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Approximation processes by weighted interpolation type operators

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1. Introduction

Let $\mathbb{N}$ denote the set of all natural numbers. Let $g$ be a real-valued continuous function on the closed unit interval $I = [0, 1]$ of the real line $\mathbb{R}$ and let $n \in \mathbb{N}$. Then $n$th Bernstein polynomial of $g$ is defined by

\[ B_n(g)(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} g\left(\frac{k}{n}\right) \quad (t \in I). \]

It is well known that the sequence $\{B_n(g)\}_{n\in \mathbb{N}}$ converges uniformly to $g$ on $I$, and the Bernstein polynomials and their generalizations play an important role in approximation theory (see, e.g., [1], [2], [7], [9], [10]).

In view of these concerns, Balázs [3] introduced and studied several approximation properties of the Bernstein type rational functions defined as follows:

Let $f$ be a real-valued function on $[0, \infty)$ and let $n \in \mathbb{N}$, and define

\[ R_n(f; x) = \frac{1}{(1 + a_n x)^n} \sum_{k=0}^{n} \binom{n}{k} (a_n x)^k f\left(\frac{k}{b_n}\right) \quad (x \in [0, \infty)), \]

where $a = \{a_n\}_{n \in \mathbb{N}}$ and $b = \{b_n\}_{n \in \mathbb{N}}$ are suitably chosen sequences of positive real numbers. To compare (1) and (2), setting

\[ q_n,k(t) = \binom{n}{k} t^k (1-t)^{n-k} \quad (t \in I, \; k = 0, 1, \ldots, n) \]
and
\[ r_{n,k}(x) = \binom{n}{k} \frac{(a_{n}x)^{k}}{(1+a_{n}x)^{n}} \quad (x \in [0, \infty), \ k = 0, 1, \ldots, n), \]
we have
\[ r_{n,k}(x) = q_{n,k} \left( \frac{a_{n}x}{1+a_{n}x} \right) \quad (x \in [0, \infty), \ k = 0, 1, \ldots, n), \]
and so
\[ R_{n}(f; x) = B_{n}(f|_{II}) \left( \frac{a_{n}x}{1+a_{n}x} \right), \]
where \( f|_{II} \) denotes the restriction of \( f \) to \( II \).

In [4], the estimate of the rate of convergence of \( R_{n}(f; x) \) to \( f(x) \) given in [3] is improved by an appropriate choice of \( a \) and \( b \) when \( f \) satisfies some more restrictive conditions. Furthermore, in [14] the saturation problem is discussed for \( \{R_{n}\}_{n \in \mathbb{N}} \) and the uniform approximation problem is considered for \( R_{n} \)-like rational functions defined by

(3) \[ R_{n}(B; a; f; x) = \frac{1}{(1+a_{n}x)^{n}} \sum_{k=0}^{n} \binom{n}{k} (a_{n}x)^{k} f(b_{n,k}) \]
\[ = \sum_{k=0}^{n} r_{n,k}(x) f(b_{n,k}) \quad (x \in [0, \infty)), \]
where \( B = (b_{n,k})_{0 \leq k \leq n(n=1,2,\ldots)} \) is a matrix whose entries satisfy
\[ 0 \leq b_{n,0} < b_{n,1} < b_{n,2} < \cdots < b_{n,n}, \]
and \( f \) is a real-valued continuous function on \([0, \infty)\) for which \( \lim_{x \to \infty} f(x) \) exists. Note that if
\[ a_{n} = 1 \quad (n \in \mathbb{N}), \]
and if
\[ b_{n,k} = \frac{k}{n-k+1} \quad (0 \leq k \leq n, \ n \in \mathbb{N}), \]
then (3) reduces to

(4) \[ L_{n}(f)(x) := \frac{1}{(1+x)^{n}} \sum_{k=0}^{n} \binom{n}{k} x^{k} f \left( \frac{k}{n-k+1} \right), \]
which was introduced by Bleimann, Butzer and Hahn [5]. In [15], the saturation properties of the sequence \( \{L_{n}\}_{n \in \mathbb{N}} \) is established
and it is showed that these operators satisfy an asymptotic relation of the Voronovskaja type, i.e.,

$$\lim_{n \to \infty} n(L_n(x_0) - f(x_0)) = f''(x_0)x_0(1 + x_0)^2$$

if $f$ is a real-valued continuous function on $[0, \infty)$ for which $\lim_{x \to \infty} f(x)$ exists and the second derivative $f''(x_0)$ exists at a point $x_0$.

