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ON FIRMLY NONEXPANSIVE TYPE MAPPINGS
IN BANACH SPACES
(パナッハ空間におけるFIRMLY NONEXPANSIVE TYPE 写像について)

FUMIAKI KOHSAKA (高阪 史明) AND WATARU TAKAHASHI (高橋 淳)

ABSTRACT. In this paper, we state the recently obtained strong convergence theorem of
Browder’s type for firmly nonexpansive-type mappings in Banach spaces.

1. INTRODUCTION

The following is Browder’s strong convergence theorem [5] for nonexpansive mappings
in Hilbert spaces; see, for instance, Takahashi [24]:

**Theorem 1.1** (Browder [5]). Let $H$ be a Hilbert space, $C$ a nonempty closed convex
subset of $H$, $T$ a nonexpansive mapping from $C$ into itself such that $F(T)$ is nonempty,
and $x \in C$. Then the following hold:

1. For each $t \in (0, 1)$, there exists a unique $u_t \in C$ such that
   
   
   $u_t = tx + (1-t)Tu_t$;

2. the net $\{u_t\}$ converges strongly to $P_{F(T)}(x)$ as $t \downarrow 0$, where $P_{F(T)}$ denotes the metric
   projection from $H$ onto $F(T)$.

This result was extended to accretive operators in Banach spaces by Reich [18] and
Takahashi and Ueda [27].

Recently, the authors [13] proposed the class of firmly nonexpansive-type mappings in
Banach spaces. It is a subclass of $D$-firm operators introduced by Bauschke, Borwein,
and Combettes [3]. This class contains the classes of firmly nonexpansive mappings in
Hilbert spaces and resolvents of maximal monotone operators in Banach spaces. In [14],
the class of nonspreading mappings in Banach spaces was also introduced. Every firmly
nonexpansive-type mapping is known to be nonspreading. Then fixed point theorems and
convergence theorems for these nonlinear operators were investigated [13, 14].

In this paper, we state a strong convergence theorem [15] of Browder’s type for firmly
nonexpansive-type mappings in Banach spaces.

2. PRELIMINARIES

Throughout the paper, every linear space is real. The set of real numbers is denoted by
$\mathbb{R}$. The conjugate space of a Banach space $E$ is denoted by $E^*$. We denote $x^*(x)$ by $\langle x, x^* \rangle$
for all $(x, x^*) \in E \times E^*$. For a sequence $\{x_n\}$ of $E$, the strong and weak convergence of
$\{x_n\}$ to $x \in E$ is denoted by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively.

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Let $E$ be a Banach space with norm $\| \cdot \|$ and let $S(E) = \{ x \in E : \|x\| = 1 \}$. Then the duality mapping $J$ from $E$ into $2^{E^*}$ is defined by

$$(2.1) \quad Jx = \{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \}$$

for all $x \in E$. It is known that $Jx \neq \emptyset$ for all $x \in E$. The space $E$ is said to be smooth if the limit

$$(2.2) \quad \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$. In this case, the norm of $E$ is said to be Gâteaux differentiable. The norm of $E$ is also said to be uniformly Gâteaux differentiable (resp. uniformly Fréchet differentiable) if the limit $(2.2)$ converges uniformly in $x \in S(E)$ for all $y \in S(E)$ (resp. uniformly in $x, y \in S(E)$). The space $E$ is said to be uniformly smooth if the norm of $E$ is uniformly Fréchet differentiable.

The space $E$ is said to be strictly convex if $\|(x + y)/2\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is also said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x - y\| \geq \varepsilon$ and $x, y \in S(E)$ imply that $\|(x + y)/2\| \leq 1 - \delta$. The space $E$ is said to have the Kadec–Klee property if $x_n \to x$ whenever $\{x_n\}$ is a sequence of $E$ such that $x_n \to x$ and $\|x_n\| \to \|x\|$. We know the following; see, for instance, [10, 24]:

1. If $E$ is smooth, then $J$ is single-valued;
2. If $E$ is strictly convex, then $Jx \cap Jy \neq \emptyset$ implies that $x = y$;
3. If $E$ is reflexive, then $J$ is onto;
4. If $E$ is uniformly smooth if and only if $E^*$ is uniformly convex;
5. If $E$ is uniformly convex, then $E$ is a strictly convex and reflexive Banach space with the Kadec–Klee property.

