数理解析研究所講究録
第 1643 巻 2009 年 17-26



# Entropy and recurrent dimensions of discrete dynamical systems given by almost periodic sequences

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## 1. Introduction

Khinchin's conjecture ( firstly considered in his paper [2]) for the sequence of partial quotients of an irrational number is as follows: The sequence of partial quotients in the continued fraction expansions of an algebraic real number of degree $\geq 3$ is unbounded and "random"(aperiodic or irregular). Recently, in [1] B.Adamczewski and Y. Bugeaud gave a class of transcendental numbers, the partial quotients sequences of which have some recurrent properties. For this class numbers with the recurrent order value $\tau_0 > 0$ we call them $\tau_0$-transcendental numbers (cf. [11]). The complement of the set of transcendental numbers, which have these recurrent properties, in the set of irrational numbers contains algebraic numbers of degree $\geq 3$ and 0-recurrent or non-recurrent transcendental numbers.

In this paper we study "symbolic dynamically" almost periodic sequences, which have recurrent properties, to investigate some unpredictable behaviors of these sequences by estimating the topological entropies and the recurrent dimensions. First we treat symbolic dynamical systems with finite alphabet spaces and we give inequality relations between recurrent dimensions and entropy of strongly almost periodic or eventually strongly almost periodic sequences. Next we extend these relations to the case of infinite alphabet spaces by using a truncation method.

The recurrent dimensions have been introduced in our previous paper [6] as the parameters, which indicate recurrent properties, defined by using $\varepsilon$-neighborhood recurrent times. For a sequence $u = \{a_i\}_{i\geq 1}$ and a shift map $\sigma$, defined by $(\sigma u)_n = u_{n+1} = a_{n+1}$, we consider a discrete orbit $\Sigma = \{u, \sigma u, \sigma^2 u, ..., \sigma^n u, ...\}$. We define the lower recurrent dimension by the following limit infimum value as $\varepsilon \to 0$, using the infimum of the first $\varepsilon$-neighborhood recurrent times in the orbit $\Sigma$, which is denoted by $\underline{M}_\Sigma(\varepsilon)$:

$$\underline{D}_r(\Sigma) = \liminf_{\varepsilon \to 0} \frac{\log \underline{M}_\Sigma(\varepsilon)}{-\log \varepsilon}$$

and we also define the upper recurrent dimension by using their supremum values:

$$\overline{D}_r(\Sigma) = \limsup_{\varepsilon \to 0} \frac{\log \overline{M}_\Sigma(\varepsilon)}{-\log \varepsilon}.$$

In our previous papers [7], [9], [10] we introduce the gap value $\mathcal{G}(\Sigma)$ of recurrent dimensions by

$$\mathcal{G}(\Sigma) = \overline{D}_r(\Sigma) - \underline{D}_r(\Sigma)$$

as the parameter which indicates the levels of unpredictability of a sequence $u$ or a discrete orbit $\Sigma$. In this paper we define these values for almost periodic sequences and show the inequality relations between the recurrent dimensions and the topological entropies of these sequences.

As an example we investigate a modified Sturmian sequence, which has an infinite alphabet space, and we show that its discrete orbit has a positive gap of recurrent dimensions if the irrational frequency of its original Sturmian sequence is a Liouville number.

The plan of this paper is as follows: In Section 2, introducing notations in symbolic dynamical systems with a finite alphabet space, we show the inequality relations between the recurrent dimensions and the topological entropies of symbolic dynamical systems given by almost periodic sequences. In Section 3 we treat the infinite alphabet space by truncating these symbol spaces. In Section 4 we estimate the gaps of recurrent dimensions of discrete orbits, which are given by modified Sturmian sequences.

## 2. Symbolic dynamical systems: finite alphabets case

In this section, introducing notations in symbolic dynamical systems, we show some inequality relations between the recurrent dimensions and the topological entropy of almost periodic sequences.

