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Existence and Multiplicity of Solutions for a Coupled Nonlinear Schrödinger System (Nonlinear Analysis and Convex Analysis)

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Existence and Multiplicity of Solutions for a Coupled Nonlinear Shr"{o}dinger System

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1. Introduction

In the present paper, we consider the multiple existence of non-radial positive solutions of coupled Schrödinger system

\[
\text{(P)} \quad \begin{cases} 
-\Delta u + \mu_1 u = u^3 + \beta uv^2 & \text{in } \mathbb{R}^3 \\
-\Delta v + \mu_2 v = v^3 + \beta u^2 v & \text{in } \mathbb{R}^3 
\end{cases}
\]

where $\mu_1, \mu_2 > 0$ and $\beta \in \mathbb{R}$.

Coupled nonlinear Schrödinger system (P) models many physical problems. In nonlinear optics, the phenomenon in Kerr-like photorefractive media is described by system (P) (cf. [2]). In this case, the solution $u$ and $v$ denote the components of the beam in Kerr-like photorefractive media, and the coupling constant $\beta$ is the interaction between two components $u$ and $v$. In case $\beta > 0$, the interaction is attractive, while the interaction is repulsive if $\beta < 0$. The bimodal pulse in optical fibers under birefringent effects is also governed by system (P) (cf. [16]). It is also known that system (P) is a model for a mixture of two Bose-Einstein condensates (cf. [7]).

Motivated by these physical interest, the existence of solutions of (P) has been investigated by several authors. In the case that $\beta > 0$, problem (P) was studied by Ambrosetti & Colorado [3], Maia, Montefusco & Pellacci [15] and Lin & Wei [12]. They proved the existence of least energy solutions $(u,v)$ of (P) with $u,v > 0$. By the result of Troy (cf. [22]), we know that in this case, all positive solutions $(u,v)$ of (P) satisfying

\[
(1.1) \quad u(x) \to 0, v(x) \to 0, \quad \text{as } |x| \to \infty
\]

are radially symmetric functions. On the other hand, in case that $\beta < 0$, it is known that there is no least energy solution of (P) (cf. [12]). Moreover, positive solutions of (P) satisfying (1.1) is not always radial. In case that $\beta < 0$ and $|\beta|$ is small, there are positive solutions with one component concentrating on the origin and the other component concentrating around a regular polygon (cf. [14]). The existence of non-radial positive solutions was also considered in [24] for the case that $\beta < 0$ and $\mu_1 = \mu_2$. In this case, there are infinitely many nonradial solutions if $\beta < -1$. Recently, Sirakov [21]...
established the existence of ground state solutions of (P) in the case that coefficients of nonlinear terms $u^3$ and $v^3$ in (P) are different.

On the other hand, the existence of sign changing solutions of nonlinear scalar elliptic problem

$$-\Delta u + u = |u|^{p-1} u, \quad u \in H^1_0(\Omega)$$

has been investigated by many authors in the last decade. Here $\Omega$ is a domain in $\mathbb{R}^N (N \geq 3)$, and $p \in (1, (N+2)/(N-2)]$. We refer to [5], [6] and [17] for related results of sign changing solutions of (1.2). The existence of sign changing solutions of (P) with $\beta > 0$ was considered in [10]. For the problem (P) with $\mathbb{R}^3$ replaced by a bounded domain, we refer to [13] and [19].

In the present paper, we first see the multiple existence of solutions of problem (P) in the case that $\beta > 0$. Next we consider the case that $\beta < 0$ and $|\beta|$ is small, i.e. the case that the interaction of two solutions are small and repulsive. We will show the multiple existence of nonradial solutions of (P) in this case with $\mu_1 \neq \mu_2$. Our results improve the results in [14]. (See Remark 2 and Remark 3).

