}{2} \left\| \tilde{V} \right\|_{\infty} \text{ for } z \in \mathbb{R}^3 \backslash B_{R_0}(0). 
\]

**Remark 3.** The existence of positive solutions of (P) of the form (1.8) was proved in [14] in the case that the spacial dimension is 2 and \( \mu_1, \mu_2 \) satisfy

\[
\sqrt{\frac{\mu_1}{\mu_2}} < \sin \frac{\pi}{k}. \]

We give a sketch of the proof of Theorem 2 for the case \( i = 2 \). The proofs of Theorem 2 for \( i \neq 2 \) and the proof of Theorem 3 are slight modifications of that of the case \( i = 2 \) of Theorem 2. The detail of the proofs can be found in [9].

2. Preliminaries

Throughout the rest of this paper, we assume that \( \sqrt{\mu_1/\mu_2} \) is irrational.

For each \( u, v \in H = H^1(\mathbb{R}^3) \), we put \( \langle u, v \rangle = \int_{\mathbb{R}^3} uv \). We denote by \( \langle \cdot, \cdot \rangle_{\mu_i} \) the inner product of \( H \) defined by \( \langle u, v \rangle_{\mu_i} = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + \mu_i uv) \) for \( u, v \in H \) and \( i \in \{0,1,2\} \). The inner product of \( H \) is defined by \( \langle U_1, U_2 \rangle_H = \langle U_1, U_2 \rangle_{\mu_1} + \langle V_1, V_2 \rangle_{\mu_2} \) for \( U_1 = (U_1, V_1), U_2 = (U_2, V_2) \in H \). For each \( u \in L^4(\mathbb{R}^3) \), we put \( \Omega(u) = \{ x \in \mathbb{R}^3 : \tilde{u}(x) \geq \frac{1}{2} |\tilde{u}|_{\infty} \} \) and

\[
B(u) = \frac{\int_{\Omega(u)} \tilde{x}(\tilde{u}(x) - \frac{1}{2} |\tilde{u}|_{\infty}) dx}{\int_{\Omega(u)} (\tilde{u}(x) - \frac{1}{2} |\tilde{u}|_{\infty}) dx}.
\]

The mapping \( B \) is called generalized barycenter, which is introduced in [18](cf. also [4]). By Sobolev's embedding theorem([1]), for \( p \in [2,6] \) there exists \( m_p > 0 \) such that

\[
|z|_{L^p(B_r(0))} \leq m_p \left( |\nabla z|_{L^2(B_r(0))}^2 + \mu_i |z|_{L^2(B_r(0))}^2 \right)^{1/2}.
\]
for \( r > 1, z \in H^1(B_r(0))\), and \( i \in \{0, 1, 2\}\). For each \( a \in R\), and a functional \( F: H \rightarrow R\), we denote by \( F^a\) the level set defined by \( F^a = \{ v \in H: F(v) \leq a \}\). The same notation is used for functionals defined on \( H\).

It is easy to see that for each \( u \in H \setminus \{0\}\) with \( u^+ \neq 0\), there exists a unique positive number \( s\) such that \( su \in S_i\) (cf. [25]). It follows from the definitions of \( U_i\) that \( U_i(x) = \sqrt{\mu}U_0(\sqrt{\mu}x)\) on \( R^3\). Then one can see that

\[ c_1 = \sqrt{\mu_1}c_0 > c_2 = \sqrt{\mu_2}c_0. \]

Let \( i \in \{0, 1, 2\}\). It is known that \( \{U_{i,x}: x \in R^3\}\) is a nondegenerate critical set of \( I_i\) (cf. [23]). More precisely, we have there exists \( \lambda > 0\) such that

\[ \|u\|_{\mu}^2 - 3 \langle U^2 u, u \rangle \geq \lambda \|u\|_{\mu}^4 \quad \text{for all } u \in U_i, \partial U_i, \partial U_i. \]

