<table>
<thead>
<tr>
<th>Title</th>
<th>Approximation processes by weighted interpolation type operators (Nonlinear Analysis and Convex Analysis)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Nishishiraho, Toshihiko</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2009), 1643: 183-192</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2009-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/140620">http://hdl.handle.net/2433/140620</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
<tr>
<td>Creator</td>
<td>Kyoto University</td>
</tr>
</tbody>
</table>
Approximation processes by weighted interpolation type operators

Toshihiko Nishishiraho (西白保 敏彦)
Department of Mathematical Science, Faculty of Science
University of the Ryukyus, Okinawa, Japan

1. Introduction

Let $\mathbb{N}$ denote the set of all natural numbers. Let $g$ be a real-valued continuous function on the closed unit interval $I = [0, 1]$ of the real line $\mathbb{R}$ and let $n \in \mathbb{N}$. Then $n$th Bernstein polynomial of $g$ is defined by

$$B_n(g)(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} g\left(\frac{k}{n}\right) \quad (t \in I).$$

It is well known that the sequence $\{B_n(g)\}_{n \in \mathbb{N}}$ converges uniformly to $g$ on $I$, and the Bernstein polynomials and their generalizations play an important role in approximation theory (see, e.g., [1], [2], [7], [9], [10]).

In view of these concernsments, Balázs [3] introduced and studied several approximation properties of the Bernstein type rational functions defined as follows:

Let $f$ be a real-valued function on $[0, \infty)$ and let $n \in \mathbb{N}$, and define

$$R_n(f; x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} \binom{n}{k} (a_n x)^k f\left(\frac{k}{b_n}\right) \quad (x \in [0, \infty)),$$

where $a = \{a_n\}_{n \in \mathbb{N}}$ and $b = \{b_n\}_{n \in \mathbb{N}}$ are suitably chosen sequences of positive real numbers. To compare (1) and (2), setting

$$q_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k} \quad (t \in I, \ k = 0, 1, \ldots, n)$$
and

\[ r_{n,k}(x) = \binom{n}{k} \frac{(a_{n}x)^{k}}{(1 + a_{n}x)^{n}} \quad (x \in [0, \infty), \ k = 0, 1, \ldots, n), \]

we have

\[ r_{n,k}(x) = q_{n,k} \left( \frac{a_{n}x}{1 + a_{n}x} \right) \quad (x \in [0, \infty), \ k = 0, 1, \ldots, n), \]

and so

\[ R_{n}(f; x) = B_{n}(f|_{II}) \left( \frac{a_{n}x}{1 + a_{n}x} \right), \]

where \( f|_{II} \) denotes the restriction of \( f \) to \( II \).

In [4], the estimate of the rate of convergence of \( R_{n}(f; x) \) to \( f(x) \) given in [3] is improved by an appropriate choice of \( a \) and \( b \) when \( f \) satisfies some more restrictive conditions. Furthermore, in [14] the saturation problem is discussed for \( \{R_{n}\}_{n \in \mathbb{N}} \) and the uniform approximation problem is considered for \( R_{n} \)-like rational functions defined by

\[ R_{n}(B; a; f; x) = \frac{1}{(1 + a_{n}x)^{n}} \sum_{k=0}^{n} \binom{n}{k} (a_{n}x)^{k} f(b_{n,k}) \]

\[ = \sum_{k=0}^{n} r_{n,k}(x) f(b_{n,k}) \quad (x \in [0, \infty)), \]

where \( B = (b_{n,k})_{0 \leq k \leq n (n = 1, 2, \ldots)} \) is a matrix whose entries satisfy

\[ 0 \leq b_{n,0} < b_{n,1} < b_{n,2} < \cdots < b_{n,n}, \]

and \( f \) is a real-valued continuous function on \([0, \infty)\) for which \( \lim_{x \to \infty} f(x) \) exists. Note that if

\[ a_{n} = 1 \quad (n \in \mathbb{N}), \]

and if

\[ b_{n,k} = \frac{k}{n - k + 1} \quad (0 \leq k \leq n, \ n \in \mathbb{N}), \]

