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CONVERGENCE THEOREMS AND CONVERGENCE RATE ESTIMATES OF ITERATIVE SCHEMES INVOLVING ϕ-STRONGLY PSEUDOCONTRACTIVE MAPPINGS IN A BANACH SPACE

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ABSTRACT. In this paper we introduce the existence theorem and convergence theorems with respect to a ϕ-strongly pseudocontractive mapping in a Banach space and give the convergence rate estimates of Mann iterative sequence involving ϕ-strongly pseudocontractive mapping. By this convergent estimate, we show how to construct an iterative scheme with a desired convergence rate.

1. INTRODUCTION

Let E be an arbitrary Banach space and let $T : E \to E$ be a nonlinear mapping such that the set $F(T)$ of fixed points of T is nonempty. Let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where E^* denotes the dual space of E, $\langle \cdot, \cdot\rangle$ denotes the duality pairing and $\|\cdot\|$ denotes the norm on E and E^* while there are no confusion. Then we give a definition of ϕ−strongly pseudocontractive mappings T as follows.

Let Φ be the set of all strictly increasing functions $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$. For any $\phi \in \Phi$, we define a ϕ−strongly pseudocontractive mapping as follows.

Definition 1. a mapping $T : D(T) \to R(T)$ is called a ϕ− strong pseudocontractive mapping, if for all $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \phi(\|x - y\|)\,\|x - y\|\,.$$

If there exists a constant $c > 1$ so that

$$\langle Tx - Ty, j(x - y)\rangle \leq (1 - \frac{1}{c})\,\|x - y\|^2\,,$$

then T is strongly pseudocontractive. If $\phi(t) = \frac{1}{c}t$ for some constant $c > 1$, then T is strongly pseudocontractive. If T is a non-expansive mapping, then by Schwartz inequality implies

$$\langle Tx - Ty, j(x - y)\rangle \leq \|Tx - Ty\|\,\|x - y\| \leq \|x - y\|^2$$

for all $x, y \in D(T)$. Let I be an identity mapping, and let $A = I - T$. Then the following are equivalent:

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(1) T is strongly pseudocontractive,
(2) A is strongly accretive.
Similarly, we obtain the following definition:

Definition 2. A is $\phi-$strongly accretive if $(I-A)$ is $\phi-$strongly pseudocontractive.

Consequently, the following statements are equivalent:
(1) A $\phi-$strongly pseudocontractive mapping T has a unique fixed point x^*, i.e., $Tx^* = x^*$.
(2) A $\phi-$strongly accretive mapping A has a unique zero x^*, i.e., $Ax^* = 0$.
In this paper, we introduce the existence theorem of fixed points by Kirk and Morales, and convergence theorems of some iterative schemes involving ϕ-pseudocontractive mapping. And then, motivated by the result of strongly pseudocontractive mapping shwed by Liu, Sastry and Babu, we give a convergence rate estimate of Mann iterative sequence involving ϕ-strongly pseudocontractive mapping. Moreover, by using this result we show how to construct an iterative sequence with a desired convergence rate estimates.

2. Preliminaries

We shall first introduce the existence theorem and convergence theorem of iterative schemes involving a ϕ-strongly pseudocontractive mapping in a Banach space.

Theorem 2.1. *(Kirk-Morales, 1980) Let $\alpha : [0,\infty) \to [0,\infty)$ be a function for which $\alpha(0) = 0$ and $\liminf_{r\to r_0} \alpha(r) > 0$ for every $r_0 > 0$, and let C be a closed convex subset in E. Suppose that for the α, a mapping $T : C \to C$ is α-strongly pseudocontractive and continuous. Then T has a unique fixed point p.*

There are a lot of convergence theorems with many kind of iterative schemes. [Examples of iterative schemes]
1. Mann type: For a coefficient sequence $\{t_n\}$ in $[0,1]$ and any $x_0 \in E$, $x_{n+1} = (1-t_n)x_n + t_nTx_n$ for all $n \geq 0$.
2. Ishikawa type: For coefficient sequences $\{t_n\}$ and $\{s_n\}$ in $[0,1]$ and $x_0 \in E$,

$$(1) \qquad \begin{cases} x_{n+1} = (1-t_n)x_n + t_nTy_n, \\ \qquad y_n = (1-s_n)x_n + s_nTx_n, \end{cases}$$

for all $n \geq 0$.
3. General type: For a coefficient sequence $\{t_n\}$ in $[0,1]$, a sequense $\{v_n\}$ in E and $x_0 \in E$,

$$(2) \qquad x_{n+1} = (1-t_n)x_n + t_nTv_n + u_n \text{ for all} n \geq 0,$$

where $\{u_n\}$ is an error term sequence.
With respect to the third type iterative scheme, we obtained a convergence theorem to a fixed point p as follows.

Theorem 2.2. *(JMAA,2006) Let E be a Banach space. Let $\phi \in \Phi$. Suppose that $T : E \to E$ is a uniformly continuous and $\phi-$strongly pseudocontractive mapping with a bounded range, and that a sequence $\{x_n\}$ defined by (2) with a coefficient sequence $\{t_n\}$ and an error term sequence $\{u_n\}$ which satisfy the following condition (a)-(c):*
(a) $\sum_{n=1}^{\infty} t_n = \infty$ *(b)* $\lim_{n\to\infty} t_n = 0$ *(c)* $\sum_{n=1}^{\infty} \|u_n\| < \infty$.
If $\lim_{n\to\infty} \|v_n - x_n\| = 0$, *then* $x_n \to p \in F(T)$.