Let $\mathcal{A} = \{a_1, a_2, ..., a_\mu\}$ be a finite set of symbols and a word $V = v_1 v_2 .. v_r$ be a finite string of elements of $\mathcal{A}$ with its length $r$, denoted by $|V| = r$. The set of nonnegative integers is denoted by $\mathbf{N}_0 = \mathbf{N} \cup \{0\} = \{0, 1, 2, ...\}$ and we consider a (one-sided) sequence of elements of $\mathcal{A}$, $u = (u_n)_{n \in \mathbf{N}_0} = u_0 u_1 u_2 ... \in \mathbf{A}^{\mathbf{N}_0}$. A word $W = w_1 w_2 ... w_r$ is called a factor of $u$ if $u_m = w_1, u_{m+1} = w_2, ..., u_{m+r-1} = w_r$ for some $m \in \mathbf{N}_0$. $\mathcal{L}(u)$ denotes the set of all factors of $u$, which is called the language of the sequence $u$ and $\mathcal{L}_n(u)$ denotes the set of all factors with its length $n$.

If $i \leq j$ are nonnegative integers, denote by $[i, j]$ the segment of $\mathbf{N}_0$ with ends $i$ and $j$. For a sequence $\omega \in \mathbf{A}^{\mathbf{N}_0}$ we also denote by $\omega[i, j]$ a substring $\omega_i \omega_{i+1} ... \omega_j$ of $\omega$. The string of the form $\omega[0, i]$ for some $i$ is called a prefix of $\omega$, and the sequence of the form $\omega_i \omega_{i+1} \omega_{i+2} ...$ for some $i$ is called a suffix of $\omega$ and is denoted by $\omega[i, \infty)$.

Following the notations introduced in [4] or [12], we give some defintions on almost periodic sequences. A sequence $\omega$ is called almost periodic if for any factor $x$ of $\omega$ occurring in it infinitely many times there exists a number $l$ such that any factor of $\omega$ with its length $l$ contains at least one occurrence of $x$. We denote the class of all almost periodic sequences by $\mathcal{AP}$.

A sequence $\omega$ is called strongly almost periodic if for any factor $x$ of $\omega$ there exists a number $l$ such that any factor of $\omega$ with its length $l$ contains at least one occurrence of $x$. We denote the class of all strongly almost periodic sequences by $\mathcal{SAP}$.

A sequence $\omega$ is called eventually strongly almost periodic if some its suffix is strongly almost periodic. We denote the class of all eventually strongly almost periodic sequences by $\mathcal{EAP}$.

First we consider the case where a sequence $u \in \mathcal{SAP}$.

We denote the complexity function of $u$ by $P_u(n) = \#\mathcal{L}_n(u)$, which is the number of different words of length $n$ occurring in $u$. We consider the following metric on $\mathcal{A}^{\mathbf{N}_0}$:

$$d(u,v) = 2^{-\min\{n\in\mathbf{N}_0 : u_n \neq v_n\}}$$

for $u, v \in \mathcal{A}^{\mathbf{N}_0} : u \neq v$. The one-sided shift $\sigma : \mathcal{A}^{\mathbf{N}_0} \to \mathcal{A}^{\mathbf{N}_0}$ is defined by

$$(\sigma u)_n = u_{n+1}, \quad n \in \mathbf{N}_0$$

and its discrete orbit is denoted by

$$\Sigma := \Sigma_u = \{u, \sigma u, \sigma^2 u, ..., \sigma^n u, ...\}.$$

Denote the recurrency function of $u$ by $R_u(n)$, which is the least integer $m(:= R_u(n))$ such that each $m$-factor of $u$ contains every $n$-factor of $u$.

We define the first $\varepsilon$-recurrent times by

$$\underline{M}_\Sigma(\varepsilon) = \inf_{l\in\mathbf{N}_0} \min\{m \in \mathbf{N} : \ d(\sigma^{m+l}u, \sigma^l u) < \varepsilon\},$$
$$\overline{M}_\Sigma(\varepsilon) = \sup_{l\in\mathbf{N}_0} \min\{m \in \mathbf{N} : \ d(\sigma^{m+l}u, \sigma^l u) < \varepsilon\}.$$

Next we extend these definitions to the case of eventually strongly almost periodic sequences.

