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# SEVERAL NONLINEAR SCALARIZATION METHODS FOR SET-VALUED MAPS*
# (集合値写像に対する様々な非線形スカラー化手法)

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Abstract

In the paper, we introduce several nonlinear scalarization methods for set-valued maps including unified nonlinear scalarizing functions for sets as images of set-valued maps. Also, we investigate their monotonicity and inherited properties on cone-convexity of parent set-valued maps.

## 1 Introduction

Recently, nonlinear scalarization methods for set-valued maps are investigated as one of important tools in set-valued optimization in an analogous fashion to linear scalarizing function like inner product. The theory of set-valued optimization means several researches on continuity and convexity of set-valued objective functions, and on existence results of solutions for certain criteria to such objective functions subject to some restriction. This kind of research has been developed as a generalization of vector-valued optimization for around thirty years. In the paper, we consider scalarizing functions which characterize sets in a vector space. There are two types of scalarizing functions like "linear" and "nonlinear." The importance of such research is as follows. If a real-valued or vector-valued function has some kinds of convexity and continuity, then we can utilize such properties as a means to solve several types of equilibrium problems including variational inequalities, minimax problems, complementarity problems and optimization problems. In vector-valued case, we apply corresponding real-valued results as long as we find suitable monotone scalarizing functions for objective vector-valued functions. However it is unclear whether any scalarizing function for set-valued maps plays such a similar role. Hence, it is important that we verify its monotonicity and inherited properties on convexity and

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semicontinuity of set-valued maps.

In [4], Nishizawa, Tanaka, and Georgiev introduce four types of nonlinear scalarizing functions for set-valued maps and show several properties of them with respect to some kinds of convexity and semicontinuity. In [1], Hamel and Löhne propose certain interesting nonlinear scalarizing functions for sets, and they give generalized results on Ekeland variational principle in an abstract space like topological vector space without such strong assumption as convexity. Moreover, by using a modified scalarizing function in [6], Shimizu and Tanaka give a similar result to a minimal element theorem in [1] under different assumptions. As seen from the above, there are several types of nonlinear scalarizing functions for set-valued maps, which preserve monotonicity and some kinds of convexity. In [7], a new unified approach on such scalarization for sets is proposed.

In the paper, we observe several nonlinear scalarization methods for set-valued maps including unified nonlinear scalarizing functions proposed in [7] for sets as images of set-valued maps, and we investigate the monotonicity of such unified scalarizing functions for sets and show their inherited properties on cone-convexity of parent set-valued maps.

## 2 Set-relation in an ordered topological vector space

Throughout the paper, we assume that the field of each vector space is the real field. We introduce the relationship between two sets in an ordered topological vector space. At first, we recall the concept of an ordered topological vector space. Let $Y$ be a topological vector space with the vector ordering $\leq_C$ induced by a nonempty convex cone $C$ as follows: $x \leq_C y$ if $y - x \in C$ for $x, y \in Y$. It is well known that $\leq_C$ is reflexive and transitive. Then, the space $Y$ is called an *ordered topological vector space.* In particular, if $C$ is pointed, then $\leq_C$ is antisymmetric, and hence $(Y, \leq_C)$ is a partially ordered topological vector space. Moreover, a convex cone characterizing the vector ordering is called an *ordering cone.*

Next, we introduce mathematical methodology on comparisons between two sets in $Y$. First, we consider comparisons between two vectors. There are two types of comparable cases and one in-comparable case. Comparable cases are as follows: for $a, b \in Y$,

$$\text{(i) } a \in b - C \text{ (that is, } a \leq_C b), \quad \text{(ii) } a \in b + C \text{ (that is, } b \leq_C a).$$

When we replace a vector $a \in Y$ with a set $A \subset Y$, that is, we consider comparisons between a vector and a set with respect to $C$, there are four types of comparable cases and one in-comparable case. Comparable cases are as follows: for $A \subset Y$ and $b \in Y$,

$$\begin{array}{ll} \text{(i) } A \subset (b - C), & \text{(ii) } A \cap (b - C) \neq \emptyset, \\ \text{(iii) } A \cap (b + C) \neq \emptyset, & \text{(iv) } A \subset (b + C). \end{array}$$