then (3) reduces to

\[ L_{n}(f)(x) := \frac{1}{(1 + x)^{n}} \sum_{k=0}^{n} \binom{n}{k} x^{k} f \left( \frac{k}{n - k + 1} \right), \]

which was introduced by Bleimann, Butzer and Hahn [5]. In [15], the saturation properties of the sequence \( \{L_{n}\}_{n \in \mathbb{N}} \) is established
and it is showed that these operators satisfy an asymptotic relation of the Voronovskaja type, i.e.,
\[
\lim_{n \to \infty} n(L_n(x_0) - f(x_0)) = f''(x_0)x_0(1 + x_0)^2
\]
if \( f \) is a real-valued continuous function on \([0, \infty)\) for which \( \lim_{x \to \infty} f(x) \) exists and the second derivative \( f''(x_0) \) exists at a point \( x_0 \).

Let \( 1 \leq p \leq \infty \) be fixed and let \( \mathbb{R}^r \) denote the metric linear space of all \( r \)-tuples of real numbers, equipped with the usual metric
\[
d_p(x, y) := \begin{cases} 
\left( \sum_{i=1}^{r} |x_i - y_i|^p \right)^{1/p} & (1 \leq p < \infty) \\
\max\{|x_i - y_i| : 1 \leq i \leq r\} & (p = \infty),
\end{cases}
\]
\((x = (x_1, x_2, \ldots, x_r), y = (y_1, y_2, \ldots, y_r) \in \mathbb{R}^r)\).

The purpose of this paper is to generalize (2) for vector-valued functions on the \( r \)-dimensional first hyperquadrant
\([0, \infty)^r := \{x = (x_1, x_2, \ldots, x_r) \in \mathbb{R}^r : x_i \geq 0, i = 1, 2, \ldots, r\}\)
and to consider their uniform convergence with rates in terms of the modulus of continuity of functions to be approximated. We refer to [13] for details.

2. Convergence theorems

Let
\[(X, d) := ([0, \infty)^r, d_p),\]
and let \((E, \| \cdot \|)\) be a normed linear space. Let \( B(X, E) \) denote the normed linear space of all \( E \)-valued bounded functions on \( X \) with the supremum norm \( \| \cdot \|_X \). Also, we denote by \( C(X, E) \) the linear space consisting of all \( E \)-valued continuous functions on \( X \) and set \( BC(X, E) = B(X, E) \cap C(X, E) \).

Let \( \{n_{\alpha, i}\}_{\alpha \in D}, i = 1, 2, \ldots, r \), be nets of positive integers and let \( \{b_{n_{\alpha, i}}\}_{\alpha \in D}, i = 1, 2, \ldots, r \), be nets of positive real numbers such that
\[
\lim_{\alpha} b_{n_{\alpha, i}} = +\infty \quad (i = 1, 2, \ldots, r).
\]
Let \( \{g_{n_{\alpha, i}}\}_{\alpha \in D} \) and \( \{h_{n_{\alpha, i}}\}_{\alpha \in D}, i = 1, 2, \ldots, r \), be nets of nonnegative functions in \( C([0, \infty), \mathbb{R}) \) such that
\[
\inf\{g_{n_{\alpha, i}}(t) + h_{n_{\alpha, i}}(t) : t \in [0, \infty)\} > 0
\]
for all $\alpha \in D$ and for $i = 1, 2, \ldots, r$. Then we define
\[ F_\alpha(f)(x) = F_\alpha(E; f; x) = \prod_{i=1}^{r} \frac{1}{(g_{n_{\alpha,i}}(x_i) + h_{n_{\alpha,i}}(x_i))^{n_{\alpha,i}}} \]
\[ \times \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}} \prod_{i=1}^{r} \rho_{n_{\alpha,i},k_i}(x_i) f \left( \frac{k_1}{b_{n_{\alpha,1}}}, \frac{k_2}{b_{n_{\alpha,2}}}, \ldots, \frac{k_r}{b_{n_{\alpha,r}}} \right) \]
\[ (\alpha \in D, f \in C(X, E), x = (x_1, x_2, \ldots, x_r) \in X) , \]
where
\[ \rho_{n_{\alpha,i},k_i}(x_i) = \left( \begin{array}{c} n_{\alpha,i} \\ k_i \end{array} \right) g_{n_{\alpha,i}}^{k_i}(x_i) h_{n_{\alpha,i}}^{n_{\alpha,i}-k_i}(x_i) \]
\[ (\alpha \in D, i = 1, 2, \ldots, r) . \]