Let $E$ be a smooth Banach space. Following [1, 12], let $\phi$ be a mapping from $E \times E$ into $\mathbb{R}$ defined by

$$(2.3) \quad \phi(x, y) = \|x\|^2 - 2 \langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. It is obvious that

$$(2.4) \quad (\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$$

for all $x, y \in E$. If $C$ is a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, then for each $x \in E$, there exists a unique $z \in C$ (denoted by $\Pi_C x$) such that $\phi(z, x) = \min_{y \in C} \phi(y, x)$. The mapping $\Pi_C$ is called the generalized projection [1] from $E$ onto $C$. Similarly, for each $x \in E$, there exists a unique $z \in C$ (denoted by $P_C x$) such that $\|z - x\| = \min_{y \in C} \|y - x\|$. The mapping $P_C$ is called the metric projection from $E$ onto $C$. It is easy to see that

$$(2.5) \quad \Pi_C(0) = P_C(0).$$

If $E$ is a Hilbert space, then $\Pi_C(x) = P_C(x)$ for all $x \in E$. For $(x, z) \in E \times C$, the following hold; see [1, 12, 24]:

1. $z = \Pi_C(x)$ if and only if $\langle y - z, Jx - Jz \rangle \leq 0$ for all $y \in C$;
2. $z = P_C(x)$ if and only if $\langle y - z, J(x - z) \rangle \leq 0$ for all $y \in C$.

Let $E$ be a smooth Banach space, $C$ a nonempty closed convex subset of $E$, and $T$ a mapping from $C$ into itself. The set of fixed points of $T$ is denoted by $F(T)$. Then $T$ is
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sion to be of firmly nonexpansive type [13] if
\begin{equation}
(Tx - Ty, Jx - JTx - (Jy - JTy)) \geq 0
\end{equation}
for all $x, y \in C$. If $E$ is a Hilbert space, then $J = I$ (the identity operator on $E$) and hence $T$ is of firmly nonexpansive type if and only if it is firmly nonexpansive in the classical sense, that is,
\begin{equation}
\|Tx - Ty\|^2 \leq (Tx - Ty, x - y)
\end{equation}
for all $x, y \in C$; see, for example, [6, 8, 9, 11, 26]. It is easy to verify that the generalized projection operator $II_C$ is of firmly nonexpansive type and $F(II_C) = C$. If $r > 0$, $C$ is a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, and $A \subset E \times E^*$ is a monotone operator such that $D(A) \subset C \subset J^{-1}R(J + rA)$, then the resolvent $Q_r$ of $A$ defined by
\begin{equation}
Q_r x = (J + rA)^{-1}Jx
\end{equation}
for all $x \in C$ is a firmly nonexpansive-type mapping from $C$ into itself and $F(Q_r) = A^{-1}0$; see [13-15] for more details. The class of firmly nonexpansive-type mappings is included in the class of $D$-firm operators [3], where $D$ stands for a Bregman distance. We also know that $T$ is of firmly nonexpansive type if and only if
\begin{equation}
\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(Tx, x) + \phi(Ty, y) \leq \phi(Tx, x) + \phi(Ty, y)
\end{equation}
for all $x, y \in C$; see [3, 13]. In particular, if a firmly nonexpansive-type mapping $T$ has a fixed point, then
\begin{equation}
\phi(u, Tx) + \phi(Tx, x) \leq \phi(u, x)
\end{equation}
for all $u \in F(T)$ and $x \in C$. The mapping $T$ is also said to be nonspreading [14] if
\begin{equation}
\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, x) + \phi(Ty, x)
\end{equation}
for all $x, y \in C$. It is easy to see that every firmly nonexpansive-type mapping is nonspreading. A point $u \in C$ is said to be asymptotic fixed point [19] of $T$ if there exists a sequence $\{x_n\}$ of $C$ such that $x_n \rightharpoonup u$ and $\|x_n - Tx_n\| \to 0$. The set of asymptotic fixed points of $T$ is denoted by $\widehat{F}(T)$. The mapping $T$ is also said to be relatively nonexpansive [16, 17] if the following conditions are satisfied:

1. $F(T)$ is nonempty;
2. $\widehat{F}(T) = F(T)$;
3. $\phi(u, Tx) \leq \phi(u, x)$ for all $u \in F(T)$ and $x \in C$.

We know the following lemmas:

**Lemma 2.1** ([14]). Let $E$ be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$, and $T$ a nonspreading mapping from $C$ into itself. Then $\widehat{F}(T) = F(T)$. Further, if $F(T)$ is nonempty, then $T$ is relatively nonexpansive.