Let $1 \leq p \leq \infty$ be fixed and let $\mathbb{R}^r$ denote the metric linear space of all $r$-tuples of real numbers, equipped with the usual metric

$$d_p(x, y) := \begin{cases} \left( \sum_{i=1}^{r} |x_i - y_i|^p \right)^{1/p} & (1 \leq p < \infty) \\ \max\{|x_i - y_i| : 1 \leq i \leq r\} & (p = \infty), \end{cases}$$

$(x = (x_1, x_2, \ldots, x_r), y = (y_1, y_2, \ldots, y_r) \in \mathbb{R}^r)$.

The purpose of this paper is to generalize (2) for vector-valued functions on the $r$-dimensional first hyperquadrant

$$[0, \infty)^r := \{x = (x_1, x_2, \ldots, x_r) \in \mathbb{R}^r : x_i \geq 0, i = 1, 2, \ldots, r\}$$

and to consider their uniform convergence with rates in terms of the modulus of continuity of functions to be approximated. We refer to [13] for details.

### 2. Convergence theorems

Let

$$(X, d) := ([0, \infty)^r, d_p),$$

and let $(E, \| \cdot \|)$ be a normed linear space. Let $B(X, E)$ denote the normed linear space of all $E$-valued bounded functions on $X$ with the supremum norm $\| \cdot \|_X$. Also, we denote by $C(X, E)$ the linear space consisting of all $E$-valued continuous functions on $X$ and set $BC(X, E) = B(X, E) \cap C(X, E)$.

Let $\{n_{\alpha,i}\}_{\alpha \in D}, i = 1, 2, \ldots, r$, be nets of positive integers and let $\{b_{n_{\alpha,i}}\}_{\alpha \in D}, i = 1, 2, \ldots, r$, be nets of positive real numbers such that

$$\lim_{\alpha} b_{n_{\alpha,i}} = +\infty \quad (i = 1, 2, \ldots, r).$$

Let $\{g_{n_{\alpha,i}}\}_{\alpha \in D}$ and $\{h_{n_{\alpha,i}}\}_{\alpha \in D}, i = 1, 2, \ldots, r$, be nets of nonnegative functions in $C([0, \infty), \mathbb{R})$ such that

$$\inf\{g_{n_{\alpha,i}}(t) + h_{n_{\alpha,i}}(t) : t \in [0, \infty)\} > 0.$$
for all $\alpha \in D$ and for $i = 1, 2, \ldots, r$. Then we define

$$F_{\alpha}(f)(x) = F_{\alpha}(E; f; x) = \prod_{i=1}^{r} \frac{1}{(g_{n_{\alpha,i}}(x_{i}) + h_{n_{\alpha,i}}(x_{i}))^{n_{\alpha,i}}}$$

$$\times \sum_{k_{1}=0}^{n_{\alpha,1}} \sum_{k_{2}=0}^{n_{\alpha,2}} \cdots \sum_{k_{r}=0}^{n_{\alpha,r}} \prod_{i=1}^{r} \rho_{n_{\alpha,i},k_{i}}(x_{i}) f \left( \frac{k_{1}}{b_{n_{\alpha,1}}}, \frac{k_{2}}{b_{n_{\alpha,2}}}, \ldots, \frac{k_{r}}{b_{n_{\alpha,r}}} \right)$$

where

$$\rho_{n_{\alpha,i},k_{i}}(x_{i}) = \binom{n_{\alpha,i}k_{i}}{k_{i}} g_{n_{\alpha,i}}^{k_{i}}(x_{i}) h_{n_{\alpha,i}}^{n_{\alpha,i}-k_{i}}(x_{i})$$

($\alpha \in D, i = 1, 2, \ldots, r$).