From now on let $K_i, i = 1, 2, \ldots, r$, be compact subsets of $[0, \infty)$ and we set
\[ X_0 = \prod_{i=1}^{r} K_i . \]

**Theorem 1.** We define
\[ I_{\alpha,i}(t) = \frac{n_{\alpha,i}g_{n_{\alpha,i}}(t)}{b_{n_{\alpha,i}}(g_{n_{\alpha,i}}(t) + h_{n_{\alpha,i}}(t))} \quad (i = 1, 2, \ldots, r, t \in [0, \infty) .) \]
If
\[ \lim_{\alpha} I_{\alpha,i}(t) = t \quad \text{uniformly in } t \in K_i \]
for $i = 1, 2, \ldots, r$, then
\[ \lim_{\alpha} \| F_\alpha(f) - f \|_{X_0} = 0 \]
for all $f \in BC(X, E) . \]

Let
\[ a_{n_{\alpha,i}} := \frac{b_{n_{\alpha,i}}}{n_{\alpha,i}} \quad (\alpha \in D, i = 1, 2, \ldots, r) , \]
and we define
\[ T_\alpha(f)(x) = T_\alpha(E; f; x) = \prod_{i=1}^{r} \frac{1}{(1 + a_{n_{\alpha,i}}x_i)^{n_{\alpha,i}}} \]
\[ \times \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}} \prod_{i=1}^{r} \rho_{n_{\alpha,i},k_i}(x_i) f \left( \frac{k_1}{b_{n_{\alpha,1}}}, \frac{k_2}{b_{n_{\alpha,2}}}, \ldots, \frac{k_r}{b_{n_{\alpha,r}}} \right) , \]
\[ (\alpha \in D, f \in C(X, E), x = (x_1, x_2, \ldots, x_r) \in X) , \]
where
\[ \rho_{n_{\alpha,i},k_i}(x_i) = \binom{n_{\alpha,i}}{k_i} (a_{n_{\alpha,i}}x_i)^{k_i} \quad (\alpha \in D, \, i = 1, 2, \ldots, r). \]

**Theorem 2.** If

\[ a_{n_{\alpha,i}} = o(1) \]

for \( i = 1, 2, \ldots, r \), then

\[ \lim_{\alpha} \| T_{\alpha}(f) - f \|_{X_0} = 0 \]

for all \( f \in BC(X, E) \).

**Remark 1.** (6) generalizes (2) to the \( r \)-dimensional Bernstein type rational vector-valued functions. Also, (3) can be extended by the following form to the \( r \)-dimensional case for vector-valued functions:

\[ R_{\alpha}(f)(x) = R_{\alpha}(E; B; A; f; x) = \prod_{i=1}^{r} \frac{1}{(1 + a_{n_{\alpha,i}}x_i)^{n_{\alpha,i}}} \]

\[ \times \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}} \prod_{i=1}^{r} \binom{n_{\alpha,i}}{k_i} f(b_{n_{\alpha,1},k_1}, b_{n_{\alpha,2},k_2}, \ldots, b_{n_{\alpha,r},k_r}) \]

\( (\alpha \in D, \, f \in C(X, E), \, x = (x_1, x_2, \ldots, x_r) \in X) \),

where

\[ A = \{ a_{n_{\alpha,i}} : \alpha \in D, \, i = 1, 2, \ldots, r \} \]

is a family of positive real numbers and

\[ B = \{ b_{n_{\alpha,i},k_i} : 0 \leq k_i \leq n_{\alpha,i}, \, \alpha \in D, \, i = 1, 2, \ldots, r \} \]

is a family of nonnegative real numbers with

\[ 0 \leq b_{n_{\alpha,i},0} < b_{n_{\alpha,i},1} < b_{n_{\alpha,i},2} < \cdots < b_{n_{\alpha,i},n_{\alpha,i}} \]

\( (\alpha \in D, \, i = 1, 2, \ldots, r) \).

