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Approximation processes by weighted interpolation type operators

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1. Introduction

Let $\mathbb{N}$ denote the set of all natural numbers. Let g be a real-valued continuous function on the closed unit interval $\mathbb{I} = [0, 1]$ of the real line $\mathbb{R}$ and let $n \in \mathbb{N}$. Then nth Bernstein polynomial of g is defined by

$$B_n(g)(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} g\left(\frac{k}{n}\right) \qquad (t \in \mathbb{I}). \tag{1}$$

It is well known that the sequence $\{B_n(g)\}_{n\in\mathbb{N}}$ converges uniformly to g on $\mathbb{I}$, and the Bernstein polynomials and their generalizations play an important role in approximation theory (see, e,g., [1], [2], [7], [9], [10]).

In view of these concernments, Balázs [3] introduced and studied several approximation properties of the Bernstein type rational functions defined as follows:

Let f be a real-valued function on $[0, \infty)$ and let $n \in \mathbb{N}$, and define

(2)

$$R_n(f;x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} \binom{n}{k} (a_n x)^k f\left(\frac{k}{b_n}\right) \qquad (x \in [0,\infty)),$$

where $a = \{a_n\}_{n\in\mathbb{N}}$ and $b = \{b_n\}_{n\in\mathbb{N}}$ are suitably chosen sequences of positive real numbers. To compare (1) and (2), setting

$$q_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k} \qquad (t \in \mathbb{I},\ k = 0, 1, \ldots, n)$$

and

$$r_{n,k}(x) = \binom{n}{k} \frac{(a_n x)^k}{(1+a_n x)^n} \qquad (x \in [0,\infty),\ k = 0,1,\ldots,n),$$

we have

$$r_{n,k}(x) = q_{n,k}\Big(\frac{a_n x}{1+a_n x}\Big) \qquad (x \in [0,\infty),\ k = 0,1,\ldots,n),$$

and so

$$R_n(f;x) = B_n(f|_{\mathbb{I}})\Big(\frac{a_n x}{1+a_n x}\Big),$$

where $f|_{\mathbb{I}}$ denotes the restriction of f to $\mathbb{I}$.

In [4], the estimate of the rate of convergence of $R_n(f;x)$ to $f(x)$ given in [3] is improved by an appropriate choice of a and b when f satisfies some more restrictive conditions. Furthermore, in [14] the saturation problem is discussed for $\{R_n\}_{n\in\mathbb{N}}$ and the uniform approximation problem is considered for R_n-like rational functions defined by

$$\text{(3)} \qquad R_n(B;a;f;x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} \binom{n}{k} (a_n x)^k f(b_{n,k})$$

$$= \sum_{k=0}^{n} r_{n,k}(x) f(b_{n,k}) \qquad (x \in [0,\infty)),$$

where $B = (b_{n,k})_{0\le k\le n\,(n=1,2,\ldots)}$ is a matrix whose entries satisfy

$$0 \le b_{n,0} < b_{n,1} < b_{n,2} < \cdots < b_{n,n},$$

and f is a real-valued continuous function on $[0,\infty)$ for which $\lim_{x\to\infty} f(x)$ exists. Note that if

$$a_n = 1 \qquad (n \in \mathbb{N}),$$

and if

$$b_{n,k} = \frac{k}{n-k+1} \qquad (0 \le k \le n,\ n \in \mathbb{N}),$$

then (3) reduces to

$$\text{(4)} \qquad L_n(f)(x) := \frac{1}{(1+x)^n} \sum_{k=0}^{n} \binom{n}{k} x^k f\Big(\frac{k}{n-k+1}\Big),$$

which was introduced by Bleimann, Butzer and Hahn [5]. In [15], the satutration properties of the sequence $\{L_n\}_{n\in\mathbb{N}}$ is established and it is showed that these operators satisfy an asymptotic relation of the Voronovskaja type, i.e.,

$$\lim_{n\to\infty} n(L_n(x_0) - f(x_0)) = f''(x_0)x_0(1+x_0)^2$$

if f is a real-valued continuous function on $[0,\infty)$ for which $\lim\limits_{x\to\infty} f(x)$ exists and the second derivative $f''(x_0)$ exists at a point x_0.