From now on let $K_{i}, i = 1, 2, \ldots, r$, be compact subsets of $[0, \infty)$ and we set

$$X_{0} = \prod_{i=1}^{r} K_{i}.$$  

**Theorem 1.** We define

$$I_{\alpha,i}(t) = \frac{n_{\alpha,i}g_{n_{\alpha,i}}(t)}{b_{n_{\alpha,i}}(g_{n_{\alpha,i}}(t) + h_{n_{\alpha,i}}(t))} \quad (i = 1, 2, \ldots, r, t \in [0, \infty)).$$

If

$$\lim_{\alpha} I_{\alpha,i}(t) = t \quad \text{uniformly in } t \in K_{i}$$

for $i = 1, 2, \ldots, r$, then

$$\lim_{\alpha} \| F_{\alpha}(f) - f \|_{X_{0}} = 0$$

for all $f \in BC(X, E)$.

Let

$$a_{n_{\alpha,i}} := \frac{b_{n_{\alpha,i}}}{n_{\alpha,i}} \quad (\alpha \in D, i = 1, 2, \ldots, r),$$

and we define

$$T_{\alpha}(f)(x) = T_{\alpha}(E; f; x) = \prod_{i=1}^{r} \frac{1}{(1 + a_{n_{\alpha,i}}x_{i})^{n_{\alpha,i}}}$$

$$\times \sum_{k_{1}=0}^{n_{\alpha,1}} \sum_{k_{2}=0}^{n_{\alpha,2}} \cdots \sum_{k_{r}=0}^{n_{\alpha,r}} \prod_{i=1}^{r} \rho_{n_{\alpha,i},k_{i}}(x_{i}) f \left( \frac{k_{1}}{b_{n_{\alpha,1}}}, \frac{k_{2}}{b_{n_{\alpha,2}}}, \ldots, \frac{k_{r}}{b_{n_{\alpha,r}}} \right),$$

($\alpha \in D, f \in C(X, E), x = (x_{1}, x_{2}, \ldots, x_{r}) \in X$),
where
\[ \rho_{n_{\alpha,i},k_{i}}(x_{i}) = \binom{n_{\alpha,i}}{k_{i}}(a_{n_{\alpha,i}}x_{i})^{k_{i}}, \quad (\alpha \in D, i = 1, 2, \ldots, r). \]

**Theorem 2.** If \( a_{n_{\alpha,i}} = o(1) \) for \( i = 1, 2, \ldots, r \), then
\[ \lim_{\alpha} \| T_{\alpha}(f) - f \|_{X_{0}} = 0 \]
for all \( f \in BC(X, E) \).

**Remark 1.** (6) generalizes (2) to the \( r \)-dimensional Bernstein type rational vector-valued functions. Also, (3) can be extended by the following form to the \( r \)-dimensional case for vector-valued functions:

\[ R_{\alpha}(f)(x) = R_{\alpha}(E; B; A; f; x) = \frac{1}{\prod_{i=1}^{r}(1 + a_{n_{\alpha,i}}x_{i})^{n_{\alpha,i}}} \]
\[ \sum_{k_{1}=0}^{n_{\alpha,1}} \sum_{k_{2}=0}^{n_{\alpha,2}} \cdots \sum_{k_{r}=0}^{n_{\alpha,r}} \prod_{i=1}^{r} \binom{n_{\alpha,i}}{k_{i}}(a_{n_{\alpha,i}}x_{i})^{k_{i}}f(b_{n_{\alpha,1},k_{1}}, b_{n_{\alpha,2},k_{2}}, \ldots, b_{n_{\alpha,r},k_{r}}) \]
\( (\alpha \in D, f \in C(X, E), x = (x_{1}, x_{2}, \ldots, x_{r}) \in X) \),

where \( A = \{a_{n_{\alpha,i}} : \alpha \in D, i = 1, 2, \ldots, r\} \) is a family of positive real numbers and
\[ B = \{b_{n_{\alpha,i},k_{i}} : 0 \leq k_{i} \leq n_{\alpha,i}, \alpha \in D, i = 1, 2, \ldots, r\} \]
is a family of nonnegative real numbers with
\[ 0 \leq b_{n_{\alpha,i},0} < b_{n_{\alpha,i},1} < b_{n_{\alpha,i},2} < \cdots < b_{n_{\alpha,i},n_{\alpha,i}} \]
(\( \alpha \in D, i = 1, 2, \ldots, r \)).