We define a functional \( \Phi: H \rightarrow R\) associated with problem (P) by

\[
\Phi(U) = \frac{1}{2}\left(\|U\|_{\mu_1}^2 + \|V\|_{\mu_2}^2\right) - \frac{1}{4}\left(|U|^4 + |V|^4\right) - \frac{\beta}{2} \int_{R^3} (U^+)^2(V^+)^2
= \Phi_1(U) + \Phi_2(U) \quad \text{for } U = (U, V) \in H,
\]

where

\[
\Phi_1(U) = \frac{1}{2}\|U\|_{\mu_1}^2 - \frac{1}{4}|U|^4 - \frac{\beta}{4} \int_{R^3} (U^+)^2(V^+)^2
\]

and

\[
\Phi_2(U) = \frac{1}{2}\|V\|_{\mu_2}^2 - \frac{1}{4}|V|^4 - \frac{\beta}{4} \int_{R^3} (U^+)^2(V^+)^2.
\]

Then a direct computation shows

\[
\langle \nabla \Phi(U), V \rangle_H = \left\langle \begin{pmatrix} \nabla_u \Phi(U) \\ \nabla_v \Phi(U) \end{pmatrix}, \begin{pmatrix} W \\ Z \end{pmatrix} \right\rangle_H
= \langle -\Delta U + \mu_1 U - (U^+)^3 - \beta U^+V^2, W \rangle
+ \langle -\Delta V + \mu_2 V - (V^+)^3 - \beta U^2V^+, Z \rangle
\]

for \( U = (U, V), V = (W, Z) \in H\). We put

\[
M_+ = \{ (U, V) \in H \setminus \{0\} : \|U\|_{\mu_1}^2 = |U|^4 + \beta \int_{R^3} (U^+)^2(V^+)^2, \|V\|_{\mu_2}^2 = |V|^4 + \beta \int_{R^3} (U^+)^2(V^+)^2 \}.
\]

Then one can see that \( U = (U, V) \in M_+\) is a critical point of \( \Phi \) if and only if \( U \) is a positive solution of problem (P). From the definition, we have

\[ \Phi_1(U) = \frac{1}{4}\|U\|_{\mu_1}^2, \Phi_2(U) = \frac{1}{4}\|V\|_{\mu_2}^2 \quad \text{and } \Phi(U) = \frac{1}{4}(\|U\|_{\mu_1}^2 + \|V\|_{\mu_2}^2)
\]

for \( U = (U, V) \in M_+\). We also have that for each \( U = (U, V) \in M_+\), there exists \( (s, t) \in R^+ \times R^+\) such that \( (su, tv) \in M_+\). In fact, for each \( U = (U, V), U \neq
0, V \not\equiv 0, (sU, tV) \in M_+ if and only if
\[
\left( \begin{array}{c} s^2 \\ t^2 \end{array} \right) = A^{-1} \left( \begin{array}{c} \|U\|_{\mu_1}^2 \\ \|V\|_{\mu_2}^2 \end{array} \right)
\]
where
\[
A = \left( \begin{array}{cc} |U^+|_4^4 & \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 |V^+|_4^4 \\ \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 & |V^+|_4^4 \end{array} \right).
\]
Since \( \beta \in (-1, 0) \), we have by the Schwartz's inequality that \( A^{-1} \) exists and then there exists a unique solution \((s, t) \in R^+ \times R^+\). For given \( U = (U, V) \in H \) with \( U \not\equiv 0, V \not\equiv 0 \), we put \( NU = N(U, V) = (N_1 U, N_2 V) = (sU, tV) \in M_+ \).