Let $1 \le p \le \infty$ be fixed and let $\mathbb{R}^r$ denote the metric linear space of all r-tuples of real numbers, equipped with the usual metric

$$d_p(x,y) := \begin{cases} \left(\sum\limits_{i=1}^{r} |x_i - y_i|^p\right)^{1/p} & (1 \le p < \infty) \\ \max\{|x_i - y_i| : 1 \le i \le r\} & (p = \infty), \end{cases}$$

$$(x = (x_1, x_2, \dots, x_r),\ y = (y_1, y_2, \dots, y_r) \in \mathbb{R}^r).$$

The purpose of this paper is to generalize (2) for vector-valued functions on the r-dimensional first hyperquadrant

$$[0,\infty)^r := \{x = (x_1, x_2, \dots, x_r) \in \mathbb{R}^r : x_i \ge 0,\ i = 1, 2, \dots, r\}$$

and to consider their uniform convergence with rates in terms of the modulus of continuity of functions to be approximated. We refer to [13] for details.

2. Convergence theorems

Let

$$(X, d) := ([0,\infty)^r, d_p),$$

and let $(E, \|\cdot\|)$ be a normed linear space. Let $B(X,E)$ denote the normed linear space of all E-valued bounded functions on X with the supremum norm $\|\cdot\|_X$. Also, we denote by $C(X,E)$ the linear space consisting of all E-valued continuous functions on X and set $BC(X,E) = B(X,E) \cap C(X,E)$.

Let $\{n_{\alpha,i}\}_{\alpha\in D}, i = 1, 2, \dots, r$, be nets of positive integers and let $\{b_{n_{\alpha,i}}\}_{\alpha\in D}, i = 1, 2, \dots, r$, be nets of positive real numbers such that

$$\lim_{\alpha} b_{n_{\alpha,i}} = +\infty \qquad (i = 1, 2, \dots, r).$$

Let $\{g_{n_{\alpha,i}}\}_{\alpha\in D}$ and $\{h_{n_{\alpha,i}}\}_{\alpha\in D}, i = 1, 2, \dots, r$, be nets of nonnegative functions in $C([0,\infty), \mathbb{R})$ such that

$$\inf\{g_{n_{\alpha,i}}(t) + h_{n_{\alpha,i}}(t) : t \in [0,\infty)\} > 0$$

for all $\alpha \in D$ and for $i = 1, 2, \ldots, r$. Then we define

$$F_\alpha(f)(x) = F_\alpha(E; f; x) = \prod_{i=1}^{r} \frac{1}{(g_{n_{\alpha,i}}(x_i) + h_{n_{\alpha,i}}(x_i))^{n_{\alpha,i}}}$$

$$\times \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}} \prod_{i=1}^{r} \rho_{n_{\alpha,i},k_i}(x_i) f\Big(\frac{k_1}{b_{n_{\alpha,1}}}, \frac{k_2}{b_{n_{\alpha,2}}}, \ldots, \frac{k_r}{b_{n_{\alpha,r}}}\Big)$$

$$(\alpha \in D,\ f \in C(X,E),\ x = (x_1, x_2, \ldots, x_r) \in X),$$

where

$$\rho_{n_{\alpha,i},k_i}(x_i) = \binom{n_{\alpha,i}}{k_i} g_{n_{\alpha,i}}{}^{k_i}(x_i) h_{n_{\alpha,i}}{}^{n_{\alpha,i}-k_i}(x_i)$$

$$(\alpha \in D,\ i = 1, 2, \ldots, r).$$

From now on let $K_i, i = 1, 2, \ldots, r$, be compact subsets of $[0, \infty)$ and we set

$$X_0 = \prod_{i=1}^{r} K_i.$$

Theorem 1. *We define*

$$I_{\alpha,i}(t) = \frac{n_{\alpha,i} g_{n_{\alpha,i}}(t)}{b_{n_{\alpha,i}}(g_{n_{\alpha,i}}(t) + h_{n_{\alpha,i}}(t))} \qquad (i = 1, 2, \ldots, r, t \in [0, \infty).$$

If

$$\lim_\alpha I_{\alpha,i}(t) = t \qquad \textit{uniformly in } t \in K_i$$

for $i = 1, 2, \ldots, r$, then

$$\lim_\alpha \|F_\alpha(f) - f\|_{X_0} = 0$$

for all $f \in BC(X, E)$.