In particular, the operator \( L_{\alpha}(f)(x) \) defined by (4) is generalized to the \( r \)-dimensional case for vector-valued functions defined as follows:

\[ L_{\alpha}(f)(x) = L_{\alpha}(E; f; x) = \prod_{i=1}^{r} \frac{1}{(1 + x_{i})^{n_{\alpha,i}}} \sum_{k_{1}=0}^{n_{\alpha,1}} \sum_{k_{2}=0}^{n_{\alpha,2}} \cdots \sum_{k_{r}=0}^{n_{\alpha,r}} \prod_{i=1}^{r} \binom{n_{\alpha,i}}{k_{i}}x_{i}^{k_{i}}f\left(\frac{k_{1}}{n_{\alpha,1} - k_{1} + 1}, \frac{k_{2}}{n_{\alpha,2} - k_{2} + 1}, \ldots, \frac{k_{r}}{n_{\alpha,r} - k_{r} + 1}\right) \]
\( (\alpha \in D, f \in C(X, E), x = (x_{1}, x_{2}, \ldots, x_{r}) \in X) \).
3. Convergence rates

Let $f \in B(X, E)$ and let $\delta \geq 0$. Then we define
\[
\omega(f, \delta) = \sup\{\|f(x) - f(y)\| : x, y \in X, d(x, y) \leq \delta\},
\]
which is called the modulus of continuity of $f$. Obviously, $\omega(f, \cdot)$ is a monotone increasing function on $[0, \infty)$ and
\[
\omega(f, 0) = 0, \quad \omega(f, \delta) \leq 2\|f\|_X \quad (\delta \geq 0).
\]
Also, $f$ is uniformly continuous on $X$ if and only if
\[
\lim_{\delta \to +0} \omega(f, \delta) = 0.
\]
Furthermore, the convexity of $d$ and $X$ yields the inequality
\[
\omega(f, \xi \delta) \leq (1 + \xi)\omega(f, \delta)
\]
for all $\xi, \delta \geq 0$ and for all $f \in B(X, E)$ (cf. [11, Lemma 1], [12, Lemma 2.4]).

We set
\[
c(p, r) := \begin{cases} r^{2/p} & (1 \leq p < \infty, \ p \neq 2) \\ 1 & (p = 2, \infty), \end{cases}
\]
and let $\{\epsilon_\alpha\}_{\alpha \in D}$ be a net of positive real numbers.

**Theorem 3.** For all $f \in BC(X, E), x = (x_1, x_2, \ldots, x_r) \in X$ and for all $\alpha \in D$,
\[
\|F_\alpha(f)(x) - f(x)\| \leq (1 + \eta_\alpha(x))\omega(f, \epsilon_\alpha),
\]
where
\[
\eta_\alpha(x) = \min\{c(p, r)\epsilon_\alpha^{-2}\theta_\alpha(x), \sqrt{c(p, r)}\epsilon_\alpha^{-1}\sqrt{\theta_\alpha(x)}\}
\]
and
\[
\theta_\alpha(x) = \sum_{i=1}^{r} \frac{1}{b_{n_{\alpha,i}}^{2}(g_{n_{\alpha,i}}(x_i) + h_{n_{\alpha,i}}(x_i))^2} \times \left( (b_{n_{\alpha,i}}x_i g_{n_{\alpha,i}}(x_i))^2 + n_{\alpha,i}g_{n_{\alpha,i}}(x_i)h_{n_{\alpha,i}}(x_i) \right. \\
+ 2b_{n_{\alpha,i}}x_i g_{n_{\alpha,i}}(x_i)(b_{n_{\alpha,i}}x_i h_{n_{\alpha,i}}(x_i) - n_{\alpha,i}g_{n_{\alpha,i}}(x_i)) \\
\left. + (b_{n_{\alpha,i}}x_i h_{n_{\alpha,i}}(x_i) - n_{\alpha,i}g_{n_{\alpha,i}}(x_i))^2 \right).
\]
Corollary 1. Let $a_{n_{\alpha,i}} \ (\alpha \in D, \ i = 1, 2, \ldots, r)$ be as in (5). Then for all $f \in BC(X, E)$, $x = (x_1, x_2, \ldots, x_r) \in X$ and for all $\alpha \in D$,
$$\|T_{\alpha}(f)(x) - f(x)\| \leq (1 + \eta_{\alpha}(x))\omega(f, \epsilon_{\alpha}),$$
where $\eta_{\alpha}(x)$ is given by (7) and
$$\theta_{\alpha}(x) = \sum_{i=1}^{r} \frac{a_{n_{\alpha,i}}^2 x_i^4 + x_i/b_{n_{\alpha,i}}}{(1 + a_{n_{\alpha,i}}x_i)^2}.$$