In particular, the operator \( L_n(f)(x) \) defined by (4) is generalized to the \( r \)-dimensional case for vector-valued functions defined as follows:

\[ L_{\alpha}(f)(x) = L_{\alpha}(E; f; x) = \prod_{i=1}^{r} \frac{1}{(1 + x_i)^{n_{\alpha,i}}} \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}} \]

\[ \prod_{i=1}^{r} \binom{n_{\alpha,i}}{k_i} x_i^{k_i} f\left( \frac{k_1}{n_{\alpha,1} - k_1 + 1}, \frac{k_2}{n_{\alpha,2} - k_2 + 1}, \ldots, \frac{k_r}{n_{\alpha,r} - k_r + 1} \right) \]

\( (\alpha \in D, \, f \in C(X, E), \, x = (x_1, x_2, \ldots, x_r) \in X) \).
3. Convergence rates

Let \( f \in B(X, E) \) and let \( \delta \geq 0 \). Then we define
\[
\omega(f, \delta) = \sup \{ \|f(x) - f(y)\| : x, y \in X, d(x, y) \leq \delta \},
\]
which is called the modulus of continuity of \( f \). Obviously, \( \omega(f, \cdot) \) is a monotone increasing function on \([0, \infty)\) and
\[
\omega(f, 0) = 0, \quad \omega(f, \delta) \leq 2\|f\|_X \quad (\delta \geq 0).
\]
Also, \( f \) is uniformly continuous on \( X \) if and only if
\[
\lim_{\delta \to +0} \omega(f, \delta) = 0.
\]
Furthermore, the convexity of \( d \) and \( X \) yields the inequality
\[
\omega(f, \xi \delta) \leq (1 + \xi) \omega(f, \delta)
\]
for all \( \xi, \delta \geq 0 \) and for all \( f \in B(X, E) \) (cf. [11, Lemma 1], [12, Lemma 2.4]).

We set
\[
c(p, r) := \begin{cases} 
    r^{2/p} & (1 \leq p < \infty, p \neq 2) \\
    1 & (p = 2, \infty),
\end{cases}
\]
and let \( \{\epsilon_{\alpha}\}_{\alpha \in D} \) be a net of positive real numbers.

**Theorem 3.** For all \( f \in BC(X, E), x = (x_1, x_2, \ldots, x_r) \in X \) and for all \( \alpha \in D \),
\[
\|F_{\alpha}(f)(x) - f(x)\| \leq (1 + \eta_{\alpha}(x)) \omega(f, \epsilon_{\alpha}),
\]
where
\[
(7) \quad \eta_{\alpha}(x) = \min \{c(p, r)\epsilon_{\alpha}^{-2} \theta_{\alpha}(x), \sqrt{c(p, r)\epsilon_{\alpha}^{-1} \sqrt{\theta_{\alpha}(x)}}\}
\]
and
\[
\theta_{\alpha}(x) = \sum_{i=1}^{r} \frac{1}{b_{n_{\alpha,i}}^{2}(g_{n_{\alpha,i}}(x_i) + h_{n_{\alpha,i}}(x_i))^2}
\times \left( (b_{n_{\alpha,i}}x_i g_{n_{\alpha,i}}(x_i))^2 + n_{\alpha,i} g_{n_{\alpha,i}}(x_i) h_{n_{\alpha,i}}(x_i) 
+ 2b_{n_{\alpha,i}} x_i g_{n_{\alpha,i}}(x_i) (b_{n_{\alpha,i}} x_i h_{n_{\alpha,i}}(x_i) - n_{\alpha,i} g_{n_{\alpha,i}}(x_i)) 
+ (b_{n_{\alpha,i}} x_i h_{n_{\alpha,i}}(x_i) - n_{\alpha,i} g_{n_{\alpha,i}}(x_i))^2 \right).
\]
Corollary 1. Let $a_{n_{\alpha,i}}$ ($\alpha \in D$, $i = 1, 2, \ldots, r$) be as in (5). Then for all $f \in BC(X, E), \ x = (x_1, x_2, \ldots, x_r) \in X$ and for all $\alpha \in D$,
\[
\|T_{\alpha}(f)(x) - f(x)\| \leq (1 + \eta_{\alpha}(x)) \omega(f, \epsilon_{\alpha}),
\]
where $\eta_{\alpha}(x)$ is given by (7) and
\[
\theta_{\alpha}(x) = \sum_{i=1}^{r} \frac{a_{n_{\alpha,i}}^2 x_i^4 + x_i/b_{n_{\alpha,i}}}{(1 + a_{n_{\alpha,i}}x_i)^2}.
\]