To state our main results, we need some notations. We denote by $B_r(x)$ the open ball in $\mathbb{R}^3$ centered at $x \in \mathbb{R}^3$ with radius $r > 0$. The inner product in $\mathbb{R}^3$ is denoted by $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$. We put $H = H^1(\mathbb{R}^3)$ and $H = H \times H$. We set $\mu_0 = 1$. We denote by $\| \cdot \|_{\mu_i}$ the norm of $H$ defined by $\| u \|_{\mu_i}^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + \mu_i |u|^2) dx$ for $u \in H$ and $i \in \{0, 1, 2\}$. For simplicity of notations, we put $|u(x)|_{\mu_i}^2 = |\nabla u(x)|^2 + \mu_i |u(x)|^2$ for $u \in H$ and $x \in \mathbb{R}^3$. For each function $u \in H$, we set $u^+(x) = \max \{u(x), 0\}$, $u^-(x) = \max \{-u(x), 0\}$. For each $p \geq 1$, we denote by $\| \cdot \|_p$ the norm of the space $L^p(\mathbb{R}^3)$. The Hilbert space $H$ is equipped with the norm defined by $\| u \|^2 = \| u \|_{\mu_1}^2 + \| u \|_{\mu_2}^2$ for $U = (u, v) \in H$. We recall that for each $i \in \{0, 1, 2\}$, problem

$$\begin{cases}
-\Delta u + \mu_i u = u^3 & \text{in } \mathbb{R}^3 \\
u(x) > 0 & \text{in } \mathbb{R}^3 \\
u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty
\end{cases}$$

(P_i)

has a radial solution, denoted by $U_i$ (cf. [8], [11]). The function $U_i$ is the unique smooth solution of (P_i) up to translation. Moreover we know that $U_i$ satisfies $I_i(U_i) = c_i = \min \{ I(v) : v \in S_i \}$, where $I_i$ is the functional associated with problem (P_i) defined by

$$I_i(v) = \frac{1}{2} \| v \|_{\mu_i}^2 - \frac{1}{4} |v^+|^4_4$$

for $v \in H$ and $S_i$ is the set defined by

$$S_i = \{ v \in H : \| v \|_{\mu_i}^2 = |v^+|^4_4 \} \quad \text{for } i = 1, 2.$$
For each \( x \in \mathbb{R}^3 \) and \( i \in \{0, 1, 2\}, \) we put \( U_{i,x} (\cdot) = U_i (\cdot - x). \) It is also known that
\[
|U_i (x)|_{\mu_i} |x| \exp (\sqrt{\mu_i} |x|) \to c > 0, \quad \text{as } |x| \to \infty \quad \text{for } i \in \{0, 1, 2\}.
\]
(cf. [11]). For each \( u \in L^4 (\mathbb{R}^3), \) we put \( \hat{u} (x) = \int_{B_1 (x)} |u(x)|^4 \, dx \) for \( x \in \mathbb{R}^3. \) Then from (1.4), we can choose \( R_0 > 0 \) such that
\[
\hat{U}_i (z) < \frac{1}{3} |\hat{U}|_{\infty} \quad \text{for all } z \in \mathbb{R}^3 \backslash B_{R_0} (0) \text{ and } i \in \{1, 2\}.
\]
Since we consider the case that \( \mu_1 \neq \mu_2, \) we may assume without any loss of generality that \( \mu_2 < \mu_1. \) We can now state our main results.

**Theorem 1.** Suppose that the following condition holds:
\[
0 < 2 \sqrt{\mu_2} < \sqrt{\mu_1}.
\]
Then there exists \( \beta_0 > 0 \) such that for each \( \beta \in (0, \beta_0), \) problem (P) possesses at least one ground state solution \( U_0 \in H^1 (\mathbb{R}^3) \times H^1 (\mathbb{R}^3) \) and one nonradial sign changing solution \( U \in H^1 (\mathbb{R}^3) \times H^1 (\mathbb{R}^3). \)