We define the eventually restrict function $e_r : \ \mathcal{EAP} \to \mathbf{N}_0$ by

$$e_r(u) = \min\{m \in \mathbf{N}_0 : \ u[m,\infty) \in \mathcal{SAP}\}$$

for $u \in \mathcal{EAP}$.

The eventually complexity function $p_u(n)$ is the number of different words of length $n$ occurring in $u[e_r(u),\infty)$ and the eventually recurrency function $r_u(n)$ is is the least integer $m(:= r_u(n))$ such that each $m$-factor of $u[e_r(u),\infty)$ contains every $n$-factor of $u[e_r(u),\infty)$.

Similarly, we can define the eventually first $\varepsilon$-recurrent times by

$$\underline{m}_\Sigma(\varepsilon) = \inf_{l\in[e_r(u),\infty)} \min\{m \in \mathbf{N} : \ d(\sigma^{m+l}u, \sigma^l u) < \varepsilon\},$$
$$\overline{m}_\Sigma(\varepsilon) = \sup_{l\in[e_r(u),\infty)} \min\{m \in \mathbf{N} : \ d(\sigma^{m+l}u, \sigma^l u) < \varepsilon\}.$$

Then we can obtain the following relations

**Lemma 2.1.** *For $\varepsilon_n = 2^{-n}$, $n = 1, 2, ...$ and $u \in \mathcal{EAP}$, we have*

$$\underline{m}_\Sigma(\varepsilon_n) \leq p_u(n), \tag{2.1}$$
$$\overline{m}_\Sigma(\varepsilon_n) = r_u(n) - n + 1. \tag{2.2}$$

*Proof.* First we prove (2.1). Assume that $\underline{m}_\Sigma(\varepsilon_n) \geq p_u(n) + 1$, then we have

$$d(\sigma^l u, \sigma^{l+m} u) \geq \varepsilon_n = 2^{-n}, \quad 1 \leq m \leq p_u(n)$$

for every $l \in [e_r(u), \infty)$. It follows that for each $m, m' : 1 \leq m, m' \leq p_u(n),\ m \neq m'$, there exists $k \in \{0, 1, ..., n\}$ such that

$$(\sigma^{l+m} u)_k \neq (\sigma^{l+m'} u)_k, \quad (\sigma^l u)_k \neq (\sigma^{l+m} u)_k.$$

Thus there exist $p_u(n) + 1$ different $n$-factors, which is a contradiction.

Next we show that $\overline{m}_\Sigma(\varepsilon_n) \geq r_u(n) - n + 1$. Let $L = r_u(n) - 1$, then there exists a $L$-factor: $u_r, u_{r+1}, ..., u_{r+L-1},\ r \in [e_r(u), \infty)$, which does not contain a $n$-factor $W : w_j, w_{j+1} ..., w_{j+1-n},\ j \in [e_r(u), \infty)$. The following case

$$u_{r-1} = w_j, u_r = w_{j+1}, ..., u_{r+n-2} = w_{j+n-1}, \cdots$$
$$u_{r+L-n+1} = w_j, u_r = w_{j+1}, ..., u_{r+L} = w_{j+n-1}$$

has the least $\varepsilon_n$-recurrent time:

$$d(\sigma^{r-1} u, \sigma^{r+L-n+1} u) < \varepsilon_n = 2^{-n},$$

which yields

$$\overline{m}_\Sigma(\varepsilon_n) \geq L - n + 1 + 1 = r_u(n) - n + 1.$$

Next we show that $\overline{m}_\Sigma(\varepsilon_n) \leq r_u(n) - n + 1$. For estimating an upper bound of $\overline{m}_\Sigma(\varepsilon_n)$ it is sufficient to consider a $r_u(n)$-factor, which contains exactly one $n$-factor $W : w_j, ..., w_{j+n-1},\ j \in [e_r(u), \infty)$, not one another $W$. Suitablly shifting gives the following typical case:

$$u_l = w_j, u_{l+1} = w_{j+1}, ..., u_{l+n-1} = w_{j+n-1}, \cdots$$
$$u_{l+r_u(n)-n+1} = w_j, ..., u_{l+r_u(n)-1} = w_{j+n-1}.$$

