Duality in Nondifferentiable Multiobjective Programming with Cone Constraints (Nonlinear Analysis and Convex Analysis)

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Duality in Nondifferentiable Multiobjective Programming with Cone Constraints

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1 Introduction

In study of duality under generalized convexity, Mond and Weir [5] proposed a number of different duals for nonlinear programming problems with nonnegative variables and established duality theorems under appropriate pseudo-convexity/quasi-convexity assumptions. Taking motivation from Bazaraa and Goode [1] and Kuk and Kim [3], Nanda and Das [6] attempted to extend the results of Mond and Weir [5] to cone domains with appropriate pseudo-invexity and quasi-invexity assumptions on objective and constraint functions. However, certain shortcomings were pointed out in the work of Nanda and Das [6] and appropriate modifications were suggested for studying duality under pseudo-invexity assumptions in Chandra and Abha [2]. Recently, Yang et al. [7] established various converse duality results for nonlinear programming with cone constraints and its four dual models introduced by Chandra and Abha [2].

In this paper, we construct nondifferentiable multiobjective dual problems with cone constraints over arbitrary closed convex cones, which are Mond-Weir type and Wolfe type. And we establish weak, strong duality theorems for a weakly efficient solution by using suitable generalized invexity conditions.
2 Preliminaries

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and let $\mathbb{R}^n_+$ be its non-negative orthant. The following convention for inequalities will be used in this talk. If $x, u \in \mathbb{R}^n$, then
\begin{align*}
x < u & \iff u - x \in \text{int}\mathbb{R}^n_+ ; \\
x \leq u & \iff u - x \in \mathbb{R}^n_+ ; \\
x \leq u & \iff u - x \in \mathbb{R}^n_+ \setminus \{0\} ; \\
x \neq u & \text{ is the negation of } x < u .
\end{align*}

**Definition 2.1** A nonempty set $C$ in $\mathbb{R}^n$ is said to be a cone with vertex zero, if $x \in C$ implies that $\lambda x \in C$ for all $\lambda \geq 0$. If, in addition, $C$ is convex, then $C$ is called a convex cone.

**Definition 2.2** The polar cone $C^*$ of $C$ is defined by
\[ C^* = \{z \in \mathbb{R}^n \mid x^T z \leq 0 \text{ for all } x \in C\}. \]

**Definition 2.3** Let $S \subseteq \mathbb{R}^n$ be open and $f : S \to \mathbb{R}$ be a differentiable function.

1. The function $f$ is said to be invex at $u \in S$, if there exists a function $\eta : S \times S \to \mathbb{R}^n$ such that
\[ f(x) - f(u) \geq \eta(x, u)^T \nabla f(u). \]

2. The function $f$ is said to be pseudoinvex at $u \in S$, if there exists a function $\eta : S \times S \to \mathbb{R}^n$ such that
\[ \eta(x, u)^T \nabla f(u) \geq 0 \Rightarrow f(x) - f(u) \geq 0. \]
(9) The function $f$ is said to be quasiinvex at $u \in S$, if there exists a function $\eta : S \times S \to \mathbb{R}^n$ such that

$$f(x) - f(u) \leq 0 \Rightarrow \eta(x, u)^T \nabla f(u) \leq 0.$$

**Definition 2.4** [4] The support function $s(x|B)$, being convex and everywhere finite, has a subdifferential, that is, there exists $z$ such that

$$s(y|B) \geq s(x|B) + z^T(y - x) \text{ for all } y \in B.$$ 

Equivalently,

$$z^Tx = s(x|B).$$

The subdifferential of $s(x|B)$ is given by

$$\partial s(x|B) := \{z \in B : z^Tx = s(x|B)\}.$$ 

For any set $S \subset \mathbb{R}^n$, the normal cone to $S$ at a point $x \in S$ is defined by

$$N_S(x) := \{y \in \mathbb{R}^n : y^T(z - x) \leq 0 \text{ for all } z \in S\}.$$ 

It is readily verified that for a compact convex set $B$, $y$ is in $N_B(x)$ if and only if $s(y|B) = x^Ty$, or equivalently, $x$ is in the subdifferential of $s$ at $y$. 
3 Mond-Weir Type Duality

We consider the following multiobjective programming problem:

(MP) Minimize $f(x) + s(x | D)$

$$= (f_1(x) + x^T w_1, \cdots, f_k(x) + x^T w_k)$$

subject to $-g(x) \in C_2^*, \ x \in C_1,$

and its Mond Weir type dual programming problem (MWD):