Remark 2. Corollary 1 sharply extends and improves [4, Theorem 1] to the very general settings.

Theorem 4. For all $f \in BC(X, E)$, $x = (x_1, x_2, \ldots, x_r) \in X$ and for all $\alpha \in D$,
$$\|R_{\alpha}(f)(x) - f(x)\| \leq (1 + \gamma_{\alpha}(x))\omega(f, \epsilon_{\alpha}),$$
where
$$\gamma_{\alpha}(x) = \min\{c(p, r)\epsilon_{\alpha}^{-2}\nu_{\alpha}(x), \ \sqrt{c(p, r)\epsilon_{\alpha}^{-1}\sqrt{\nu_{\alpha}(x)}}\}$$
and
$$\nu_{\alpha}(x) = \sum_{i=1}^{r} \sum_{k_{i}=0}^{n_{\alpha,i}} \binom{n_{\alpha,i}}{k_{i}} \frac{(a_{n_{\alpha,i}}x_i)^{k_{i}}}{(1 + a_{n_{\alpha,i}}x_i)^{n_{\alpha,i}}} (x_i - b_{n_{\alpha,i},k_{i}})^2.$$ 

Theorem 5. For all $f \in BC(X, E)$, $x = (x_1, x_2, \ldots, x_r) \in X$ and for all $\alpha \in D$,
$$\|L_{\alpha}(f)(x) - f(x)\| \leq (1 + \zeta_{\alpha}(x))\omega(f, \epsilon_{\alpha}),$$
where
$$\zeta_{\alpha}(x) = \min\{c(p, r)\epsilon_{\alpha}^{-2}\psi_{\alpha}(x), \ \sqrt{c(p, r)\epsilon_{\alpha}^{-1}\sqrt{\psi_{\alpha}(x)}}\}$$
and
$$\psi_{\alpha}(x) = \sum_{i=1}^{r} \sum_{k_{i}=2}^{n_{\alpha,i}+1} \binom{n_{\alpha,i}+1}{k_{i}} \frac{x_i^{1-k_{i}}}{k_{i}-1}.$$ 

Remark 3. By [6, Remark 3] (cf. [8, (6)]), we have the following more explicit expression for the second (absolute) moment (8) of $L_{\alpha}$:
$$\psi_{\alpha}(x) = \sum_{i=1}^{r} \frac{(x_i - n_{\alpha,i})x_i^{n_{\alpha,i}+1}}{(1 + x_i)^{n_{\alpha,i}}} + \frac{x_i^{n_{\alpha,i}+1}}{(1 + x_i)^{n_{\alpha,i}}} \sum_{k_{i}=2}^{n_{\alpha,i}+1} \binom{n_{\alpha,i}+1}{k_{i}} \frac{x_i^{1-k_{i}}}{k_{i}-1}.$$
Theorem 6. For all \( f \in BC(X) \), \( x = (x_1, x_2, \ldots, x_r) \in X \) and for all \( \alpha \in D \),

\[
\|L_\alpha(f)(x) - f(x)\| \leq (1 + \kappa_\alpha(x))\omega(f, \epsilon_\alpha),
\]

where

\[
\kappa_\alpha(x) = \min\{c(p, r)\epsilon_\alpha^{-2}\sigma_\alpha(x), \sqrt{c(p, r)}\epsilon_\alpha^{-1}\sqrt{\sigma_\alpha(x)}\}
\]

and

\[
\sigma_\alpha(x) = 4 \sum_{i=1}^{r} \frac{x_i(1+x_i)^2}{n_{\alpha,i}}.
\]