Let

$$a_{n_{\alpha,i}} := \frac{b_{n_{\alpha,i}}}{n_{\alpha,i}} \qquad (\alpha \in D,\ i = 1, 2, \ldots, r), \tag{5}$$

and we define

$$T_\alpha(f)(x) = T_\alpha(E; f; x) = \prod_{i=1}^{r} \frac{1}{(1 + a_{n_{\alpha,i}} x_i)^{n_{\alpha,i}}} \tag{6}$$

$$\times \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}} \prod_{i=1}^{r} \rho_{n_{\alpha,i},k_i}(x_i) f\Big(\frac{k_1}{b_{n_{\alpha,1}}}, \frac{k_2}{b_{n_{\alpha,2}}}, \ldots, \frac{k_r}{b_{n_{\alpha,r}}}\Big),$$

$$(\alpha \in D,\ f \in C(X,E),\ x = (x_1, x_2, \ldots, x_r) \in X),$$

where

$$\rho_{n_{\alpha,i},k_i}(x_i) = \binom{n_{\alpha,i}}{k_i}\left(a_{n_{\alpha,i}}x_i\right)^{k_i} \qquad (\alpha \in D,\ i = 1, 2, \ldots, r).$$

Theorem 2. *If*

$$a_{n_{\alpha,i}} = o(1)$$

for $i = 1, 2, \ldots, r$, *then*

$$\lim_{\alpha} \|T_\alpha(f) - f\|_{X_0} = 0$$

for all $f \in BC(X, E)$.

Remark 1. (6) generalizes (2) to the r-dimensional Bernstein type rational vector-valued functions. Also, (3) can be extended by the following form to the r-dimensional case for vector-valued functions:

$$R_\alpha(f)(x) = R_\alpha(E; \mathcal{B}; \mathcal{A}; f; x) = \prod_{i=1}^{r} \frac{1}{(1 + a_{n_{\alpha,i}}x_i)^{n_{\alpha,i}}}$$

$$\times \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}} \prod_{i=1}^{r} \binom{n_{\alpha,i}}{k_i}\left(a_{n_{\alpha,i}}x_i\right)^{k_i} f\left(b_{n_{\alpha,1},k_1}, b_{n_{\alpha,2},k_2}, \ldots, b_{n_{\alpha,r},k_r}\right)$$

$$(\alpha \in D,\ f \in C(X, E),\ x = (x_1, x_2, \ldots, x_r) \in X),$$

where

$$\mathcal{A} = \{a_{n_{\alpha,i}} : \alpha \in D,\ i = 1, 2, \ldots, r\}$$

is a family of positive real numbers and

$$\mathcal{B} = \{b_{n_{\alpha,i},k_i} : 0 \le k_i \le n_{\alpha,i},\ \alpha \in D,\ i = 1, 2, \ldots, r\}$$

is a family of nonnegative real numbers with

$$0 \le b_{n_{\alpha,i},0} < b_{n_{\alpha,i},1} < b_{n_{\alpha,i},2} < \cdots < b_{n_{\alpha,i},n_{\alpha,i}}$$

$$(\alpha \in D,\ i = 1, 2, \ldots, r).$$

In particular, the operator $L_n(f)(x)$ defined by (4) is generalized to the r-dimensional case for vector-valued functions defined as follows:

$$L_\alpha(f)(x) = L_\alpha(E; f; x) = \prod_{i=1}^{r} \frac{1}{(1 + x_i)^{n_{\alpha,i}}} \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}}$$

$$\prod_{i=1}^{r} \binom{n_{\alpha,i}}{k_i} x_i^{k_i} f\left(\frac{k_1}{n_{\alpha,1} - k_1 + 1}, \frac{k_2}{n_{\alpha,2} - k_2 + 1}, \ldots, \frac{k_r}{n_{\alpha,r} - k_r + 1}\right)$$

$$(\alpha \in D,\ f \in C(X, E),\ x = (x_1, x_2, \ldots, x_r) \in X).$$

3. Convergence rates

Le $f \in B(X, E)$ and let $\delta \geq 0$. Then we define

$$\omega(f, \delta) = \sup\{\|f(x) - f(y)\| : x, y \in X, d(x, y) \leq \delta\},$$

which is called the modulus of continuity of f. Obviously, $\omega(f, \cdot)$ is a monotone increasing function on $[0, \infty)$ and

$$\omega(f, 0) = 0, \quad \omega(f, \delta) \leq 2\|f\|_X \qquad (\delta \geq 0).$$

Also, f is uniformly continuous on X if and only if

$$\lim_{\delta \to +0} \omega(f, \delta) = 0.$$

Furthermore, the convexity of d and X yields the inequality

$$\omega(f, \xi\delta) \leq (1 + \xi)\omega(f, \delta)$$

for all $\xi, \delta \geq 0$ and for all $f \in B(X, E)$ (cf. [11, Lemma 1], [12, Lemma 2.4]).

We set

$$c(p, r) := \begin{cases} r^{2/p} & (1 \leq p < \infty,\ p \neq 2) \\ 1 & (p = 2,\ \infty), \end{cases}$$

and let $\{\epsilon_\alpha\}_{\alpha \in D}$ be a net of positive real numbers.