In the same way, when we replace a vector $b \in Y$ with a set $B \subset Y$, that is, we consider comparisons between two sets with respect to $C$, there are twelve types of comparable cases and one in-comparable case. For two sets $A, B \subset Y$, $A$ would be inferior to $B$ if we have one of the following situations:

(i) $A \subset (\cap_{b\in B}(b-C))$, (ii) $A \cap (\cap_{b\in B}(b-C) \neq \emptyset$,
(iii) $(\cup_{a\in A}(a+C)) \supset B$, (iv) $((\cup_{a\in A}(a+C)) \cap B) \neq \emptyset$,
(v) $(\cap_{a\in A}(a+C)) \supset B$, (vi) $((\cap_{a\in A}(a+C)) \cap B) \neq \emptyset$,
(vii) $A \subset (\cup_{b\in B}(b-C))$, (viii) $(A \cap (\cup_{b\in B}(b-C))) \neq \emptyset$.

Also, there are eight converse situations in which $B$ would be inferior to $A$. Actually relationships (i) and (iv) coincide with relationships (v) and (viii), respectively. Therefore, we define the following six kinds of classification for set-relationships; see Figure 1.

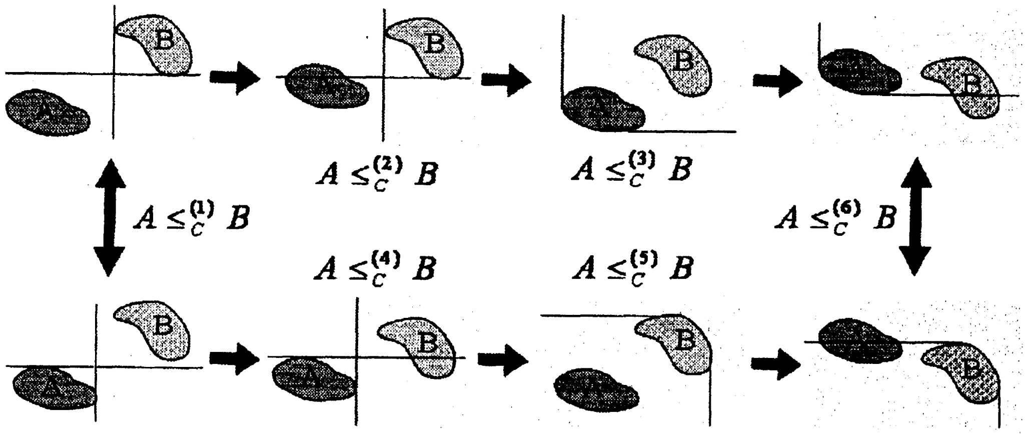


Figure 1. Six kinds of classification for set-relationship.

**Definition 2.1.** (set-relation, [2].) For nonempty sets $A$, $B \subset Y$, we write

$A \le_C^{(1)} B$ by $A \subset \bigcap_{b\in B}(b-C)$, equivalently $B \subset \bigcap_{a\in A}(a+C)$;
$A \le_C^{(2)} B$ by $A \cap (\bigcap_{b\in B}(b-C)) \neq \emptyset$;
$A \le_C^{(3)} B$ by $B \subset (A+C)$;
$A \le_C^{(4)} B$ by $(\bigcap_{a\in A}(a+C)) \cap B \neq \emptyset$;
$A \le_C^{(5)} B$ by $A \subset (B-C)$;
$A \le_C^{(6)} B$ by $A \cap (B-C) \neq \emptyset$, equivalently $(A+C) \cap B \neq \emptyset$.

**Proposition 2.1.** ([2].) *For nonempty sets $A, B \subset Y$, the following statements hold.*

$A \le_C^{(1)} B$ *implies* $A \le_C^{(2)} B$; $A \le_C^{(1)} B$ *implies* $A \le_C^{(4)} B$;
$A \le_C^{(2)} B$ *implies* $A \le_C^{(3)} B$; $A \le_C^{(4)} B$ *implies* $A \le_C^{(5)} B$;
$A \le_C^{(3)} B$ *implies* $A \le_C^{(6)} B$; $A \le_C^{(5)} B$ *implies* $A \le_C^{(6)} B$.