(MWD)

Maximize $f(u) + u^T w$

subject to $\lambda^T [\nabla f(u) + w] = \nabla y^T g(u)$,  

$g(u) \in C_2^*$,  

$w_i \in D_i, \ i = 1, \cdots, k$,  

$y \in C_2, \ \lambda \geq 0, \ \lambda^T e = 1$,  

where

(i) $f : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^k$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are differentiable functions,  

(ii) $C_1$ and $C_2$ are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ with nonempty interiors, respectively,  

(iii) $C_1^*$ and $C_2^*$ are polar cones of $C_1$ and $C_2$, respectively,  

(iv) $e = (1, \cdots, 1)^T$ is vector in $\mathbb{R}^k$,  

(v) $w_i(i = 1, \cdots, k)$ is vector in $\mathbb{R}^n$ and $D_i(i = 1, \cdots, k)$ is compact convex set in $\mathbb{R}^n$, respectively,  

(vi) $u^T w = (u^T w_1, \cdots, u^T w_k)^T$. 
Now we establish the duality theorems of (MP) and (MWD).

**Theorem 3.1 (Weak Duality)** Let $x$ and $(u, y, \lambda, w)$ be feasible solutions of (MP) and (MWD), respectively. Assume that

(a) $f_i(\cdot) + (\cdot)^T w_i, i = 1, \cdots, k$, is invex at $u$ and $-y^T g(\cdot)$ is invex at $u$ or

(b) $\lambda^T [f(\cdot) + (\cdot)^T w]$ is pseudoinvex at $u$ and $-y^T g(\cdot)$ is quasiinvex at $u$.

Then

$$f(x) + s(x|D) \not\simeq f(u) + u^T w.$$  

**Proof.** Assume to the contrary that

$$f(x) + s(x|D) < f(u) + u^T w.$$  

Since $\lambda \geq 0$, we have

$$\lambda^T [f(x) + s(x|D)] < \lambda^T [f(u) + u^T w].$$  

(3)

(a) From the assumption (a), we get

$$\lambda^T [f(x) + x^T w] - \lambda^T [f(u) + u^T w] \geq \eta(x, u)^T [\lambda^T (\nabla f(u) + w)]$$  

(4)

and

$$-y^T g(x) + y^T g(u) \geq -\eta(x, u)^T \nabla y^T g(u).$$  

(5)

Adding (4) and (5), we obtain

$$\lambda^T [f(x) + x^T w] - y^T g(x) - \lambda^T [f(u) + u^T w] + y^T g(u)$$

$$\geq \eta(x, u)^T [\lambda^T (\nabla f(u) + w) - \nabla y^T g(u)].$$

Also, by $-y^T g(x) \leq 0, y^T g(u) \leq 0$ and the dual constraint (1), it follows that

$$\lambda^T [f(x) + x^T w] - \lambda^T [f(u) + u^T w] \geq 0.$$
Using the fact that \( s(x|D) \geq x^Tw \), the above inequality becomes
\[
\lambda^T[f(x) + s(x|D)] - \lambda^T[f(u) + u^Tw] \geq 0,
\]
which contradicts (3). Hence,
\[
f(x) + s(x|D) \not\prec f(u) + u^Tw.
\]

(b) From the assumption (b), (3) implies that
\[
\eta(x, u)^T[\lambda^T(\nabla f(u) + w)] < 0.
\]
From the dual constraint (1), it yields
\[
\eta(x, u)^T \nabla y^Tg(u) < 0.
\]
By the quasiinvexity of \(-y^Tg(\cdot)\), the above inequality becomes
\[
-y^Tg(x) > -y^Tg(u). \tag{6}
\]
Since \(-y^Tg(x) \leq 0\) and \(y^Tg(u) \leq 0\), we get \(-y^Tg(x) \leq -y^Tg(u)\), which contradicts (6). Thus,
\[
f(x) + s(x|D) \not\prec f(u) + u^Tw.
\]

\[\square\]

By using the necessary optimality condition due to Bazaraa and Goode [1], we can obtain the following lemma.