Remark 2. Corollary 1 sharply extends and improves [4, Theorem 1] to the very general settings.

Theorem 4. For all $f \in BC(X, E), \ x = (x_1, x_2, \ldots, x_r) \in X$ and for all $\alpha \in D,$
\[
\|R_{\alpha}(f)(x) - f(x)\| \leq (1 + \gamma_{\alpha}(x)) \omega(f, \epsilon_{\alpha}),
\]
where
\[
\gamma_{\alpha}(x) = \min\{c(p, r)\epsilon_{\alpha}^{-2} \nu_{\alpha}(x), \sqrt{c(p, r)} \epsilon_{\alpha}^{-1} \sqrt{\nu_{\alpha}(x)}\}
\]
and
\[
\nu_{\alpha}(x) = \sum_{i=1}^{r} \sum_{k_i=0}^{n_{\alpha,i}} \binom{n_{\alpha,i}}{k_i} \frac{(a_{n_{\alpha,i}}x_i)^{k_i}}{(1 + a_{n_{\alpha,i}}x_i)^{n_{\alpha,i}}}(x_i - b_{n_{\alpha,i},k_i})^2.
\]

Theorem 5. For all $f \in BC(X, E), \ x = (x_1, x_2, \ldots, x_r) \in X$ and for all $\alpha \in D,$
\[
\|L_{\alpha}(f)(x) - f(x)\| \leq (1 + \zeta_{\alpha}(x)) \omega(f, \epsilon_{\alpha}),
\]
where
\[
\zeta_{\alpha}(x) = \min\{c(p, r)\epsilon_{\alpha}^{-2} \psi_{\alpha}(x), \sqrt{c(p, r)} \epsilon_{\alpha}^{-1} \sqrt{\psi_{\alpha}(x)}\}
\]
and
\[
(8) \quad \psi_{\alpha}(x) = \sum_{i=1}^{r} \sum_{k_i=0}^{n_{\alpha,i}} \binom{n_{\alpha,i}}{k_i} \frac{x_i^{k_i}}{(1 + x_i)^{n_{\alpha,i}}}(x_i - \frac{k_i}{n_{\alpha,i} + 1})^2.
\]

Remark 3. By [6, Remark 3] (cf. [8, (6)]), we have the the following more explicit expression for the second (absolute) moment (8) of $L_{\alpha}$:
\[
\psi_{\alpha}(x) = \sum_{i=1}^{r} \frac{(x_i - n_{\alpha,i})x_i^{n_{\alpha,i}+1}}{(1 + x_i)^{n_{\alpha,i}}} + \frac{x_i^{n_{\alpha,i}+1}}{(1 + x_i)^{n_{\alpha,i}} \sum_{k_i=2}^{n_{\alpha,i}+1} \binom{n_{\alpha,i}+1}{k_i} \frac{x_i^{1-k_i}}{k_i - 1}}.
\]
Theorem 6. For all $f \in BC(X), x = (x_1, x_2, \ldots, x_r) \in X$ and for all $\alpha \in D$,