**Remark 2.1.** Since $C$ is a convex cone including the zero vector of $Y$, the following elementary properties hold for nonempty sets $A, B \subset Y$:

(i) $\bigcap_{a\in A}(a+C)=\bigcap_{y\in A-C}(y+C)$ and $\bigcap_{b\in B}(b-C)=\bigcap_{y\in B+C}(y-C)$;
(ii) $A+C=A+C+C$ and $B-C=B-C-C$;
(iii) $B\subset(A+C)$ if and only if $(B+C)\subset(A+C)$;
(iv) $A\subset(B-C)$ if and only if $(A-C)\subset(B-C)$.

Moreover, for sets $A, B\subset X$, conditions $A\subset B$ and $A\cap B\neq\emptyset$ are invariant under translation and scalar multiplication, that is, for $y\in Y$ and $\alpha>0$,

(v) $A\subset B$ implies $(A+y)\subset(B+y)$ and $(\alpha A)\subset(\alpha B)$;
(vi) $A\cap B\neq\emptyset$ implies $(A+y)\cap(B+y)\neq\emptyset$ and $(\alpha A)\cap(\alpha B)\neq\emptyset$.

Hence, for nonempty sets $A, B, D, E\subset Y$ and $\alpha\in\mathbb{R}$, the following property holds:

(vii) $A\cap B\neq\emptyset$ and $D\cap E\neq\emptyset$ implies $(A+D)\cap(B+E)\neq\emptyset$.

**Proposition 2.2.** ([3].) *For nonempty sets $A, B\subset Y$, the following equivalences hold.*

(i) $A\leq_C^{(1)} B\Leftrightarrow A\leq_C^{(1)}(B+C)\Leftrightarrow(A-C)\leq_C^{(1)} B\Leftrightarrow(A-C)\leq_C^{(1)}(B+C)$
(ii) $A\leq_C^{(2)} B\Leftrightarrow A\leq_C^{(2)}(B+C)\Leftrightarrow(A+C)\leq_C^{(2)} B\Leftrightarrow(A+C)\leq_C^{(2)}(B+C)$
(iii) $A\leq_C^{(3)} B\Leftrightarrow A\leq_C^{(3)}(B+C)\Leftrightarrow(A+C)\leq_C^{(3)} B\Leftrightarrow(A+C)\leq_C^{(3)}(B+C)$
(iv) $A\leq_C^{(4)} B\Leftrightarrow A\leq_C^{(4)}(B-C)\Leftrightarrow(A-C)\leq_C^{(4)} B\Leftrightarrow(A-C)\leq_C^{(4)}(B-C)$
(v) $A\leq_C^{(5)} B\Leftrightarrow A\leq_C^{(5)}(B-C)\Leftrightarrow(A-C)\leq_C^{(5)} B\Leftrightarrow(A-C)\leq_C^{(5)}(B-C)$
(vi) $A\leq_C^{(6)} B\Leftrightarrow A\leq_C^{(6)}(B-C)\Leftrightarrow(A+C)\leq_C^{(6)} B\Leftrightarrow(A+C)\leq_C^{(6)}(B-C)$

**Proposition 2.3.** ([3].) *For nonempty sets $A, B\subset Y$, the following statements hold.*

(i) *For each $j=1,\dots,6$,*
  $A\leq_C^{(j)} B$ *implies* $(A+y)\leq_C^{(j)}(B+y)$ *for* $x\in Y$, *and*
  $A\leq_C^{(j)} B$ *implies* $\alpha A\leq_C^{(j)}\alpha B$ *for* $\alpha>0$;
(ii) *For each $j=1,\dots,5$, $\leq_C^{(j)}$ is transitive;*
(iii) *For each $j=3,5,6$, $\leq_C^{(j)}$ is reflexive.*