**Lemma 3.1** If \( \bar{x} \) is a weakly efficient solution of (MP) at which constraint qualification be satisfied. Then there exist \( \bar{w}_i \in D_i (i = 1, \cdots, k), \bar{\lambda} \geq 0 \) and \( \bar{y} \in C_2 \) with \((\bar{\lambda}, \bar{y}) \neq 0\) such that
\[
[\bar{\lambda}^T(\nabla f(\bar{x}) + \bar{w}) - \bar{y}^T \nabla g(\bar{x})]^T(x - \bar{x}) \geq 0, \quad \text{for all} \quad x \in C_1,
\]
\[
\bar{y}^Tg(\bar{x}) = 0,
\]
\[
\bar{w}_i \in D_i, \quad s(\bar{x}|D_i) = \bar{x}^T \bar{w}_i, \quad i = 1, \cdots, k.
\]
Theorem 3.2 (Strong Duality) If $\overline{x}$ is a weakly efficient solution of (MP) at which constraint qualification be satisfied. Then there exist $\overline{\lambda} \geq 0$, $\overline{y} \in C_2$ and $\overline{w}_i \in D_i (i = 1, \cdots, k)$ such that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is feasible for (MWD) and the corresponding values of (MP) and (MWD) are equal. If the assumption of Theorem 3.1 are satisfied, then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is weakly efficient for (MWD).

Proof. Since $\overline{x}$ is a weakly efficient solution of (MP), then there exist $w_i \in D_i$, $i = 1, \cdots, k$, $\overline{\lambda} \geq 0$ and $\overline{y} \in C_2$ with $(\overline{\lambda}, \overline{y}) \neq 0$ such that

\[
[\overline{\lambda}^T(\nabla f(\overline{x}) + w) - \overline{y}^T \nabla g(\overline{x})]^T(x - \overline{x}) \geq 0, \text{ for all } x \in C_1, \quad (7)
\]
\[
\overline{y}^T g(\overline{x}) = 0, \quad (8)
\]
\[
w_i \in D_i, \ s(\overline{x} | D_i) = \overline{x}^T w_i, \ i = 1, \cdots, k. \quad (9)
\]

Since $x \in C_1$, $\overline{x} \in C_1$ and $C_1$ is a closed convex cone, we have $x + \overline{x} \in C_1$ and thus the inequality (7) implies

\[
[\overline{\lambda}^T(\nabla f(\overline{x}) + w) - \overline{y}^T \nabla g(\overline{x})]^T x \geq 0, \text{ for all } x \in C_1,
\]
i.e.,
\[
\overline{\lambda}^T(\nabla f(\overline{x}) + w) - \overline{y}^T \nabla g(\overline{x}) = 0.
\]

And (8) implies $\overline{y}^T g(\overline{x}) \leq 0$, then $g(\overline{x}) \in C_2^*$. Taking $\overline{w}_i = w_i \in D_i$, $i = 1, \cdots, k$, we find that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is feasible for (MWD) and corresponding values of (MP) and (MWD) are equal, by (9). If the assumptions of Theorem 3.1 are satisfied, then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is a weakly efficient solution of (MWD). $\square$
4 Wolfe Type Duality

We propose the following Wolfe Type multiobjective dual problem to the primal problem (MP):

\[(WD)\]

Maximize $f(u) + u^T w - y^T g(u)e$
subject to $\lambda^T [\nabla f(u) + w] = \nabla y^T g(u)$, \hspace{1cm} (10)
   $w_i \in D_i, \ i = 1, \cdots, k,$
   $y \in C_2, \ \lambda \geq 0, \ \lambda^T e = 1,$

where
(i) $f : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^k$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are differentiable functions,
(ii) $C_1$ and $C_2$ are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ with nonempty interiors, respectively,
(iii) $C_1^*$ and $C_2^*$ are polar cones of $C_1$ and $C_2$, respectively,
(iv) $e = (1, \cdots, 1)^T$ is vector in $\mathbb{R}^k$,
(v) $w_i (i = 1, \cdots, k)$ is vector in $\mathbb{R}^n$ and $D_i (i = 1, \cdots, k)$ is compact convex set in $\mathbb{R}^n$, respectively,
(vi) $u^T w = (u^T w_1, \cdots, u^T w_k)^T$.

Now we establish the duality theorems of (MP) and (WD).
Theorem 4.1 (Weak Duality) Let \(x\) and \((u, y, \lambda, w)\) be feasible solutions of (MP) and (WD), respectively. Assume that

(a) \(f_i(\cdot) + (\cdot)^T w_i, i = 1, \cdots, k\), is invex at \(u\) and \(-y^T g(\cdot)\) is invex at \(u\) or

(b) \(\lambda^T [f(\cdot) + (\cdot)^T w] - y^T g(\cdot)\) is pseudoinvex at \(u\).