Note that, if $u_{l+r_u(n)-1} = w_i : i < j + n - 1$, there exists a $r_u(n)$-factor, which does not contain the $n$-factor $W$. Thus we have

$$d(\sigma^l u, \sigma^{l+r_u(n)-n+1} u) < \varepsilon_n = 2^{-n},$$

which yields $\overline{m}_\Sigma(\varepsilon_n) \leq r_u(n) - n + 1$. $\square$

In [3] Morse and Hedlund have given the following inequality

$$p_u(n) + n \leq r_u(n). \tag{2.3}$$

Now we have the following sequence of estimates:

$$\underline{m}_\Sigma(\varepsilon_n) \leq p_u(n), \quad p_u(n) + n \leq r_u(n), \quad r_u(n) - n + 1 = \overline{m}_\Sigma(\varepsilon_n).$$

It follows that

$$\underline{m}_\Sigma(\varepsilon_n) \leq p_u(n) \leq \overline{m}_\Sigma(\varepsilon_n). \tag{2.4}$$

For $u \in \mathcal{SAP}$, the topological entropy $\mathcal{H}_P(u)$ is given by the complexity function:

$$\mathcal{H}_P(u) = \lim_{n\to\infty} \frac{\log_\mu P_u(n)}{n}.$$

Here we also put

$$\mathcal{H}_R(u) = \lim_{n\to\infty} \frac{\log_\mu R_u(n)}{n}$$

and we define the recurrent dimensions by

$$\overline{D}_r(\Sigma) = \limsup_{\varepsilon\to 0} \frac{\log \overline{M}_\Sigma(\varepsilon)}{-\log\varepsilon},$$
$$\underline{D}_r(\Sigma) = \liminf_{\varepsilon\to 0} \frac{\log \underline{M}_\Sigma(\varepsilon)}{-\log\varepsilon}.$$

In [7] we have shown that these recurrent dimensions are given by

$$\overline{D}_r(\Sigma) = \limsup_{n\to\infty} \sup_{\varepsilon_{n+1}\le\varepsilon\le\varepsilon_n} \frac{\log \overline{M}_\Sigma(\varepsilon)}{-\log\varepsilon},$$
$$\underline{D}_r(\Sigma) = \liminf_{n\to\infty} \inf_{\varepsilon_{n+1}\le\varepsilon\le\varepsilon_n} \frac{\log \underline{M}_\Sigma(\varepsilon)}{-\log\varepsilon}.$$

for any sequence $\{\varepsilon_n\} : \varepsilon_n \downarrow 0$ as $n \to \infty$.

Here we can extend these definitions to the case of eventually strongly almost periodic functions.

For $u \in \mathcal{EAP}$, the topological entropy $h_p(u)$ is given by the complexity function:

$$h_p(u) = \lim_{n\to\infty} \frac{\log_\mu p_u(n)}{n}.$$

Here we also put

$$h_r(u) = \lim_{n\to\infty} \frac{\log_\mu r_u(n)}{n}$$

and we define the recurrent dimensions by

$$\overline{d}_r(\Sigma) = \limsup_{\varepsilon\to 0} \frac{\log \overline{m}_\Sigma(\varepsilon)}{-\log\varepsilon},$$
$$\underline{d}_r(\Sigma) = \liminf_{\varepsilon\to 0} \frac{\log \underline{m}_\Sigma(\varepsilon)}{-\log\varepsilon}.$$

We can also show that these recurrent dimensions are given by

$$\overline{d}_r(\Sigma) = \limsup_{n\to\infty} \sup_{\varepsilon_{n+1}\le\varepsilon\le\varepsilon_n} \frac{\log \overline{m}_\Sigma(\varepsilon)}{-\log\varepsilon},$$
$$\underline{d}_r(\Sigma) = \liminf_{n\to\infty} \inf_{\varepsilon_{n+1}\le\varepsilon\le\varepsilon_n} \frac{\log \underline{m}_\Sigma(\varepsilon)}{-\log\varepsilon}.$$

for any sequence $\{\varepsilon_n\} : \varepsilon_n \downarrow 0$ as $n \to \infty$.