*Proof.* By Remark 2.1, (i) is clear, and also by the definition of each set-relation and $0\in C$, (iii) is clear. We show the statement (ii). First we prove the case of $j=2$. For nonempty sets $A, B, D\subset Y$, let $A\leq_C^{(2)} B$ and $B\leq_C^{(2)} D$. Then, by the definition of $\leq_C^{(2)}$,

$$A\cap\left\{\bigcap_{b\in B}(b-C)\right\}\neq\emptyset; \tag{2.1}$$

$$B\cap\left\{\bigcap_{d\in D}(d-C)\right\}\neq\emptyset. \tag{2.2}$$

By (2.1), there exists $a\in A$ such that $a\in b-C$ for any $b\in B$. By (2.2), there exists $b'\in B$ such that $b'\in d-C$ for any $d\in D$. Hence, $a\in d-C$ for any $d\in D$, that is,

$A \leq_C^{(2)} B$. Consequently, $\leq_C^{(2)}$ are transitive. The case of $j = 4$ is proved in the same way and the other cases are obvious. □

# 3 Cone-convexity and cone-concavity of set-valued maps

In this section, we introduce some definitions of cone-convexity for sets and set-valued maps, and we define cone-concavity as the dual notion of cone-convexity for set-valued maps in the sense of set-relations. We do not need a topology to mention results on convexities, and hence we suppose that $X$ is a vector space and $Y$ is an ordered vector space with the vector ordering $\leq_C$.

At first, we recall some definitions of convexity for real-valued maps.

**Definition 3.1.** (convexity) Let $f$ be a real-valued map from $X$ to $\mathbb{R}$. Then $f$ is called a *convex function* if for each $x, y \in X$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

If $-f$ is a convex function, then $f$ is called a *concave function.*

**Definition 3.2.** (quasiconvexity) Let $f$ be a real-valued map from $X$ to $\mathbb{R}$. Then $f$ is called a *quasiconvex function* if for each $x, y \in X$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}.$$

If $-f$ is a quasiconvex function, then $f$ is called a *quasiconcave function.*

Next, we consider natural extensions of the convexities above into vector-valued maps with respect to $\leq_C$; such generalized convexities are called *cone-convexity* collectively.

**Definition 3.3.** ($C$-convexity, [8].) Let $f$ be a vector-valued map from $X$ to $Y$. Then $f$ is called a *$C$-convex function* if for each $x, y \in X$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq_C \lambda f(x) + (1 - \lambda)f(y).$$

If $-f$ is a $C$-convex function, then $f$ is called a $C$-concave function.

**Definition 3.4.** (properly quasi $C$-convexity, [8].) Let $f$ be a vector-valued map from $X$ to $Y$. Then $f$ is called a *properly quasi $C$-convex function* if for each $x, y \in X$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq_C f(x) \quad \text{or} \quad f(\lambda x + (1 - \lambda)y) \leq_C f(y).$$

If $-f$ is a properly quasi $C$-convex function, then $f$ is called a properly quasi $C$-concave function.

**Definition 3.5.** (naturally quasi $C$-convexity, [8].) Let $f$ be a vector-valued map from $X$ to $Y$. Then $f$ is called a *naturally quasi $C$-convex function* if for each $x, y \in X$ and $\lambda \in [0,1]$, there exists $\mu \in [0,1]$ such that

$$f(\lambda x + (1-\lambda)y) \leq_C \mu f(x) + (1-\mu)f(y).$$

If $-f$ is a naturally quasi $C$-convex function, then $f$ is called a naturally quasi $C$-concave function.

**Proposition 3.1.** ([8].) *The following statements hold:*

(i) *if a vector-valued map $f$ is $C$-convex, then $f$ is naturally quasi $C$-convex, and*
(ii) *if a vector-valued map $f$ is properly quasi $C$-convex, then $f$ is naturally quasi $C$-convex.*

Then, we consider natural extensions on several kinds of cone-convexity for vector-valued maps into set-valued maps with respect to the set-relation $\leq_C^{(j)}$ with $j = 1, \ldots, 6$; such generalized convexities are called *type $(j)$ cone-convexity* ($j = 1, \ldots, 6$).

**Definition 3.6.** ($C$-convex set, [9].) Let $V$ be a subset of $Y$. Then $V$ is called a *$C$-convex set* if $V + C$ is a convex set.