Now to prove Theorem 1 for the case that \( i = 2 \), we define \( H_2 \subset H, H_2 \subset H \) and \( M_2 \subset M_+ \) by
\[
H_2 = \{ u \in H : u(x) = u(-x) \quad \text{for} \quad x \in \mathbb{R}^3 \}, \quad \mathbb{H}_2 = H_2 \times H_2,
\]
and
\[
\mathcal{M}_2 = \mathcal{M}_+ \cap \mathbb{H}_2.
\]
Since \( \sqrt{\mu_1/\mu_2} \) is irrational, we can choose \( \delta_2 \in (0, c_2) \) so small that
\[
(2.4) \quad c_1 + kc_2 \notin [2c_1 + c_2 - \delta_2, 2c_1 + c_2 + \delta_2] \quad \text{for all} \quad k \in \mathbb{N}.
\]

The following Lemmata are crucial for our argument.

**Lemma 1.** (1) There exists \( \beta_1 \in (-1, 0) \) such that for each \( \beta \in (\beta_1, 0) \) and each critical point \( U \in M_2 \cap \Phi^{2c_1+2c_2} \) of \( \Phi \),
\[
\Phi(U) \in \bigcup_{i \geq 1, j \geq 1} [ic_1 + jc_2 - \delta_2/2, ic_1 + jc_2 + \delta_2/2].
\]
(2) Let \( \beta \in (\beta_1, 0) \) and \( \{U_n\} \subset M_2 \) such that \( \lim_{n \to \infty} \nabla \Phi(U_n) = 0 \) and \( \lim_{n \to \infty} \Phi(U_n) = 2c_1 + c_2 + \epsilon \) with \( \epsilon \in (0, \delta_2/2) \). Then there exists a convergence subsequence \( \{U_{n_i}\} \subset \{U_n\} \).

**Lemma 2.** For given \( \epsilon > 0 \), there exists \( \beta_\epsilon \in (\beta_1, 0) \) such that for each \( \beta \in (\beta_\epsilon, 0) \) and for each \( x \in \mathbb{R}^3 \setminus \{0\} \),
\[
(2.5) \quad \Phi(N(U_{1,x} + U_{1,-x}, U_2)) < 2c_1 + c_2 + \epsilon
\]
3. Sketch of the Proof of Theorem 2 for $i = 2$. Throughout this section we assume that $\beta \in (\beta_1, 0)$. We put

$$b_R(U) = \int_{\mathbb{R}^3 \setminus B_R(0)} |U|_{\mu_1}^2$$

for $U \in H$ and $R > 0$ and

$$\Lambda_{2, \epsilon}(R) = \{ \mathcal{U} = (u, v) \in \Phi^{2c_1 + c_2 + \epsilon} \cap \mathcal{M}_2 : b_R(U) \geq 8c_1 - \min \left\{ \frac{1}{2m_4}, c_1 \right\} \}$$

for each $\epsilon > 0$ and $R > 0$.

**Proposition 1.** For $\epsilon > 0$ sufficiently small, there exists $(R_\epsilon, \delta_\epsilon, \alpha_\epsilon, \gamma_\epsilon) \in (\mathbb{R}^+)^4$ such that

$$\lim_{\epsilon \to 0} \delta_\epsilon = \lim_{\epsilon \to 0} \alpha_\epsilon = \lim_{\epsilon \to 0} \gamma_\epsilon = 0$$

and each $U = (U, V) \in \Lambda_{2, \epsilon}(R_\epsilon)$ has the form

$$U = (\alpha(U_{1,x} + U_{1,-x}) + u, \gamma U_2 + v)$$

where $\alpha \in (1 - \alpha_\epsilon, 1 + \alpha_\epsilon), \gamma \in (1 - \gamma_\epsilon, 1 + \gamma_\epsilon)$,

$$|x| \geq R_\epsilon, \quad x = B(U|_{B_{R_0}(x)}), \quad \hat{U}(z) < \frac{1}{2} |\hat{U}|_{\infty} \quad \text{for } z \in \mathbb{R}^3 \setminus \bigcup_{i=\pm 1} B_{R_0}(ix),$$

$$\hat{V}(z) < \frac{1}{2} |\hat{V}|_{\infty} \quad \text{for } z \in \mathbb{R}^3 \setminus B_{R_0}(0),$$