**Theorem 2.** Suppose that \( \sqrt{\mu_1 / \mu_2} \) is irrational. Then for each \( i \in \{2, 4, 6, 8, 12, 20\}, \) there exists \( \beta_i \in (-1, 0) \) such that for each \( \beta \in (\beta_i, 0), \) there exists a positive solution \( U_i \in H \) of (P) such that \( ic_1 + c_2 < \Phi (U_i) < ic_1 + 2c_2 \) and \( U_i \) has the form
\[
U_i = (U, V) = \left( \sum_{j=1}^{i} U_{1, x_j} + u, U_2 + v \right)
\]
where \( \{x_1, x_2, \ldots, x_i\} \) forms a regular \( i \)-polyhedra in \( \mathbb{R}^3 \) in case \( i \neq 2 \) and \( x_1 = -x_2 \) in case that \( i = 2, \) and \( u, v \in H \) such that \( \| (u, v) \| \) is so small that
\[
\hat{U} (z) < \frac{1}{2} \left| \hat{U} \right|_{\infty} \quad \text{for } z \in \mathbb{R}^3 \backslash (\bigcup_{j=1}^{i} B_{R_0} (x_j)) \quad \text{and} \quad \hat{V} (z) < \frac{1}{2} \left| \hat{V} \right|_{\infty} \quad \text{for } z \in \mathbb{R}^3 \backslash B_{R_0} (0).
\]

**Remark 1.** The expression (1.6) of \( U_i = (U, V) \) is unique when \( \| (u, v) \| \) is so small that condition (1.7) holds, i.e., for each \( U_i, (x_1, x_2, \ldots, x_i) \) is uniquely determined.

**Remark 2.** The assertion of Theorem 1 implies that for \( \beta < 0 \) with \( |\beta| \) sufficiently small, problem (P) possesses at least 6 nonradial positive solutions. In [14], the existence of positive solutions of (P) of the form (1.6) was established in the case that \( \{x_1, x_2, \ldots, x_i\} \) forms regular cube or tetrahedra under the assumption
\[
\sqrt{\frac{\mu_1}{\mu_2}} < \left\{ \begin{array}{ll}
\frac{\sqrt{3}}{3} & \text{for the cube} \\
\frac{\sqrt{2}}{3} & \text{for the tetrahedra}
\end{array} \right.
\]
Our argument employed in this paper does not require the ratio of $\sqrt{\mu_1}$ and $\sqrt{\mu_2}$.

**Theorem 3.** Suppose that $\sqrt{\frac{\mu_1}{\mu_2}}$ is irrational. Then for each $k \in \mathbb{N}$, there exists $\vec{\beta}_k \in (-1,0)$ such that for each $\beta \in (\vec{\beta}_k,0)$, the problem (P) has a positive solution $\tilde{\mathcal{U}}_k$ such that $kc_1 + 2c_2 < \Phi(\tilde{\mathcal{U}}_k) < kc_1 + c_2$ and $\tilde{\mathcal{U}}_k$ has the form

\begin{equation}
\tilde{\mathcal{U}}_k = (U,V) = \left( \sum_{j=1}^{k} U_1 x_j + u, U_2 + v \right)
\end{equation}

where $\{x_1,x_2,\ldots,x_k\} \subset \mathbb{R}^3$ form a regular $k$-polygon in a two-dimensional subspace of $\mathbb{R}^3$ and $u,v \in H$ such that $\|(u,v)\|$ is so small that

\begin{align*}
\hat{U}(z) < \frac{1}{2} |\hat{U}|_{\infty} & \quad \text{for } z \in \mathbb{R}^3 \setminus (\cup_{j=1}^{k} B_{R_0}(x_j)) \\
\hat{V}(z) < \frac{1}{2} |\hat{V}|_{\infty} & \quad \text{for } z \in \mathbb{R}^3 \setminus B_{R_0}(0).
\end{align*}

**Remark 3.** The existence of positive solutions of (P) of the form (1.8) was proved in [14] in the case that the spatial dimension is 2 and $\mu_1, \mu_2$ satisfy

$$\sqrt{\frac{\mu_1}{\mu_2}} < \sin \frac{\pi}{k}.$$

We give a sketch of the proof of Theorem 2 for the case $i = 2$. The proofs of Theorem 2 for $i \neq 2$ and the proof of Theorem 3 are slight modifications of that of the case $i = 2$ of Theorem 2. The detail of the proofs can be found in [9].