Throughout the paper, $F$ is a set-valued map from $X$ to $2^Y \setminus \{\emptyset\}$. Then, we introduce some definitions of $C$-convexity and its modifications for set-valued maps.

**Definition 3.7.** (type $(j)$ $C$-convexity, [2].) For each $j = 1, \ldots, 6$, a set-valued map $F$ is called a *type $(j)$ $C$-convex function* if for each $x, y \in X$ and $\lambda \in (0,1)$,

$$F(\lambda x + (1-\lambda)y) \leq_C^{(j)} \lambda F(x) + (1-\lambda)F(y).$$

**Definition 3.8.** (type $(j)$ properly quasi $C$-convexity, [2].) For each $j = 1, \ldots, 6$, a set-valued map $F$ is called a *type $(j)$ properly quasi $C$-convex function* if for each $x, y \in X$ and $\lambda \in (0,1)$,

$$F(\lambda x + (1-\lambda)y) \leq_C^{(j)} F(x) \quad \text{or} \quad F(\lambda x + (1-\lambda)y) \leq_C^{(j)} F(y).$$

**Definition 3.9.** (type $(j)$ naturally quasi $C$-convexity, [2].) For each $j = 1, \ldots, 6$, a set-valued map $F$ is called a *type $(j)$ naturally quasi $C$-convex function* if for each $x, y \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$F(\lambda x + (1-\lambda)y) \leq_C^{(j)} \mu F(x) + (1-\mu)F(y).$$

The relationships between these cone-convexities with $j = 1, \ldots, 6$ are as follows:

**Proposition 3.2.** ([2].) *For each $j = 1, \ldots, 6$,*

(i) *if a set-valued map $F$ is type $(j)$ $C$-convex, then $F$ is type $(j)$ naturally quasi $C$-convex, and*

(ii) *if a set-valued map $F$ is type $(j)$ properly quasi $C$-convex, then $F$ is type $(j)$ naturally quasi $C$-convex.*

Moreover, we propose cone-concavity as the dual notion of cone-convexity for set-valued maps in the sense of set-relations. Usually, the dual notion of convexity (resp. cone-convexity) for a real-valued (resp. vector-valued) function $F$ is defined by the convexity (resp. cone-convexity) of $-F$, and then $F$ is called concave (resp. cone-concave). However in the case of set-valued maps, it is not necessarily formed. For example, if $F$ is a type (3) $C$-concave function in the usual definition (e.g., [2]), then $-F$ is a type (3) $C$-convex function and hence $F$ satisfies

$$\lambda F(x) + (1-\lambda)F(y) \leq_C^{(5)} F(\lambda x + (1-\lambda)y)$$

for each $x, y \in X$ and $\lambda \in (0,1)$, but this condition seems to be the cone-concavity of $F$ in the set-relation $\leq_C^{(5)}$. Hence, we define the dual notion of cone-convexity for set-valued maps directly in a different way.

**Definition 3.10.** (type $(j)$ $C$-concavity, [3].) For each $j = 1, \ldots, 6$, a set-valued map $F$ is called a *type $(j)$ $C$-concave function* if for each $x, y \in X$ and $\lambda \in (0,1)$,

$$\lambda F(x) + (1-\lambda)F(y) \leq_C^{(j)} F(\lambda x + (1-\lambda)y).$$

**Definition 3.11.** (type $(j)$ properly quasi $C$-concavity, [3].) For each $j = 1, \ldots, 6$, a set-valued map $F$ is called a *type $(j)$ properly quasi $C$-concave function* if for each $x, y \in X$ and $\lambda \in (0,1)$,

$$F(x) \leq_C^{(j)} F(\lambda x + (1-\lambda)y) \quad \text{or} \quad F(y) \leq_C^{(j)} F(\lambda x + (1-\lambda)y).$$

**Definition 3.12.** (type $(j)$ naturally quasi $C$-concavity, [3].) For each $j = 1, \ldots, 6$, a set-valued map $F$ is called a *type $(j)$ naturally quasi $C$-concave function* if for each $x, y \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$\mu F(x) + (1-\mu)F(y) \leq_C^{(j)} F(\lambda x + (1-\lambda)y).$$