and

$$(u, v) \in \{U_{1,x}, U_{1,-x}\}^\perp \times \{U_2\}^\perp \text{ with } \|u\|_{\mu_1}^2 + \|v\|_{\mu_2}^2 \leq \delta_\epsilon.$$
Now for $x \in R^3 \setminus \{0\}$, we define a class $\Gamma_2(x) \subset C([0,1], M_2)$ by

$$\Gamma_2(x) = \{p \in C([0,1], M_2) : p(0) = N(U_1, U_2), p(1) = N(U_1 x + U_1, -x, U_2)\}$$

and put

$$c_2(x) = \inf_{p \in \Gamma_2(x)} \sup_{t \in [0,1]} \Phi(p(t)).$$

We also note that from the definitions of $N$ and $\Phi$, we have that $N(U_1 x + U_1, -x, U_2) - (U_1, x + U_1, -x, U_2) \to 0$ in $H$ as $|x| \to \infty$ and then

$$\lim_{|x| \to \infty} \Phi(N(U_1, x + U_1, -x, U_2)) = 2I_1(U_1) + I_2(U_2) = 2c_1 + c_2.$$

Based on the preliminary results above, we can prove Theorem 2 for $i = 2$.

**Proof of Theorem 2.** Let $\epsilon \in (0, \delta_2/2)$ sufficiently small. Let $\beta \in (\beta_\epsilon, 0)$. To complete the proof, it is sufficient to show that there exists $\delta > 0$ and $R > 0$ such that

$$2c_1 + c_2 + \delta < c_2(x) < 2c_1 + c_2 + \delta_2/2$$

for $|x| > R$.

In fact, if the inequalities above hold, we have by (3.6) that we can choose $x \in R^3$ such that $|x| > R$ and

$$\Phi(N(U_1, x + U_1, -x, U_2)) < 2c_1 + c_2 + \delta.$$

That is $\Phi(p(1)) < c_2(x)$ for all $p \in \Gamma_2(x)$. We also have $\Phi(p(0)) < \frac{1}{2} c_1 + c_2$. Then since the Palais-Smale condition holds by (2) of Lemma 1 on $\Phi(2c_1 + \delta)$, we have by a standard mountain pass argument that there exists a critical point $U$ of $\Phi$ with $\Phi(U) = c_2(x)$.

From the definition of $\epsilon$ and Lemma 2, one can see the pass $p \in \Gamma_2(x)$ defined by

$$p(s) = N(U_1, x + U_1, -x, U_2), \quad s \in [0,1]$$

satisfies $\max_{s \in [0,1]} \Phi(p(s)) \leq 2c_1 + c_2 + \epsilon$. Then the second inequality of (3.7) holds. We now show that the first inequality of (3.7) holds. We first see that there exists $\bar{R} > 2R$ such that

$$b_{R_\epsilon}(U) \geq 8c_1 - \frac{1}{2} \min \left\{ \frac{1}{2m_4}, c_1 \right\} \quad \text{for} \ U = (U,V) \in \Lambda_{2,\epsilon}(R_\epsilon) \text{ with } \eta(U) \geq \bar{R},$$

where $\eta$ is the function defined by (3.5). By Proposition 1, each $U = (U,V) \in \Lambda_{2,\epsilon}(R_\epsilon)$ has the form

$$U = (\alpha(U_1, x + U_1, -x) + u, \gamma U_2 + v)$$

satisfies $\max_{s \in [0,1]} \Phi(p(s)) \leq 2c_1 + c_2 + \epsilon$. Then the second inequality of (3.7) holds. We now show that the first inequality of (3.7) holds. We first see that there exists $\bar{R} > 2R$ such that

$$b_{R_\epsilon}(U) \geq 8c_1 - \frac{1}{2} \min \left\{ \frac{1}{2m_4}, c_1 \right\} \quad \text{for} \ U = (U,V) \in \Lambda_{2,\epsilon}(R_\epsilon) \text{ with } \eta(U) \geq \bar{R},$$