2. Preliminaries

Throughout the rest of this paper, we assume that $\sqrt{\frac{\mu_1}{\mu_2}}$ is irrational.

For each $u,v \in H = H^1(\mathbb{R}^3)$, we put $\langle u,v \rangle = \int_{\mathbb{R}^3} uv$. We denote by $\langle \cdot, \cdot \rangle_{\mu_i}$ the inner product of $H$ defined by $\langle u,v \rangle_{\mu_i} = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + \mu_i uv)$ for $u,v \in H$ and $i \in \{0,1,2\}$. The inner product of $H$ is defined by $\langle U_1, U_2 \rangle_H = \langle U_1, U_2 \rangle_{\mu_1} + \langle V_1, V_2 \rangle_{\mu_2}$ for $U_1 = (U_1, V_1), U_2 = (U_2, V_2) \in H$. For each $u \in L^4(\mathbb{R}^3)$, we put $\Omega(u) = \{ x \in \mathbb{R}^3 : \hat{u}(x) \geq \frac{1}{2} |\hat{u}|_{\infty} \}$ and

$$B(u) = \frac{\int_{\Omega(u)} x(\hat{u}(x) - \frac{|\hat{u}|_{\infty}}{2}) dx}{\int_{\Omega(u)} (\hat{u}(x) - \frac{|\hat{u}|_{\infty}}{2}) dx}.$$

The mapping $B$ is called generalized barycenter, which is introduced in [18] (cf. also [4]). By Sobolev's embedding theorem ([1]), for $p \in [2,6]$ there exists $m_p > 0$ such that

$$|z|_{L^p(B_r(0))} \leq m_p \left( |\nabla z|_{L^2(B_r(0))}^2 + \mu_i |z|_{L^2(B_r(0))}^2 \right)^{1/2}.$$
for $r > 1, z \in H^1(B_r(0))$, and $i \in \{0, 1, 2\}$. For each $a \in R$ and a functional $F : H \to R$, we denote by $F^a$ the level set defined by $F^a = \{ v \in H : F(v) \leq a \}$. The same notation is used for functionals defined on $H$.

It is easy to see that for each $u \in H \setminus \{0\}$ with $u^+ \neq 0$, there exists a unique positive number $s$ such that $su \in S_i$ (cf. [25]). It follows from the definitions of $U_i$ that $U_i(x) = \sqrt{\mu_i}U_0(\sqrt{\mu_i}x)$ on $R^3$. Then one can see that

\[ c_1 = \sqrt{\mu_1}c_0 > c_2 = \sqrt{\mu_2}c_0. \]

Let $i \in \{0, 1, 2\}$. It is known that $\{U_{i,x} : x \in R^3\}$ is a nondegenerate critical set of $I_i$ (cf. [23]). More precisely, we have there exists $\lambda > 0$ such that

\[ \|u\|^2_{\mu_i} - 3\langle U^2u, u \rangle \geq \lambda \|u\|^2_{\mu_i} \quad \text{for all } u \in \left\{ U_i, \frac{\partial U_i}{\partial x_1}, \frac{\partial U_i}{\partial x_2}, \frac{\partial U_i}{\partial x_3} \right\}^\perp. \]

We define a functional $\Phi : H \to R$ associated with problem (P) by

\[ \Phi(U) = \frac{1}{2}(\|U\|^2_{\mu_1} + \|V\|^2_{\mu_2}) - \frac{1}{4}(\|U^+\|^4_{4} + \|V^+\|^4_{4}) - \frac{\beta}{2} \int_{R^3} (U^+)^2(V^+)^2 \]

\[ = \Phi_1(U) + \Phi_2(U) \quad \text{for } U = (U, V) \in H, \]

where

\[ \Phi_1(U) = \frac{1}{2} \|U\|^2_{\mu_1} - \frac{1}{4} \|U^+\|^4_{4} - \frac{\beta}{4} \int_{R^3} (U^+)^2(V^+)^2 \]

and

\[ \Phi_2(U) = \frac{1}{2} \|V\|^2_{\mu_2} - \frac{1}{4} \|V^+\|^4_{4} - \frac{\beta}{4} \int_{R^3} (U^+)^2(V^+)^2. \]