(9) \[ \|L_\alpha(f)(x) - f(x)\| \leq (1 + \kappa_\alpha(x))\omega(f, \epsilon_\alpha), \]

where \[ \kappa_\alpha(x) = \min\{c(p, r)\epsilon_\alpha^{-2}\sigma_\alpha(x), \sqrt{c(p, r)}\epsilon_\alpha^{-1}\sqrt{\sigma_\alpha(x)}\} \]

and \[ \sigma_\alpha(x) = 4\sum_{i=1}^{r} \frac{x_i(1+x_i)^2}{n_{\alpha,i}}. \]

Remark 4. By putting $\epsilon_\alpha\sqrt{\sigma_\alpha(x)}$ instead of $\epsilon_\alpha$ in (9), we get the following inequality for all $f \in BC(X, E), x \in X$ and for all $\alpha \in D$:

(10) \[ \|L_\alpha(f)(x) - f(x)\| \leq (1 + \min\{c(p, r)\epsilon_\alpha^{-2}, \sqrt{c(p, r)}\epsilon_\alpha^{-1}\}) \times \omega\left(f, 2\epsilon_\alpha\sqrt{\sum_{i=1}^{r} \frac{x_i(x_i + 1)^2}{n_{\alpha,i}}}\right). \]

In particular, if $p = 2, \infty$, then (10) reduces to

\[ \|L_\alpha(f)(x) - f(x)\| \leq (1 + \min\{\epsilon_\alpha^{-1}, \epsilon_\alpha^{-2}\}) \times \omega\left(f, 2\epsilon_\alpha\sqrt{\sum_{i=1}^{r} \frac{x_i(x_i + 1)^2}{n_{\alpha,i}}}\right), \]

which generalizes the estimate given by Khan [8, Theorem 1].

Remark 5. We set

\[ M(x) = \max\{p_i(x)(1 + p_i(x))^2 : i = 1, 2, \ldots, r\} \quad (x \in X). \]

Then (10) yields the following estimate for all $f \in BC(X, E), x \in X$ and for all $\alpha \in D$:

(11) \[ \|L_\alpha(f)(x) - f(x)\| \leq \left(1 + \min\left\{\frac{4c(p, r)M(x)}{\epsilon_\alpha^2}, \frac{2\sqrt{c(p, r)\sqrt{M(x)}}}{\epsilon_\alpha}\right\}\right)\omega\left(f, \epsilon_\alpha\sqrt{\sum_{i=1}^{r} \frac{1}{n_{\alpha,i}}}\right), \]

which particularly reduces to

\[ \|L_\alpha(f)(x) - f(x)\| \]
\[
\leq \left( 1 + \min \left\{ \frac{4M(x)}{\epsilon_\alpha^2}, \frac{2\sqrt{M(x)}}{\epsilon_\alpha} \right\} \right) \omega \left( f, \epsilon_\alpha \sqrt{\sum_{i=1}^{r} \frac{1}{n_{\alpha,i}}} \right)
\]

if \( p = 2, \infty \).

**Remark 6.** If

\[
n_{\alpha,i} = n_\alpha \quad (\alpha \in D, \ i = 1, 2, \ldots, r),
\]

where \( \{n_\alpha\}_{\alpha \in D} \) is a net of natural numbers, then by (11) we obtain the following estimate for all \( f \in BC(X, E), x \in X \) and for all \( \alpha \in D \):

(12)

\[
\|L_\alpha(f)(x) - f(x)\| \leq \left( 1 + \min \left\{ 4rc(p, r)M(x), 2\sqrt{rc(p, r)}\sqrt{M(x)} \right\} \right) \omega \left( f, \sqrt{\frac{1}{n_\alpha}} \right).
\]

In particular, if \( p = 2, \infty \), then (12) reduces to

\[
\|L_\alpha(f)(x) - f(x)\| \leq \left( 1 + \min \left\{ 4rM(x), 2\sqrt{r}\sqrt{M(x)} \right\} \right) \omega \left( f, \sqrt{\frac{1}{n_\alpha}} \right).
\]

**References**