Remark 4. By putting \( \epsilon_\alpha \sqrt{\sigma_\alpha(x)} \) instead of \( \epsilon_\alpha \) in (9), we get the following inequality for all \( f \in BC(X, E) \), \( x \in X \) and for all \( \alpha \in D \):

\[
\|L_\alpha(f)(x) - f(x)\| \leq (1 + \min\{c(p, r)\epsilon_\alpha^{-2}, \sqrt{c(p, r)}\epsilon_\alpha^{-1}\})
\]

\[
\times \omega\left(f, 2\epsilon_\alpha \sqrt{\sum_{i=1}^{r} \frac{x_i(x_i + 1)^2}{n_{\alpha,i}}} \right).
\]

In particular, if \( p = 2, \infty \), then (10) reduces to

\[
\|L_\alpha(f)(x) - f(x)\| \leq (1 + \min\{\epsilon_\alpha^{-1}, \epsilon_\alpha^{-2}\})
\]

\[
\times \omega\left(f, 2\epsilon_\alpha \sqrt{\sum_{i=1}^{r} \frac{x_i(x_i + 1)^2}{n_{\alpha,i}}} \right),
\]

which generalizes the estimate given by Khan [8, Theorem 1].

Remark 5. We set

\[
M(x) = \max\{p_i(x)(1 + p_i(x))^2 : i = 1, 2, \ldots, r\} \quad (x \in X).
\]

Then (10) yields the following estimate for all \( f \in BC(X, E) \), \( x \in X \) and for all \( \alpha \in D \):

\[
\|L_\alpha(f)(x) - f(x)\|
\]

\[
\leq \left(1 + \min\left\{\frac{4c(p, r)M(x)}{\epsilon_\alpha^2}, \frac{2\sqrt{c(p, r)\sqrt{M(x)}}}{\epsilon_\alpha}\right\}\right) \omega\left(f, \epsilon_\alpha \sqrt{\sum_{i=1}^{r} \frac{1}{n_{\alpha,i}}} \right),
\]

which particularly reduces to

\[
\|L_\alpha(f)(x) - f(x)\|
\]

\[
\leq \left(1 + \min\left\{\frac{4c(p, r)M(x)}{\epsilon_\alpha^2}, \frac{2\sqrt{c(p, r)\sqrt{M(x)}}}{\epsilon_\alpha}\right\}\right) \omega\left(f, \epsilon_\alpha \sqrt{\sum_{i=1}^{r} \frac{1}{n_{\alpha,i}}} \right).
\]
\[
\leq \left(1 + \min\left\{ \frac{4M(x)}{\epsilon_{\alpha}^2}, \frac{2\sqrt{M(x)}}{\epsilon_{\alpha}} \right\}\right) \omega\left(f, \epsilon_{\alpha}, \sqrt{\frac{1}{\sum_{i=1}^{r} \frac{1}{n_{\alpha,i}}}}\right)
\]

if \( p = 2, \infty \).

**Remark 6.** If

\[n_{\alpha,i} = n_{\alpha} \quad (\alpha \in D, \ i = 1, 2, \ldots, r),\]

where \( \{n_{\alpha}\}_{\alpha \in D} \) is a net of natural numbers, then by (11) we obtain the following estimate for all \( f \in BC(X, E), x \in X \) and for all \( \alpha \in D \):

\[(12) \quad ||L_{\alpha}(f)(x) - f(x)|| \leq \left(1 + \min\left\{ 4rc(p, r)M(x), 2\sqrt{rc(p, r)}\sqrt{M(x)} \right\}\right) \omega\left(f, \sqrt{\frac{1}{n_{\alpha}}}\right).\]

In particular, if \( p = 2, \infty \), then (12) reduces to

\[
||L_{\alpha}(f)(x) - f(x)|| \leq \left(1 + \min\left\{ 4rM(x), 2\sqrt{r}\sqrt{M(x)} \right\}\right) \omega\left(f, \sqrt{\frac{1}{n_{\alpha}}}\right).
\]

**References**