The relationships between these cone-concavities with $j = 1, \ldots, 6$ are as follows:

**Proposition 3.3.** ([3].) *For each $j = 1, \ldots, 6$,*

(i) *if a set-valued map $F$ is type $(j)$ $C$-concave, then $F$ is type $(j)$ naturally quasi $C$-concave, and*
(ii) *if a set-valued map $F$ is type $(j)$ properly quasi concave, then $F$ is type $(j)$ naturally quasi $C$-concave.*

*Proof.* In the same way in [2], the statements are proved straightforward. $\square$

**Example 3.1.** Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+ := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x, y \geq 0 \right\}$. We consider the following three set-valued maps $F_k : \mathbb{R} \to 2^{\mathbb{R}^2}$ $(k = 1, 2, 3)$ defined by $F_1(x) :=$

$\left\{\begin{pmatrix} x \\ -x \end{pmatrix}\right\}$, $F_2(x) := \left\{\begin{pmatrix} x^3 \\ x^3 \end{pmatrix}\right\}$, and $F_3(x) := \left\{\begin{pmatrix} x^3 \\ -x^3 \end{pmatrix}\right\}$. They can be regarded as vector-valued maps but they show essential feature of cone-convexity and cone-concavity for multicriteria situations with partial orderings. Then, for each $j$, $F_1$ is both type $(j)$ $C$-convex and -concave, but neither type $(j)$ properly quasi $C$-convex nor -concave. Also, for each $j$, $F_2$ is both type $(j)$ properly quasi $C$-convex and -concave but neither type $(j)$ $C$-convex nor -concave. On the other hand, $F_3$ is both type $(j)$ naturally quasi $C$-convex and -concave for each $j$, but it is neither type $(j)$ $C$-convex nor type $(j)$ properly quasi $C$-convex for each $j$, and neither type $(j)$ $C$-concave nor type $(j)$ properly quasi $C$-concave for each $j$.

# 4 Several types of scalarizing functions for sets

At first, we recall the method proposed in [1] with four types of nonlinear scalarizing functions for sets.

**Theorem 4.1.** ([1].) *Let $k_0 \in C \setminus (-\mathrm{cl}\, C)$ and $\mathcal{V} \subseteq 2^Y$. We assume that $\mathcal{V}$ is nonempty and $\leq_C^{(3)}$-bounded, that is, there is a topological bounded set $V' \subseteq Y$ and a nonempty set $V'' \subseteq Y$ such that*

$$V' \leq_C^{(3)} V \leq_C^{(3)} V'' \quad \textit{for all} \quad V \in \mathcal{V}.$$

*Then, the functional $z^l : 2^Y \to \mathbb{R} \cup \{\pm\infty\}$, defined by*

$$z^l := \inf\{t \in \mathbb{R} : tk^0 + V'' \subseteq V + \mathrm{cl}\, C\},$$

*has the following properties:*

(i) *$z^l$ is bounded on $\mathcal{V}$;*
(ii) *$V \in \mathcal{V}$, $\alpha \in \mathbb{R}$ implies $z^l(V + \alpha k^0) = z^l(V) + \alpha$;*
(iii) *$z^l$ is $\leq_C^{(3)}$-monotone, that is, for $A, B \subset Y$ with $A$ and $B$ are nonempty sets,*

$$A \leq_C^{(3)} B \quad \textit{implies} \quad z^l(A) \leq z^l(B).$$

**Theorem 4.2.** ([1].) *Let $k_0 \in C \setminus (-\mathrm{cl}\, C)$ and $\mathcal{V} \subseteq 2^Y$. We assume that $\mathcal{V}$ is nonempty and $\leq_C^{(5)}$-bounded, that is, there is a nonempty set $W' \subseteq Y$ and a topological bounded set $W'' \subseteq Y$ such that*

$$W' \leq_C^{(5)} V \leq_C^{(5)} W'' \quad \textit{for all} \quad V \in \mathcal{V}.$$