Then we have the following inequality relations.

**Theorem 2.2.** *For a sequence $u \in \mathcal{EAP}$ and $\Sigma = \{\sigma^n u : n \in \mathbf{N}_0\}$ we have*

$$\frac{\log 2}{\log\mu} \cdot \underline{d}_r(\Sigma) \le h_p(u) \le h_r(u) = \frac{\log 2}{\log\mu} \cdot \overline{d}_r(\Sigma). \tag{2.5}$$

*Proof.* The first (left side) inequality can be estimated by the definitions and (2.1) in Lemma 2.1 for $\varepsilon_n = 2^{-n}$.

$$\begin{aligned}
\underline{d}_r(\Sigma) &= \liminf_{n\to\infty} \inf_{\varepsilon_{n+1}\le\varepsilon\le\varepsilon_n} \frac{\log \underline{m}_\Sigma(\varepsilon)}{-\log\varepsilon} \\
&\le \liminf_{n\to\infty} \frac{\log \underline{m}_\Sigma(\varepsilon_n)}{-\log\varepsilon_n} \\
&\le \frac{\log\mu}{\log 2}\lim_{n\to\infty}\frac{\log p_u(n)}{n\log\mu} = \frac{\log\mu}{\log 2}\cdot h_p(u).
\end{aligned}$$

The second inequality is obvious from the definitions and (2.3). For the right side equality we have the following estimates by using (2.2).

$$\begin{aligned}
\overline{d}_r(\Sigma) &= \limsup_{n\to\infty} \sup_{\varepsilon_{n+1}\le\varepsilon\le\varepsilon_n} \frac{\log \overline{m}_\Sigma(\varepsilon)}{-\log\varepsilon} \\
&\le \limsup_{n\to\infty} \frac{\log \overline{m}_\Sigma(\varepsilon_{n+1})}{-\log\varepsilon_n} \\
&= \limsup_{n\to\infty} \frac{\log(r_u(n+1)-(n+1)+1)}{n\log 2} \\
&\le \frac{\log\mu}{\log 2}\lim_{n\to\infty}\frac{\log r_u(n+1)}{(n+1)\log\mu}\cdot\frac{n+1}{n} = \frac{\log\mu}{\log 2}\cdot h_r(u).
\end{aligned}$$

If $u$ is periodic or eventually periodic, then we have $\overline{d}_r(\Sigma) = h_r(u) = 0$. Thus it is sufficient to consider the case where the sequence $u$ is neither periodic nor eventually periodic and then we have $r_u(n) \ge 2n$. Using this inequality and (2.2), we have the following estimates.

$$\begin{aligned}
\overline{d}_r(\Sigma) &= \limsup_{n\to\infty} \sup_{\varepsilon_{n+1}\le\varepsilon\le\varepsilon_n} \frac{\log \overline{m}_\Sigma(\varepsilon)}{-\log\varepsilon} \\
&\ge \limsup_{n\to\infty} \frac{\log \overline{m}_\Sigma(\varepsilon_n)}{-\log\varepsilon_n} \\
&= \lim_{n\to\infty} \frac{\log(r_u(n)-n+1)}{n\log 2} \\
&\ge \frac{\log\mu}{\log 2}\lim_{n\to\infty}\frac{\log \frac{1}{2}r_u(n)}{n\log\mu} = \frac{\log\mu}{\log 2}\cdot h_r(u).
\end{aligned}$$

Thus we obtain the equality in (2.5).