Then

\[ f(x) + s(x|D) \neq f(u) + u^T w - y^T g(u)e. \]

Proof. Assume to the contrary that

\[ f(x) + s(x|D) < f(u) + u^T w - y^T g(u)e. \]

Since \(\lambda \geq 0\), we have

\[ \lambda^T [f(x) + s(x|D)] < \lambda^T [f(u) + u^T w - y^T g(u)e]. \] (11)

(a) By the assumption (a), we obtain

\[ \lambda^T [f(x) + x^T w] - \lambda^T [f(u) + u^T w] \geq \eta(x, u)^T [\lambda^T (\nabla f(u) + w)] \]

and

\[-y^T g(x) + y^T g(u) \geq -\eta(x, u)^T \nabla y^T g(u). \]

So, we get

\[ \lambda^T [f(x) + x^T w] - y^T g(x) - \lambda^T [f(u) + u^T w] + y^T g(u) \geq \eta(x, u)^T [\lambda^T (\nabla f(u) + w) - \nabla y^T g(u)]. \]

Also, by \(-y^T g(x) \leq 0\) and the dual constraint (10), it follows that

\[ \lambda^T [f(x) + x^T w] - \lambda^T [f(u) + u^T w] + y^T g(u) \geq 0. \]

Using the fact that \(s(x|D) \geq x^T w\), the above inequality becomes

\[ \lambda^T [f(x) + s(x|D)] - \lambda^T [f(u) + u^T w] + y^T g(u) \geq 0, \]
which contradicts (11). Hence,

$$f(x) + s(x|D) \not\in f(u) + u^T w - y^T g(u)e.$$

(b) Since $-y^T g(x) \leq 0$, (11) implies that

$$\lambda^T [f(x) + s(x|D)] - y^T g(x) < \lambda^T [f(u) + u^T w] - y^T g(u).$$

By the assumption (b), it yields

$$\eta(x, u)^T [\nabla f(u) + w - \nabla y^T g(u)] < 0,$$

which contradicts (10). Thus,

$$f(x) + s(x|D) \not\in f(u) + u^T w - y^T g(u)e.$$

\[ \square \]

**Theorem 4.2 (Strong Duality)** If $\bar{x}$ is a weakly efficient solution of (MP) at which constraint qualification be satisfied. Then there exist $\bar{\lambda} \geq 0$, $\bar{y} \in C_2$ and $\bar{w}_i \in D_i$ ($i = 1, \cdots, k$) such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w})$ is feasible for (WD) and the corresponding values of (MP) and (WD) are equal. If the assumption of Theorem 4.1 are satisfied, then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w})$ is weakly efficient for (WD).

**Proof.** Since $\bar{x}$ is a weakly efficient solution of (MP), then there exist $w_i \in D_i, i = 1, \cdots, k$, $\bar{\lambda} \geq 0$ and $\bar{y} \in C_2$ with $(\bar{\lambda}, \bar{y}) \neq 0$ such that

\[
[\bar{\lambda}^T (\nabla f(\bar{x}) + w) - \bar{y}^T \nabla g(\bar{x})]^T (x - \bar{x}) \geq 0, \quad \text{for all } x \in C_1, \tag{12}
\]

\[
\bar{y}^T g(\bar{x}) = 0, \tag{13}
\]

\[
w_i \in D_i, \quad s(\bar{x}|D_i) = \bar{x}^T w_i, \quad i = 1, \cdots, k. \tag{14}
\]
Since $x \in C_1$, $\overline{x} \in C_1$ and $C_1$ is a closed convex cone, we have $x + \overline{x} \in C_1$ and thus the inequality (12) implies
\[
[\overline{\lambda}^T(\nabla f(\overline{x}) + w) - \overline{y}^T \nabla g(\overline{x})]^T x \geq 0, \quad \text{for all } x \in C_1,
\]
i.e.,
\[
\overline{\lambda}^T(\nabla f(\overline{x}) + w) - \overline{y}^T \nabla g(\overline{x}) = 0.
\]
Taking $\overline{w}_i = w_i \in D_i$, $i = 1, \cdots, k$, we find that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is feasible for (WD) and corresponding values of (MP) and (WD) are equal, by (13) and (14). If the assumptions of Theorem 4.1 are satisfied, then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is a weakly efficient solution of (WD). \qed

References