*Then, the functional $z^u : 2^Y \to \mathbb{R} \cup \{\pm\infty\}$, defined by*

$$z^u := -\inf\{t \in \mathbb{R} : t(-k^0) + W' \subseteq V - \mathrm{cl}\, C\},$$

*has the following properties:*

(i) $z^u$ *is bounded on* $\mathcal{V}$;
(ii) $V \in \mathcal{V}$, $\alpha \in \mathbb{R}$ *implies* $z^u(V + \alpha k^0) = z^u(V) + \alpha$;
(iii) $z^u$ *is* $\leq_C^{(5)}$*-monotone, that is, for* $A, B \subset Y$ *with* $A$ *and* $B$ *are nonempty sets,*

$$A \leq_C^{(5)} B \quad \text{implies} \quad z^u(A) \leq z^u(B).$$

In [6], they proposed another type of nonlinear scalarizing function for sets under different assumptions from Theorem 4.1.

**Theorem 4.3.** ([6].) *Let* $k_0 \in C \setminus (-\mathrm{cl}\, C)$ *and* $\mathcal{V} \subseteq 2^Y$. *We assume that* $\mathcal{V}$ *is nonempty and* $\leq_C^{(3)}$*-bounded below, that is, there is a topological bounded set* $V' \subseteq Y$ *such that*

$$V' \leq_C^{(3)} V \quad \text{for all} \quad V \in \mathcal{V}.$$

*Then, the functional* $g_{k^0}^l : 2^Y \to \mathbb{R} \cup \{\pm\infty\}$, *defined by*

$$g_{k^0}^l(V) := \inf\{t \in \mathbb{R} : tk^0 + V' \subseteq V + \mathrm{cl}\, C\},$$

*has the following properties:*

(i) $g_{k^0}^l$ *is bounded on* $\mathcal{V}$;
(ii) $V \in \mathcal{V}$, $\alpha \in \mathbb{R}$ *implies* $g_{k^0}^l(V + \alpha k^0) = g_{k^0}^l(V) + \alpha$;
(iii) $g_{k^0}^l$ *is* $\leq_C^{(3)}$*-monotone.*

Now, we introduce two types of nonlinear scalarizing functions for sets proposed by a unified approach in [7].

**Definition 4.1.** (unified types of scalarizing functions, [7].) Let $V$ and $V'$ be nonempty subsets of $Y$, and direction $k \in \mathrm{int}\, C$. For each $j = 1, \ldots, 6$, $I_{k,V'}^{(j)} : 2^Y \setminus \{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$ and $S_{k,V'}^{(j)} : 2^Y \setminus \{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$ are defined by

$$I_{k,V'}^{(j)}(V) := \inf\left\{t \in \mathbb{R} \;\middle|\; V \leq_C^{(j)} (tk + V')\right\},$$

$$S_{k,V'}^{(j)}(V) := \sup\left\{t \in \mathbb{R} \;\middle|\; (tk + V') \leq_C^{(j)} V\right\},$$

respectively.

If $V' = \{\theta_Y\}$ (a singleton set consisting of the zero vector of $Y$), then these nonlinear scalarizing functions coincide with the four types of scalarizing functions introduced in [4]. In addition, these functions are essentially coincident with scalarizing functions which are introduced in [1] in the cases of $j = 3, 5$, respectively. Hence, these functions become extensions of the four functions in [4] as well as unified forms including all functions introduced in [1]. Accordingly we call these functions *unified types of scalarizing functions* for sets.

In general, the monotonicity of a function $f : Y \to \mathbb{R}$ is defined as follows:

$$\text{for any } x, y \in Y,\ x \leq_C y \text{ implies } f(x) \leq f(y).$$

Based on the approach of [1], when $f$ is a scalarizing function for nonempty sets in $Y$, its monotonicity is defined in the following:

$$\text{for nonempty sets } A, B \subset Y,\ A \leq_C^{(j)} B \text{ implies } f(A) \leq f(B)$$

for each set-relation $j = 1, \ldots, 6$. Under this definition, we consider the monotonicity of nonlinear scalarizing functions for set-valued maps.