Theorem 3. *For all $f \in BC(X, E), x = (x_1, x_2, \dots, x_r) \in X$ and for all $\alpha \in D$,*

$$\|F_\alpha(f)(x) - f(x)\| \leq (1 + \eta_\alpha(x))\omega(f, \epsilon_\alpha),$$

where

$$\eta_\alpha(x) = \min\{c(p, r)\epsilon_\alpha^{-2}\theta_\alpha(x),\ \sqrt{c(p, r)}\epsilon_\alpha^{-1}\sqrt{\theta_\alpha(x)}\} \tag{7}$$

and

$$\theta_\alpha(x) = \sum_{i=1}^{r} \frac{1}{b_{n_{\alpha,i}}^2 (g_{n_{\alpha,i}}(x_i) + h_{n_{\alpha,i}}(x_i))^2}$$
$$\times \Big(\big(b_{n_{\alpha,i}} x_i g_{n_{\alpha,i}}(x_i)\big)^2 + n_{\alpha,i} g_{n_{\alpha,i}}(x_i) h_{n_{\alpha,i}}(x_i)$$
$$+ 2 b_{n_{\alpha,i}} x_i g_{n_{\alpha,i}}(x_i) \big(b_{n_{\alpha,i}} x_i h_{n_{\alpha,i}}(x_i) - n_{\alpha,i} g_{n_{\alpha,i}}(x_i)\big)$$
$$+ \big(b_{n_{\alpha,i}} x_i h_{n_{\alpha,i}}(x_i) - n_{\alpha,i} g_{n_{\alpha,i}}(x_i)\big)^2 \Big).$$

Corollary 1. *Let* $a_{n_{\alpha,i}}$ $(\alpha \in D,\ i = 1, 2, \ldots, r)$ *be as in (5). Then for all* $f \in BC(X, E), x = (x_1, x_2, \ldots, x_r) \in X$ *and for all* $\alpha \in D$,

$$\|T_\alpha(f)(x) - f(x)\| \le (1 + \eta_\alpha(x))\omega(f, \epsilon_\alpha),$$

where $\eta_\alpha(x)$ *is given by (7) and*

$$\theta_\alpha(x) = \sum_{i=1}^{r} \frac{a_{n_{\alpha,i}}^2 x_i^4 + x_i/b_{n_{\alpha,i}}}{(1 + a_{n_{\alpha,i}} x_i)^2}.$$

Remark 2. Corollary 1 sharply extends and improves [4, Theorem 1] to the very general settings.

Theorem 4. *For all* $f \in BC(X, E), x = (x_1, x_2, \ldots, x_r) \in X$ *and for all* $\alpha \in D$,

$$\|R_\alpha(f)(x) - f(x)\| \le (1 + \gamma_\alpha(x))\omega(f, \epsilon_\alpha),$$

where

$$\gamma_\alpha(x) = \min\{c(p, r)\epsilon_\alpha^{-2}\nu_\alpha(x),\ \sqrt{c(p, r)}\epsilon_\alpha^{-1}\sqrt{\nu_\alpha(x)}\}$$

and

$$\nu_\alpha(x) = \sum_{i=1}^{r} \sum_{k_i=0}^{n_{\alpha,i}} \binom{n_{\alpha,i}}{k_i} \frac{(a_{n_{\alpha,i}} x_i)^{k_i}}{(1 + a_{n_{\alpha,i}} x_i)^{n_{\alpha,i}}} (x_i - b_{n_{\alpha,i},k_i})^2.$$

Theorem 5. *For all* $f \in BC(X, E), x = (x_1, x_2, \ldots, x_r) \in X$ *and for all* $\alpha \in D$,

$$\|L_\alpha(f)(x) - f(x)\| \le (1 + \zeta_\alpha(x))\omega(f, \epsilon_\alpha),$$

where

$$\zeta_\alpha(x) = \min\{c(p, r)\epsilon_\alpha^{-2}\psi_\alpha(x),\ \sqrt{c(p, r)}\epsilon_\alpha^{-1}\sqrt{\psi_\alpha(x)}\}$$

and

$$\psi_\alpha(x) = \sum_{i=1}^{r} \sum_{k_i=0}^{n_{\alpha,i}} \binom{n_{\alpha,i}}{k_i} \frac{x_i^{k_i}}{(1 + x_i)^{n_{\alpha,i}}} \left(x_i - \frac{k_i}{n_{\alpha,i} - k_i + 1}\right)^2. \tag{8}$$