**Lemma 2.2** ([17]). Let $E$ be a smooth and strictly convex Banach space, $C$ a nonempty closed convex subset of $E$, and $T$ a mapping from $C$ into itself such that $F(T)$ is nonempty and $\phi(u, Tx) \leq \phi(u, x)$ for all $u \in F(T)$ and $x \in C$. Then $F(T)$ is closed and convex.

Motivated by the technique in [23, 24], the following fixed point theorem for nonspreading mappings in Banach spaces was shown:
**Theorem 2.3** ([14]). Let $E$ be a smooth, strictly convex, and reflexive Banach space, $C$ a nonempty closed convex subset of $E$, and $T$ a nonspringing mapping from $C$ into itself. Then $F(T)$ is nonempty if and only if there exists $x \in C$ such that $\{T^n x\}$ is bounded.

As a direct consequence of Theorem 2.3, we obtain the following:

**Corollary 2.4** ([13]). Let $E$ be a smooth, strictly convex, and reflexive Banach space, $C$ a nonempty closed convex subset of $E$, and $T$ a firmly nonexpansive-type mapping from $C$ into itself. Then $F(T)$ is nonempty if and only if there exists $x \in C$ such that $\{T^n x\}$ is bounded.

The following lemma implies that the class of firmly nonexpansive-type mappings is coincident with that of resolvents of monotone operators:

**Lemma 2.5** ([14]). Let $E$ be a smooth, strictly convex, and reflexive Banach space, $C$ a nonempty closed convex subset of $E$, and $T$ a mapping from $C$ into itself. Then $T$ is of firmly nonexpansive type if and only if there exists a monotone operator $A \subset E \times E^*$ such that $D(A) \subset C \subset J^{-1}R(J + A)$ and $Tx = (J + A)^{-1}Jx$ for all $x \in C$.  

3. Results

Using Lemmas 2.1, 2.2 and Corollary 2.4, we can show the following strong convergence theorem of Browder's type for firmly nonexpansive-type mappings in Banach spaces:

**Theorem 3.1** ([15]). Let $E$ be a smooth, strictly convex, and reflexive Banach space, $C$ a nonempty bounded closed convex subset of $E$ with $0 \in C$, and $T$ a firmly nonexpansive-type mapping from $C$ into itself. Then the following hold:

1. For each $t \in (0, 1)$, there exists a unique $u_t \in C$ such that
   \[ u_t = (1 - t)Tu_t; \]
2. if $E$ has the Kadec-Klee property and the norm of $E$ is uniformly Gâteaux differentiable, then the net $\{u_t\}$ converges strongly to $P_{F(T)}(0)$ as $t \downarrow 0$, where $P_{F(T)}$ denotes the metric projection from $E$ onto $F(T)$.

The following is a direct consequence of Theorem 3.1 and Lemma 2.5:

**Theorem 3.2** ([15]). Let $E$ be a smooth, strictly convex, and reflexive Banach space and $C$ a nonempty bounded closed convex subset of $E$ with $0 \in C$. Let $r$ be a positive real number, $A \subset E \times E^*$ a monotone operator such that $D(A) \subset C \subset J^{-1}R(J + rA)$, and $Q_rx = (J + rA)^{-1}Jx$ for all $x \in C$. Then the following hold:

1. For each $t \in (0, 1)$, there exists a unique $u_t \in C$ such that
   \[ u_t = (1 - t)Q_ru_t; \]
2. if $E$ has the Kadec-Klee property and the norm of $E$ is uniformly Gâteaux differentiable, then the net $\{u_t\}$ converges strongly to $P_{A^{-1}0}(0)$ as $t \downarrow 0$, where $P_{A^{-1}0}$ denotes the metric projection from $E$ onto $A^{-1}0$.

**Corollary 3.3.** Let $E$ be a smooth, strictly convex, and reflexive Banach space and $A \subset E \times E^*$ a maximal monotone operator such that $D(A)$ is bounded and $0 \in D(A)$, where $D(A)$ denotes the norm closure of $D(A)$. Let $r$ be a positive real number and $Q_rx = (J + rA)^{-1}Jx$ for all $x \in D(A)$. Then the following hold:
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(1) For each $t \in (0, 1)$, there exists a unique $u_t \in \overline{D(A)}$ such that

$$u_t = (1 - t)Q_ru_t;$$

(2) if $E$ has the Kadec–Klee property and the norm of $E$ is uniformly Gâteaux differentiable, then the net $\{u_t\}$ converges strongly to $P_{A^{-1}0}(0)$ as $t \downarrow 0$, where $P_{A^{-1}0}$ denotes the metric projection from $E$ onto $A^{-1}0$.