Then a direct computation shows

\[ \langle \nabla \Phi(U), V \rangle_H = \left\langle \begin{pmatrix} \nabla_U \Phi(U) \\ \nabla_V \Phi(U) \end{pmatrix}, \begin{pmatrix} W \\ Z \end{pmatrix} \right\rangle_H \]

\[ = \langle -\Delta U + \mu_1 U - (U^+)^3 - \beta U^+U^+V^+^2, W \rangle \]

\[ + \langle -\Delta V + \mu_2 V - (V^+)^3 - \beta (U^+)^2V^+, Z \rangle \]

for $U = (U, V), V = (W, Z) \in H$. We put

\[ \mathcal{M}_+ = \{(U, V) \in H \setminus \{0\} : \|U\|^2_{\mu_1} = \|U^+\|^4_{4} + \beta \int_{R^3} (U^+)^2(V^+)^2, \]

\[ \|V\|^2_{\mu_2} = \|V^+\|^4_{4} + \beta \int_{R^3} (U^+)^2(V^+)^2 \}. \]

Then one can see that $U = (U, V) \in M_+$ is a critical point of $\Phi$ if and only if $U$ is a positive solution of problem (P). From the definition, we have

\[ \Phi_1(U) = \frac{1}{4} \|U\|^2_{\mu_1}, \quad \Phi_2(U) = \frac{1}{4} \|V\|^2_{\mu_2} \quad \text{and} \quad \Phi(U) = \frac{1}{4} (\|U\|^2_{\mu_1} + \|V\|^2_{\mu_2}) \]

for $U = (U, V) \in M_+$. We also have that for each $U = (U, V) \in M_+$, there exists $(s, t) \in R^+ \times R^+$ such that $(sU, tV) \in M_+$. In fact, for each $U = (U, V), U \neq \]
$0, V \not\equiv 0,(sU, tV) \in M_+ \text{if and only if}$

\[
\begin{pmatrix}
  s^2 \\
  t^2
\end{pmatrix} = A^{-1}
\begin{pmatrix}
  \|U\|_{\mu_1}^2 \\
  \|V\|_{\mu_2}^2
\end{pmatrix}
\]

where

\[
A = \begin{pmatrix}
  |U^+|^4_4 & \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 |V^+|^4_4 \\
  \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 & |V^+|^4_4
\end{pmatrix}.
\]

Since $\beta \in (-1, 0)$, we have by the Schwartz's inequality that $A^{-1}$ exists and then there exists a unique solution $(s, t) \in \mathbb{R}^+ \times \mathbb{R}^+$. For given $U = (U, V) \in H$ with $U \not\equiv 0, V \not\equiv 0$, we put $NU = N(U, V) = (N_1 U, N_2 V) = (sU, tV) \in M_+$.

Now to prove Theorem 1 for the case that $i = 2$, we define $H_2 \subset H, H_2 \subset H$ and $M_2 \subset M_+$ by

\[H_2 = \{u \in H : u(x) = u(-x) \text{ for } x \in \mathbb{R}^3\}, \quad \mathbb{H}_2 = H_2 \times H_2,\]

and

\[M_2 = M_+ \cap \mathbb{H}_2.\]

Since $\sqrt{\mu_1/\mu_2}$ is irrational, we can choose $\delta_2 \in (0, c_2)$ so small that

\[
(2.4) \quad c_1 + kc_2 \notin [2c_1 + c_2 - \delta_2, 2c_1 + c_2 + \delta_2] \quad \text{for all } k \in \mathbb{N}.
\]

The following Lemmata are crucial for our argument.