**Theorem 4.4.** ([3].) *For nonempty subsets $V, V' \subset Y$ and direction $k \in \operatorname{int} C$, $I_{k,V'}^{(j)}$ and $S_{k,V'}^{(j)}$ satisfy the following properties:*

(i) *For each $j = 1, \ldots, 6$, $I_{k,V'}^{(j)}(V + \alpha k) = I_{k,V'}^{(j)}(V) + \alpha$ for any $\alpha \geq 0$;*
(ii) *For each $j = 1, \ldots, 6$, $S_{k,V'}^{(j)}(V + \alpha k) = S_{k,V'}^{(j)}(V) + \alpha$ for any $\alpha \geq 0$.*

**Theorem 4.5.** ([3].) *Let $V'$ be a nonempty subset of $Y$ and $k \in \operatorname{int} C$. For each $j = 1, \ldots, 5$, $I_{k,V'}^{(j)}(\cdot)$ and $S_{k,V'}^{(j)}(\cdot)$ are monotone on $2^Y \setminus \{\emptyset\}$ with respect to $\leq_C^{(j)}$, that is, for $A, B \subset Y$ with $A$ and $B$ are nonempty sets,*

$$A \leq_C^{(j)} B \quad \text{implies} \quad I_{k,V'}^{(j)}(A) \leq I_{k,V'}^{(j)}(B) \quad \text{and} \quad S_{k,V'}^{(j)}(A) \leq S_{k,V'}^{(j)}(B).$$

# 5 Inherited properties on cone-convexity of set-valued maps

In the previous section, we prove the monotonicity of unified types of scalarizing functions with respect to set-relations. In this section, we use this property to prove that some kinds of cone-convexity for set-valued maps are inherited to unified types of scalarizing functions.

At first, we remark that the unified types of scalarizing functions have an important merit on the inheritance properties in contrast with the approach of [4].

Let $V' \in 2^Y \setminus \{\emptyset\}$, and direction $k \in \operatorname{int} C$ and $F : X \to 2^Y \setminus \{\emptyset\}$ a set-valued map. For any $x \in X$ and for each $j = 1, \ldots, 6$, we consider the following composite functions:

$$(I_{k,V'}^{(j)} \circ F)(x) := I_{k,V'}^{(j)}(F(x)),$$

$$(S_{k,V'}^{(j)} \circ F)(x) := S_{k,V'}^{(j)}(F(x)).$$

Then, we can directly discuss inherited properties on cone-convexity of parent set-valued map $F$ to $I_{k,V'}^{(j)} \circ F$ and $S_{k,V'}^{(j)} \circ F$ in an analogous fashion to linear scalarizing function like inner product. Thus, we show how some kinds of cone-convexity of parent set-valued maps are inherited to unified types of scalarizing functions.

**Theorem 5.1.** ([3].) *Let $F$ be a type $(j)$ naturally quasi $C$-convex function. Then, the following statements hold.*

(i) *For each* $j = 1, 2, 3$, $I^{(j)}_{k,V'} \circ F$ *is a quasiconvex function,*

(ii) *For each* $j = 4, 5$, *if* $V'$ *is a* $(-C)$*-convex set, that is,* $V' - C$ *is a convex set, then* $I^{(j)}_{k,V'} \circ F$ *is a quasiconvex function.*

**Remark 5.1.** By Proposition 3.2, we obtain the same results in Theorem 5.1 for type $(j)$ $C$-convexity and type $(j)$ properly quasi $C$-convexity, respectively.

**Theorem 5.2.** ([3].) *Let* $F$ *be a type* $(j)$ *naturally quasi* $C$*-concave function. Then, the following statements hold.*

(i) *For each* $j = 1, 4, 5$, $S^{(j)}_{k,V'} \circ F$ *is a quasiconcave function,*

(ii) *For each* $j = 2, 3$, *if* $V'$ *is a* $C$*-convex set then* $S^{(j)}_{k,V'} \circ F$ *is a quasiconcave function.*

**Remark 5.2.** By Proposition 2.3, we obtain the same results in Theorem 5.2 for type $(j)$ $C$-concavity and type $(j)$ properly quasi $C$-concavity, respectively.

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