□

## 3. Symbolic dynamical systems II: infinite alphabets case

In this section we consider the infinite alphabet space $\mathcal{A} = \{a_1, a_2, \dots, a_n, \dots\}$ and we use the same notations and definitions as in the previous section if these are well defined. Since the complexity function $P_u(n)$, the recurrency function $R_u(n)$ and the upper first $\varepsilon$-recurrent time $\overline{M}_\Sigma(\varepsilon)$ become infinite for the infinite alphabet space, we need some truncations of the symbols to define these functions.

For a large integer $N$ we define a finite alphabet space $\mathcal{A}_N$ by

$$\mathcal{A}_N = \{a_1, a_2, ..., a_N\}$$

and for a sequence $u \in \mathcal{A}^{\mathbf{N}_0}$ we define $u^{(N)} \in \mathcal{A}_N^{\mathbf{N}_0}$, substituting any symbols $a_j$, $j \geq N+1$ in the sequence $u$ by $a_N$, that is, $u^{(N)}$ is obtained by putting

$$a_N = a_{N+1} = a_{N+2} = \cdots$$

in $u$.

Here we assume that $u^{(N)} \in \mathcal{EAP}$, then we can define the the complexity function $p_{u^{(N)}}(n)$, the recurrency function $r_{u^{(N)}}(n)$ for the finite alphabet space $\mathcal{A}_N$ as in the previous section. Let

$$\Sigma^{(N)} = \{u^{(N)}, \sigma u^{(N)}, \sigma^2 u^{(N)}, ..., \sigma^n u^{(N)}, ...\},$$

then we can also define the first $\varepsilon$-recurrent times $\overline{m}_{\Sigma^{(N)}}(\varepsilon), \underline{m}_{\Sigma^{(N)}}(\varepsilon)$ and Lemma 2.1 holds for the sequence $u^{(N)}$.

Furthermore, we can give the entropies and recurrent dimensions for $u^{(N)} \in \mathcal{A}_N$ as follows:

$$\begin{aligned}
h_p(u^{(N)}) &= \lim_{n\to\infty} \frac{\log_N p_{u^{(N)}}(n)}{n}, \\
h_r(u^{(N)}) &= \lim_{n\to\infty} \frac{\log_N r_{u^{(N)}}(n)}{n}, \\
\overline{d}_r(\Sigma^{(N)}) &= \limsup_{\varepsilon\to 0} \frac{\log \overline{m}_{\Sigma^{(N)}}(\varepsilon)}{-\log\varepsilon}, \\
\underline{d}_r(\Sigma^{(N)}) &= \liminf_{\varepsilon\to 0} \frac{\log \underline{m}_{\Sigma^{(N)}}(\varepsilon)}{-\log\varepsilon}.
\end{aligned}$$

Applying Theorem 2.2 we can obtain the following inequality relations.

$$\frac{\log 2}{\log N} \cdot \underline{d}_r(\Sigma^{(N)}) \leq h_p(u^{(N)}) \leq h_r(u^{(N)}) = \frac{\log 2}{\log N} \cdot \overline{d}_r(\Sigma^{(N)}). \tag{3.1}$$

If the limits of these functions as $N \to \infty$ exist, these limit values are denoted as follows.

$$\begin{aligned}
&\lim_{N\to\infty} \overline{d}_r(\Sigma^{(N)}) = \overline{d}_r^{\infty}(\Sigma), \quad \lim_{N\to\infty} \underline{d}_r(\Sigma^{(N)}) = \overline{d}_r^{\infty}(\Sigma), \\
&\lim_{N\to\infty} h_r(u^{(N)}) = h_r^{\infty}(u), \quad \lim_{N\to\infty} h_p(u^{(N)}) = h_p^{\infty}(u).
\end{aligned}$$

It follows from (3.1) that, if $\limsup_{N\to\infty} \overline{d}_r(\Sigma^{(N)}) < \infty$, then $h_p^{\infty}(u) = 0$ and also, if $\liminf_{N\to\infty} \dfrac{\underline{d}_r(\Sigma^{(N)})}{\log N} \geq c$ holds for some constant $c > 0$, then we have $\liminf_{N\to\infty} h_p(u^{(N)}) > 0$. Furthermore, $\limsup_{N\to\infty} \dfrac{\underline{d}_r(\Sigma^{(N)})}{\log N} \geq c > 0$ yields $h_p^{\infty}(u) > 0$ if $h_p^{\infty}(u)$ exists.