**Lemma 1.**

1. There exists $\beta_1 \in (-1, 0)$ such that for each $\beta \in (\beta_1, 0)$ and each critical point $U \in M_2 \cap \Phi^{c_1 + 2c_2}$ of $\Phi$,

\[\Phi(U) \in \bigcup_{i \geq 1, j \geq 1} [ic_1 + jc_2 - \delta_2/2, ic_1 + jc_2 + \delta_2/2].\]

2. Let $\beta \in (\beta_1, 0)$ and $\{U_n\} \subset M_2$ such that $\lim_{n \to \infty} \nabla \Phi(U_n) = 0$ and $\lim_{n \to \infty} \Phi(U_n) = 2c_1 + c_2 + \varepsilon$ with $\varepsilon \in (0, \delta_2/2)$. Then there exists a convergence subsequence $\{U_n\} \subset \{U_n\}$.

**Lemma 2.** For given $\varepsilon > 0$, there exists $\beta_2 \in (\beta_1, 0)$ such that for each $\beta \in (\beta_2, 0)$ and for each $x \in \mathbb{R}^3 \setminus \{0\}$,

\[
(2.5) \quad \Phi(N(U_{1,x} + U_{1,-x}, U_2)) < 2c_1 + c_2 + \varepsilon
\]
3. Sketch of the Proof of Theorem 2 for $i = 2$.

Throughout this section we assume that $\beta \in (\beta_1, 0)$. We put

$$b_R(U) = \int_{\mathbb{R}^3 \setminus B_R(0)} |U|_{\mu_1}^2$$

for $U \in H$ and $R > 0$.

and

$$\Lambda_{2,\epsilon}(R) = \{ \mathcal{U} = (u, v) \in \mathcal{M}_2 : b_R(U) \geq 8c_1 - \min \left\{ \frac{1}{2m_4}, c_1 \right\} \}$$

for each $\epsilon > 0$ and $R > 0$.

**Proposition 1.** For $\epsilon > 0$ sufficiently small, there exists $(R_\epsilon, \delta_\epsilon, \alpha_\epsilon, \gamma_\epsilon) \in (\mathbb{R}^+)^4$ such that $\lim_{\epsilon \to 0} \delta_\epsilon = \lim_{\epsilon \to 0} \alpha_\epsilon = \lim_{\epsilon \to 0} \gamma_\epsilon = 0$ and each $U = (U, V) \in \Lambda_{2,\epsilon}(R_\epsilon)$ has the form

(3.1) \quad $U = (\alpha(U_{1, +x} + U_{1, -x}) + u, \gamma U_2 + v)$

where $\alpha \in (1 - \alpha_\epsilon, 1 + \alpha_\epsilon), \gamma \in (1 - \gamma_\epsilon, 1 + \gamma_\epsilon)$,

(3.2) \quad $|x| \geq R_\epsilon, \quad x = B(U|_{B_{R_0}(x)}), \quad \tilde{U}(z) < \frac{1}{2} |\tilde{U}|_{\infty}$ for $z \in \mathbb{R}^3 \setminus \bigcup_{i=\pm 1} B_{R_0}(ix),$

(3.3) \quad $\hat{V}(z) < \frac{1}{2} |\hat{V}|_{\infty}$ for $z \in \mathbb{R}^3 \setminus B_{R_0}(0),$

and

(3.4) \quad $(u, v) \in \{U_{1, x}, U_{1, -x}\}^\perp \times \{U_2\}^\perp$ with $\|u\|_{\mu_1}^2 + \|v\|_{\mu_2}^2 \leq \delta_\epsilon.$

**Remark 4.** By (3.2) and the definition of $B$, one can see that for each $U \in \Lambda_{2,\epsilon}(R_\epsilon), (x, -x) \in \mathbb{R}^3 \times \mathbb{R}^3$ in (3.1) is uniquely determined, and the mapping $U \in \Lambda_{2,\epsilon}(R_\epsilon) \to (x, -x) \in \mathbb{R}^3 \times \mathbb{R}^3$ is continuous. We define a continuous mapping $\eta : \Lambda_{2,\epsilon}(R_\epsilon) \to \mathbb{R}^+$ by

(3.5) \quad $\eta(U) = |x|$ for $U \in \Lambda_{2,\epsilon}(R_\epsilon).

We also need the following Proposition.