Remark 1. In this theorem, we prove the stability of the iterative scheme defined by (2) and obtain the previous result.

Next we shall show the theorem of convergence rate estimates, which motivated us.

Theorem 2.3. *(Liu, Proc. Amer. Math. Soc., 1997) Let C be a non-empty closed convex and bounded subset in E, and suppose that $T : C \to C$ is strong pseudo-contractive and Lipschitz continuous with a Lipschitz constant L. Let $\{t_n\}$ be a coefficient sequence defined by*

$$t_n = \frac{c}{2(3+3L+L^2)}$$

for all $n \geq 1$.

Then Mann iterative sequence $\{x_n\}$with $\{t_n\}$ has the following convergence estimate: For all $n \geq 0$,

$$\|x_{n+1} - p\| \leq \rho^n \|x_0 - p\|,$$

where

$$\rho = 1 - \frac{c^2}{4(3+3L+L^2)}.$$

Furthere, Sastry and Babu gave better estimate with

$$\rho = 1 - \frac{c^2}{4(L+1)(L+2-c)+2c}.$$

[Prc. Amer. Math. Soc., 2002]

3. Main results

We obtain the following estimate of Mann iterative scheme involving $\phi-$strongly pseudocontractive mapping T.

Theorem 3.1. *(YMJ, 2008) Let C be a bounded closed convex subset in E and the diameter of C be $M > 0$. Assume that $\phi(t) = \psi(t)t$, where ψ is an increasing function on [0,M] to [0,1] with $\lim_{t\to 0+}\psi(t) = 0$. Let $T : C \to C$ be $\phi-$strongly pseudocontractive and Lipschitz continuous with a Lipschitz constant L.*

Then Mann iterative sequence $\{x_n\}$ has the following estimates: For all $n \geq 0$,

$$\|x_{n+1} - p\| \leq (1 - \gamma_n t_n + \tilde{L}t_n^2) \|x_n - p\|,$$

where $\gamma_n = \psi(\|x_{n+1} - p\|)$, and $\tilde{L} = 3 + 3L + L^2$.

Having the previous theorem of convergence rate estimate, we consider the following problem:

[Problem] For any $\beta \in (0,1)$ and any positive integer $K \geq 0$, find an approximative point x_K of $p \in F(T)$ such that

$$\|x_K - p\| \leq M\beta^K.$$

Furthermore, detemine a number $n(K)$ for β and $K \geq 0$, such that

$$\|x_n - p\| \leq M\beta^K \quad \text{for all } n \geq n(K).$$

In order to obtain the approximative sequence $\{x_n\}$ with the above convergence rate estimate, we shall construct it by the following steps.

Steps for the construction of the approximative sequence $\{x_n\}$:

(1) For any $\beta \in (0,1)$ and $\phi(t) = \psi(t)t$, determine $\{n(K)\}_K$ as follows:

$$n(K) = \sum_{j=1}^{K} m_j,$$

where

$$m_j = \min\{m \in \mathbf{N} : (C_T(\beta^j))^m \le \beta\}$$

and

$$C_T(\alpha) = 1 - \frac{1}{4\tilde{L}}(\psi(M\alpha))^2 \quad \text{for all} \quad \alpha \in (0,1].$$

(2) Since $\{n(K)\}_K$ is an increasing number sequence, for each $n \in N_0$, there exists K such that $n(K-1) \le n < n(K)$, and then define t_n by

$$t_n = \frac{1}{2\tilde{L}}\psi(M\beta^K).$$

(3) Determine a Mann sequence $\{x_n\}$ with $\{t_n\}$ as follows: For any $x_0 \in C$

$$x_{n+1} = (1-t_n)x_n + t_n T x_n.$$

Then we show that this sequence $\{x_n\}$ satisfies the desired convergence estimate, by the following theorem.

Theorem 3.2. *(YMJ, 2008) Let β be in (0,1). Let C be a bounded closed convex subset of E. Suppose $T : C \to C$ is $\phi-$strongly pseudocontractive mapping which satisfies the condition of the previous theorem. Let $\{x_n\}$ be defined by the steps of (1)-(3). Then $\{x_n\}$ satisfies the following rate estimate: For each $K \ge 0$,*

$$\|x_{n+1} - p\| \le M\beta^K \quad \textit{for all } n \ge n(K).$$

Moreover, we give the smallest number $n(K)$ which is large enough to obtain the approximative point desired to be closed to a fixed point. For the sake of simplification, we assume that the diameter $M = 1$.

Theorem 3.3. *(YMJ, 2008) Under the assumption of the previous theorem 3.2 , the following holds:*

$$n(K) \le (1+\log\beta)K - 8\tilde{L}(\log\beta)\sum_{j=1}^{K}\frac{1}{(\psi(\beta^j))^2}, \quad K \in N.$$

Moreover, in the case of $\beta = \frac{1}{2}$, the following hold:
(1) If $\psi(t) = t$, then

$$n(K) \le (1-\log 2)K + \frac{32\tilde{L}}{3}4^K \log 2, \quad K \in N;$$

(2) if

$$\psi(t) = \frac{1}{1-\log t}, \; t \in (0,1],$$

then for any $K \in N$,

$$n(K) \le \{1 + (8\tilde{L}-1)(\log 2)\}K + 8\tilde{L}(\log 2)^2 K(K+1) + \frac{4}{3}\tilde{L}(\log 2)^3 K(K+1)(2K+1).$$

It seems that the convergence rate of the $\{x_n\}$ with $\phi(t) = \frac{t}{1-\log t}$ give a faster convergence than one with $\phi(t) = t^2$.

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