## 4. Example of Sturmian sequences

Let $\mathcal{A} = \{1, 2\}$ and $w = w_0w_1w_2...$ be a 1-type sturmian sequence, which does not contain a word 22. Then it is well known that the complexity function satisfies $P_w(n) = n + 1$ and the frequency value of the letter 1 is given by

$$\tau = \lim_{n\to\infty} \frac{|w_0w_1...w_{n-1}|_1}{n}$$

where $|W|_1$ is the number of occurrences of the letter 1 in a word $W$.

Next we consider an infinite alphabet space $\mathcal{B} = \{1, a_1, a_2, ..., a_n, ...\}$ where $\{a_j\}$ is an increasing sequence of integers and $a_1 \geq 2$. Here we define a sequence $u \in \mathcal{B}^{\mathbf{N}_0}$, substituting the letters 2 in $w$ sequencially by the integer numbers of $\mathcal{B}$ in order, that is, the first appeared letter 2 in $w$ is substituted by $a_1$ and the next appeared letter 2 by $a_2$, ... and so on.

We define the truncated alphabet space for a large number $N$

$$\mathcal{B}_N = \{1, a_1, a_2, ..., a_N\},$$

the sequence $u^{(N)}$, the discrete orbit $\Sigma^{(N)}$ and the other notations as the same manner in the previous section.

Let $u_{\mathcal{E}_N} = a_N$ in the sequence $u$, then $u^{(N)}_{[\mathcal{E}_N,\infty)}$ becomes a 1-type Sturmian sequence of the alphabet space $\{1, a_N\}$. We can show that $\underline{d}_r(\Sigma^{(N)}) = 0$ by using the definition of the topological entropy $h_p(u^{(N)})$ and the inequality relation in Theorem 2.2, since the complexity function satisfies $p_{u^{(N)}}(n) = n + 1$.

Here we estimate its upper recurrent dimensions $\overline{d}_r(\Sigma^{(N)})$ according to the algebraic properties, parametrized Diophantine conditions, of the frequency value $\tau$. In our previous papers [7],[8] we introduce $d_0$-(D) condition, which specifies the (good or bad) levels of approximation by rational numbers.

If an irrational number $\tau$ satisfies $d_0$-(D) condition for $0 \leq d_0 < \infty$, then $\tau$ is a Roth number with its order $d_0 + \varepsilon$ for every $\varepsilon > 0$ and also $\tau$ is a weak Liouville number with its order $d_0 - \varepsilon$ for every $\varepsilon > 0$. If an irrational number $\tau$ does not satisfy the Diophantine condition for a finite value $d_0$, we say that $\tau$ is a Liouville number or $d_0 = \infty$.

**Theorem 4.1.** *For a large number $N$, let the sequences $u \in \mathcal{B}^{\mathbf{N}_0}$ and $u^{(N)} \in \mathcal{B}_N^{\mathbf{N}_0}$ be defined by using a Sturmian sequence $w$ as above. Assume that the frequency $\tau$ of $w$ satisfies $d_0$-(D) condition for $0 \leq d_0 < \infty$. Then we have*

$$\overline{d}_r(\Sigma^{(N)}) = 0 \tag{4.1}$$

*and consequently, we have*

$$\mathcal{G}(\Sigma^{(N)}) = 0 \tag{4.2}$$

*and furthermore, as $N \to \infty$, we obtain*

$$h_p^\infty(u) = 0. \tag{4.3}$$

*Proof.* Let $\zeta = \dfrac{\tau}{1+\tau}$, that is, $\zeta^{-1} = \tau^{-1} + 1$. Since $\zeta$ and $\tau$ have the same c.f.s. from the second partial quotient, $\zeta$ also satisfies the same order $d_0$-(D) condition as $\tau$. Denote the convergents of $\zeta$ by $\{r_j/s_j\}$, then in [3] Morse and Hedlund have shown that