Remark 3. By [6, Remark 3] (cf. [8, (6)]), we have the the following more explicit expression for the second (absolute) moment (8) of L_α:

$$\psi_\alpha(x) = \sum_{i=1}^{r} \frac{(x_i - n_{\alpha,i}) x_i^{n_{\alpha,i}+1}}{(1 + x_i)^{n_{\alpha,i}}} + \frac{x_i^{n_{\alpha,i}+1}}{(1 + x_i)^{n_{\alpha,i}}} \sum_{k_i=2}^{n_{\alpha,i}+1} \binom{n_{\alpha,i} + 1}{k_i} \frac{x_i^{1-k_i}}{k_i - 1}.$$

Theorem 6. *For all $f \in BC(X), x = (x_1, x_2, \dots, x_r) \in X$ and for all $\alpha \in D$,*

$$\|L_\alpha(f)(x) - f(x)\| \leq (1 + \kappa_\alpha(x))\omega(f, \epsilon_\alpha), \tag{9}$$

where

$$\kappa_\alpha(x) = \min\{c(p,r)\epsilon_\alpha^{-2}\sigma_\alpha(x),\ \sqrt{c(p,r)}\epsilon_\alpha^{-1}\sqrt{\sigma_\alpha(x)}\}$$

and

$$\sigma_\alpha(x) = 4\sum_{i=1}^{r} \frac{x_i(1+x_i)^2}{n_{\alpha,i}}.$$

Remark 4. By putting $\epsilon_\alpha\sqrt{\sigma_\alpha(x)}$ instead of ϵ_α in (9), we get the following inequality for all $f \in BC(X, E), x \in X$ and for all $\alpha \in D$:

$$\|L_\alpha(f)(x) - f(x)\| \leq (1 + \min\{c(p,r)\epsilon_\alpha^{-2}, \sqrt{c(p,r)}\epsilon_\alpha^{-1}\}) \times \omega\Big(f, 2\epsilon_\alpha\sqrt{\sum_{i=1}^{r} \frac{x_i(x_i+1)^2}{n_{\alpha,i}}}\Big). \tag{10}$$

In particular, if $p = 2, \infty$, then (10) reduces to

$$\|L_\alpha(f)(x) - f(x)\| \leq (1 + \min\{\epsilon_\alpha^{-1},\ \epsilon_\alpha^{-2}\}) \times \omega\Big(f, 2\epsilon_\alpha\sqrt{\sum_{i=1}^{r} \frac{x_i(x_i+1)^2}{n_{\alpha,i}}}\Big),$$

which generalizes the estimate given by Khan [8, Theorem 1].

Remark 5. We set

$$M(x) = \max\{p_i(x)(1+p_i(x))^2 : i = 1, 2, \dots, r\} \qquad (x \in X).$$

Then (10) yields the following estimate for all $f \in BC(X, E), x \in X$ and for all $\alpha \in D$:

$$\|L_\alpha(f)(x) - f(x)\| \leq \Big(1 + \min\Big\{\frac{4c(p,r)M(x)}{\epsilon_\alpha^2},\ \frac{2\sqrt{c(p,r)}\sqrt{M(x)}}{\epsilon_\alpha}\Big\}\Big)\omega\Big(f, \epsilon_\alpha\sqrt{\sum_{i=1}^{r} \frac{1}{n_{\alpha,i}}}\Big), \tag{11}$$

which particularly reduces to

$$\|L_\alpha(f)(x) - f(x)\|$$

$$\leq \Big(1+\min\Big\{\frac{4M(x)}{\epsilon_\alpha^2},\ \frac{2\sqrt{M(x)}}{\epsilon_\alpha}\Big\}\Big)\omega\Big(f,\epsilon_\alpha\sqrt{\sum_{i=1}^{r}\frac{1}{n_{\alpha,i}}}\Big)$$

if $p=2,\infty$.

Remark 6. If

$$n_{\alpha,i}=n_\alpha \qquad (\alpha\in D,\ i=1,2,\dots,r),$$

where $\{n_\alpha\}_{\alpha\in D}$ is a net of natural numbers, then by (11) we obtain the following estimate for all $f\in BC(X,E), x\in X$ and for all $\alpha\in D$:

$$\|L_\alpha(f)(x)-f(x)\| \tag{12}$$

$$\leq \Big(1+\min\Big\{4rc(p,r)M(x),\ 2\sqrt{rc(p,r)}\sqrt{M(x)}\Big\}\Big)\omega\Big(f,\sqrt{\frac{1}{n_\alpha}}\Big).$$

In particular, if $p=2,\infty$, then (12) reduces to

$$\|L_\alpha(f)(x)-f(x)\| \leq \Big(1+\min\Big\{4rM(x),\ 2\sqrt{r}\sqrt{M(x)}\Big\}\Big)\omega\Big(f,\sqrt{\frac{1}{n_\alpha}}\Big).$$

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