**Proposition 2.** There exists $M_0 > 0$ satisfying that for $\epsilon > 0$ sufficiently small,

$$\Phi(U) \geq 2c_1 + c_2 - \beta M_0 e^{-2\sqrt{|x|}}$$

for each $U \in \Lambda_{2,\epsilon}(R_\epsilon)$, where $x \in \mathbb{R}^3$ such that $U$ has the form (3.1).
Now for \( x \in \mathbb{R}^3 \setminus \{0\}, \) we define a class \( \Gamma_2(x) \subset C([0,1], \mathcal{M}_2) \) by

\[
\Gamma_2(x) = \{ p \in C([0,1], \mathcal{M}_2) : p(0) = \mathcal{N}(U_{1,x} + U_{1,-x}, U_2), p(1) = \mathcal{N}(U_{1,-x} + U_{1,x}, U_2) \}
\]

and put

\[
c_2(x) = \inf_{p \in \Gamma_2(x)} \sup_{t \in [0,1]} \Phi(p(t)).
\]

We also note that from the definitions of \( N \) and \( \Phi \), we have that \( N(U_{1,x} + U_{1,-x}, U_2) - (U_{1,x} + U_{1,-x}, U_2) \to 0 \) in \( H \) as \( |x| \to \infty \) and then

\[
\lim_{|x| \to \infty} \Phi(N(U_{1,x} + U_{1,-x}, U_2)) = 2I_1(U_1) + I_2(U_2) = 2c_1 + c_2.
\]

Based on the preliminary results above, we can prove Theorem 2 for \( i = 2 \).

**Proof of Theorem 2.** Let \( \varepsilon \in (0, \delta_2/2) \) sufficiently small. Let \( \beta \in (\beta_\varepsilon, 0) \). To complete the proof, it is sufficient to show that there exists \( \delta > 0 \) and \( R > 0 \) such that

\[
2c_1 + c_2 + \delta < c_2(x) < 2c_1 + c_2 + \delta_2/2 \quad \text{for } |x| > R.
\]

In fact, if the inequalities above hold, we have by (3.6) that we can choose \( x \in \mathbb{R}^3 \) such that \( |x| > R \) and

\[
\Phi(N(U_{1,x} + U_{1,-x}, U_2)) < 2c_1 + c_2 + \delta.
\]

That is \( \Phi(p(1)) < c_2(x) \) for all \( p \in \Gamma_2(x) \). We also have \( \Phi(p(0)) < \frac{\delta_2}{4} c_1 + c_2 \). Then since the Palais-Smale condition holds by (2) of Lemma 1 on \( \Phi(2c_1 + c_2, 2c_1 + c_2 + \delta_2/2) \), we have by a standard mountain pass argument that there exists a critical point \( U \) of \( \Phi \) with \( \Phi(U) = c_2(x) \).

From the definition of \( \varepsilon \) and Lemma 2, one can see that the pass \( p \in \Gamma_2(x) \) defined by

\[
p(s) = N(U_{1,sx} + U_{1,-sx}, U_2), \quad s \in [0,1]
\]

satisfies \( \max_{s \in [0,1]} \Phi(p(s)) \leq 2c_1 + c_2 + \varepsilon \). Then the second inequality of (3.7) holds. We now show that the first inequality of (3.7) holds. We first see that there exists \( \overline{R} > 2R_\varepsilon \) such that

\[
b_{R_\varepsilon}(U) \geq 8c_1 - \frac{1}{2} \min \left\{ \frac{1}{2m_4^2}, c_1 \right\} \quad \text{for } U = (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon) \text{ with } \eta(U) \geq \overline{R},
\]

where \( \eta \) is the function defined by (3.5). By Proposition 1, each \( U = (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon) \) has the form

\[
U = (\alpha(U_{1,x} + U_{1,-x}) + u, \gamma U_2 + v)
\]
with $\alpha \in (1-\alpha_{\epsilon}, 1+\alpha_{\epsilon})$ and $(u,v) \in \{u_{1,x}, u_{1,-x}\}^\perp \times \{u_{2}\}^\perp$ with $\|u\|_{\mu_{1}}^{2} + \|v\|_{\mu_{2}}^{2} \leq \delta_{\epsilon}$. Since $\lim_{\epsilon \to 0} \delta_{\epsilon} = \lim_{\epsilon \to 0} \alpha_{\epsilon} = 0$, we may assume that $\epsilon > 0$ is sufficiently small that

$$8\alpha_{\epsilon}^{2}c_{1} - \delta_{\epsilon} > 8c_{1} - \frac{1}{2}\min\left\{\frac{1}{2m_{4}^{4}}, c_{1}\right\}. \tag{3.10}$$