where $\eta$ is the function defined by (3.5). By Proposition 1, each $U = (U,V) \in \Lambda_{2,\epsilon}(R_\epsilon)$ has the form

$$U = (\alpha(U_1, x + U_1, -x) + u, \gamma U_2 + v)$$
with \( \alpha \in (1-\alpha_{\epsilon}, 1+\alpha_{\epsilon}), \gamma \in (1-\gamma_{\epsilon}, 1+\gamma_{\epsilon}) \) and \( (u, v) \in \{U_{1,x}, U_{1,-x}\}^\perp \times \{U_{2}\}^\perp \) with \( \|u\|_{\mu_{1}}^{2} + \|v\|_{\mu_{2}}^{2} \leq \delta_{\epsilon} \). Since \( \lim_{\epsilon \to 0} \delta_{\epsilon} = \lim_{\epsilon \to 0} \alpha_{\epsilon} = 0 \), we may assume that \( \epsilon > 0 \) is sufficiently small that

\[
\alpha_{\epsilon}^{2} c_{1} - \delta_{\epsilon} > 8c_{1} - \frac{1}{2} \min\left\{ \frac{1}{2m_{4}^{4}}, c_{1} \right\}.
\]

Then noting that

\[
b_{R_{\epsilon}}(U) \geq \frac{\alpha^{2} \|U_{1,x} + U_{1,-x}\|_{\mu_{1}}^{2}}{2m_{4}^{4}} - \|u\|_{\mu_{1}}^{2} - 2 \int_{B_{R_{\epsilon}}(0)} |U_{1,x} + U_{1,-x}|_{\mu_{1}}^{2},
\]

we find by (3.10) that there exists \( R \) such that for each \( U = (U, V) \in \Lambda_{2,\epsilon}(R_{\epsilon}) \) with \( \eta(U) \geq \bar{R} \), (3.8) holds. Now we choose \( x \in R^{3} \) so large that \( |x| > \bar{R} \).

Let \( p = (p_{1}, p_{2}) \in \Gamma_{2}(x) \) such that \( \sup_{t \in [0,1]} \Phi(p(t)) \leq 2c_{1} + c_{2} + \epsilon \).

From the definition,

\[
\eta(p_{1}(1)) = \eta(\mathcal{N}(U_{1,x} + U_{1,-x})) > \bar{R} \text{ and } b_{R_{\epsilon}}(p_{1}(1)) \geq 8c_{1} - \frac{1}{2} \min\left\{ \frac{1}{2m_{4}^{4}}, c_{1} \right\}.
\]

On the other hand, recalling that \( \Phi_{2}(U) \geq c_{2} \), we have that \( \Phi_{1}(U) \leq \frac{7}{4} c_{1} \). Then by the definition of \( \epsilon, b_{R_{\epsilon}}(p_{1}(1)) < 7c_{1} - \min\left\{ \frac{1}{2m_{4}^{4}}, c_{1} \right\} \), there exists \( t \in (0, 1) \) such that \( b_{R_{\epsilon}}(p_{1}(t)) = 8c_{1} - \min\left\{ \frac{1}{2m_{4}^{4}}, c_{1} \right\} \). Then by (3.8), \( \eta(p_{1}(t)) < \bar{R} \). Therefore by the continuity of \( \eta \), we have that there exists \( t_{0} \in (0, t) \) such that \( \eta(p_{1}(t_{0})) = \bar{R} \). By Proposition 2, we have

\[
\Phi(p(t_{0})) \geq 2c_{1} + c_{2} + \beta M_{0} e^{-2\sqrt{\mu_{2}} \bar{R}}.
\]

Therefore we obtain that \( \sup_{t \in [0,1]} \Phi(p(t)) > 2c_{1} + c_{2} + \beta M_{0} e^{-2\sqrt{\mu_{2}} \bar{R}} \). Thus by the mountain pass theorem (cf. [20]), we find that there exists a critical point \( U \) of \( \Phi \) with \( \Phi(U) = c_{2}(x) \). 

\[
\square
\]

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