$$r_{u^{(N)}}(s_j) = s_{j+1} + 2s_j - 1.$$

It follows from Lemma 2.1 that

$$\overline{m}_{\Sigma^{(N)}}(\varepsilon_{s_n}) = r_{u^{(N)}}(s_n) - s_n + 1 = s_{n+1} + s_n.$$

Thus we have

$$\begin{aligned}
\overline{d}_r(\Sigma^{(N)}) &= \limsup_{n\to\infty} \sup_{\varepsilon_{s_{n+1}}\le\varepsilon\le\varepsilon_{s_n}} \frac{\log \overline{m}_{\Sigma^{(N)}}(\varepsilon)}{-\log\varepsilon} \\
&\le \limsup_{n\to\infty} \frac{\log \overline{m}_{\Sigma^{(N)}}(\varepsilon_{s_{n+1}})}{-\log\varepsilon_{s_n}} \\
&= \lim_{n\to\infty} \frac{\log(s_{n+2}+s_{n+1})}{s_n\log 2} \\
&\le \lim_{n\to\infty} \frac{\log c s_n^{(1+d_0+\delta)^2}}{s_n\log 2} = 0
\end{aligned}$$

where we use the properties of $(d_0+\delta)$-Roth numbers for all $\delta > 0$:

$$s_{n+2} \le c s_{n+1}^{1+d_0+\delta} \le c s_n^{(1+d_0+\delta)^2}$$

and the notation $c$ as an appropriate constant in each estimate.

□

As in the proof of Theorem 4.1, let $\{r_n/s_n\}$ be the convergents of $\zeta = \tau/(\tau+1)$. Since $\zeta$ and $\tau$ have the same c.f.s. from the second partial quotient, they have the same increasing rates of denominators of the convergents. For simple descriptions and argument we use the convergents of $\zeta$ instead of $\tau$ to describe the assumption on Liouville numbers.

**Theorem 4.2.** *Under the same setting as in Theorem 4.1, assume that the frequency $\tau$ is a Liouville number such that there exists a subsequence $\{s_{n_j}\}$, which satisfies*

(4.4)
$$s_{n_j+1} \ge L^{s_{n_j}}$$

*for a constant $L > 1$. Then we have*

(4.5)
$$\overline{d}_r(\Sigma^{(N)}) \ge \frac{\log L}{\log 2}$$

*and consequently, we obtain*

(4.6)
$$\mathcal{G}(\Sigma^{(N)}) \ge \frac{\log L}{\log 2} > 0.$$

*Remark* 4.3. It follows from(4.4) that

$$|\zeta - \frac{r_{n_j}}{s_{n_j}}| \le \frac{1}{s_{n_j}L^{s_{n_j}}},$$

which gives the extremely good approximation property ($d_0 = \infty$) by rational numbers.

**Theorem 4.4.** *Under the same setting as in Theorem 4.1, assume that the frequency $\tau$ is a Liouville number such that there exists a subsequence $\{s_{l_j}\}$, which satisfies*

$$s_{l_{j+1}+1} \leq L^{s_{l_j}} \tag{4.7}$$

*for a constant $L > 1$. Then we have*

$$\overline{d}_r(\Sigma^{(N)}) \leq \frac{\log L}{\log 2} \tag{4.8}$$

*and consequently, we obtain*

$$h_p^\infty(u) = 0. \tag{4.9}$$

*Remark* 4.5. The condition (4.4) and (4.7) are incompatible. In fact, let $s_{l_i} < s_{n_j} \leq s_{l_{i+1}}$, then we can obtain the contradiction:

$$L^{s_{n_j}} \leq s_{n_j+1} \leq s_{l_{i+1}+1} \leq L^{s_{l_i}}.$$

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