Then noting that

$$b_{R_{\epsilon}}(U) \geq \alpha^{2}\|U_{1,x} + U_{1,-x}\|_{\mu_{1}}^{2} - \|u\|_{\mu_{1}}^{2} - 2\int_{B_{R_{\epsilon}}(0)}|U_{1_{1}x} + U_{1-x}|_{\mu_{1}}^{2}$$

and

$$\|U_{1,x} + U_{1,-x}\|_{\mu_{1}}^{2} \to 8c_{1} \text{ and } \int_{B_{R_{\epsilon}}(0)}|U_{1,x} + U_{1,-x}|_{\mu_{1}}^{2} \to 0, \text{ as } |x| \to \infty,$$

we find by (3.10) that there exists $\overline{R}$ such that for each $U = (U,V) \in \Lambda_{2,\epsilon}(R_{\epsilon})$ with $\eta(U) \geq \overline{R}$, (3.8) holds. Now we choose $x \in \mathbb{R}^3$ so large that $|x| > \overline{R}$. Then

$$b_{R_{\epsilon}}(N_{1}(U_{1,x} + U_{1,-x})) \geq 8c_{1} - \frac{1}{2}\min\left\{\frac{1}{2m_{4}^{4}}, c_{1}\right\}.$$

Let $p = (p_{1}, p_{2}) \in \Gamma_{2}(x)$ such that $\sup_{t \in [0,1]} \Phi(p(t)) \leq 2c_{1} + c_{2} + \epsilon$. From the definition,

$$\eta(p_{1}(1)) = \eta(N_{1}(U_{1,x} + U_{1,-x})) > \overline{R} \text{ and } b_{R_{\epsilon}}(p_{1}(1)) \geq 8c_{1} - \frac{1}{2}\min\left\{\frac{1}{2m_{4}^{4}}, c_{1}\right\}.$$

On the other hand, recalling that $\Phi_{2}(U) \geq c_{2}$, we have that $\Phi_{1}(U) \leq \frac{7}{4}c_{1}$. Then by the definition of $\epsilon$, $b_{R_{\epsilon}}(p_{1}(0)) < 7c_{1} - \min\left\{\frac{1}{2m_{4}^{4}}, c_{1}\right\}$, there exists $t \in (0,1)$ such that $b_{R_{\epsilon}}(p_{1}(t)) = 8c_{1} - \min\left\{\frac{1}{2m_{4}^{4}}, c_{1}\right\}$. Then by (3.8), $\eta(p_{1}(t)) < \overline{R}$. Therefore by the continuity of $\eta$, we have that there exists $t_{0} \in (0,t)$ such that $\eta(p_{1}(t_{0})) = \overline{R}$. By Proposition 2, we have

$$\Phi(p(t_{0})) \geq 2c_{1} + c_{2} + \beta M_{0}e^{-2\sqrt{\mu_{2}}\overline{R}}.$$ 

Therefore we obtain that $\sup_{t \in [0,1]} \Phi(p(t)) > 2c_{1} + c_{2} + \beta M_{0}e^{-2\sqrt{\mu_{2}}\overline{R}}$. Thus by the mountain pass theorem (cf. [20]), we find that there exists a critical point $U$ of $\Phi$ with $\Phi(U) = c_{2}(x)$. \hfill \Box

References


21. B. Sirakov, \textit{Least energy solitary waves for a system of nonlinear Schrodinger equations in $\mathbb{R}^n$}, Communications in Mathematical Physics \textbf{271} (2007), 199–221.