Proof. We know that $\overline{D(A)}$ is closed and convex. In fact,

$$\lim_{t \downarrow 0} J_t x = x$$

for all $x \in \overline{co}D(A)$, where $\overline{co}D(A)$ denotes the closed convex hull of $D(A)$ and $J_t$ is defined by $J_t = (I + tJ^{-1}A)^{-1}$ for all $t > 0$; see [2, 25] for more details. Thus we have $\overline{co}D(A) \subset \overline{D(A)}$. This implies that $\overline{co}D(A) = \overline{D(A)}$ and hence $\overline{D(A)}$ is closed and convex.

Since $A$ is maximal monotone, we know that $R(J+rA) = E^*$ see [2, 7, 22, 25]. Putting $C = \overline{D(A)}$, we know that $C$ is a bounded closed convex subset of $E$ with $0 \in C$,

$$D(A) \subset C \subset E = J^{-1}E^* = J^{-1}R(J+rA),$$

and $Q_r$ is a firmly nonexpansive-type mapping from $C$ into itself. Thus, by Theorem 3.2, we obtain the conclusion. $\square$

Let $E$ be a Banach space and $f$ a function from $E$ into $(-\infty, \infty]$. Then $f$ is said to be proper if the effective domain $D(f) = \{x \in E : f(x) \in \mathbb{R}\}$ of $f$ is nonempty. It is said to be convex if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

whenever $x, y \in E$ and $\alpha \in (0, 1)$. It is also said to be lower semicontinuous if $\{x \in E : f(x) \leq r\}$ is closed in $E$ for all $r \in \mathbb{R}$. Let $x \in E$ be given. Then a point $x^* \in E^*$ is said to be a subgradient of $f$ at $x$ if

$$f(x) + \langle y - x, x^* \rangle \leq f(y)$$

for all $y \in E$. The set of subgradients of $f$ at $x$ is said to be the subdifferential of $f$ at $x$ and denoted by $\partial f(x)$. The mapping $\partial f \subset E \times E^*$ is called the subdifferential mapping of $f$.

Using Corollary 3.3, we can also show the following corollary:

Corollary 3.4 ([15]). Let $E$ be a smooth, strictly convex, and reflexive Banach space, $r$ a positive real number, and $f$ a proper lower semicontinuous convex function from $E$ into $(-\infty, \infty]$ such that $D(f)$ is bounded and $0 \in \overline{D(f)}$. Then the following hold:

1. For each $t \in (0, 1)$, there exists a unique $u_t \in \overline{D(f)}$ such that

$$u_t = (1 - t) \cdot \arg\min_{y \in E} \left\{ f(y) + \frac{1}{2r} \phi(y, u_t) \right\};$$

2. if $E$ has the Kadec–Klee property and the norm of $E$ is uniformly Gâteaux differentiable, then the net $\{u_t\}$ converges strongly to $P(0)$ as $t \downarrow 0$, where $P$ denotes the metric projection from $E$ onto $\arg\min_{y \in E} f(y)$.  


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Proof. Brøndsted and Rockafellar's theorem [4] implies that $D(\partial f)$ is norm dense in $D(f)$, that is, $D(f) \subset \overline{D(\partial f)}$; see also [25]. This gives us that $D(\partial f) = D(f)$. Rockafellar's theorem [20,21] also ensures that the subdifferential $\partial f$ of $f$ is maximal monotone; see also [25]. We also know that

$$Q_r x = \arg\min_{y \in E} \{ f(y) + \frac{1}{2r} \phi(y, x) \}$$

for all $x \in C = \overline{D(f)}$, where $Q_r x = (J + r \partial f)^{-1} J$ for all $x \in C$; see, for instance, [12,25]. It is also known that $(\partial f)^{-1}(0) = \arg\min_{y \in E} f(y)$ and $D(\partial f) \subset D(f)$. Thus, by Corollary 3.3, we obtain the conclusion.

We do not know the answers to the following problems:

Problem 3.5. Is it possible to prove Theorem 3.1 without assuming that $C$ is bounded?

Problem 3.6. Is it possible to prove Theorem 3.1 for a net of the form: $x \in C$ and

$$u_t = tx + (1-t) Tu_t$$

for all $t \in (0, 1)$?

Problem 3.7. Is it possible to obtain an analogous result of Browder's strong convergence theorem for nonspreadning mappings in Banach spaces?

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