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A BRIEF HISTORY OF THE KKM THEORY

Sehie Park

*The National Academy of Sciences, ROK, Seoul 137–044; and
Department of Mathematical Sciences, Seoul National University,
Seoul 151–747, KOREA*
shpark@math.snu.ac.kr

ABSTRACT. We review briefly the history of the KKM theory from the original KKM theorem in 1929 to the birth of the new KKM spaces. We recall Fan's works on the KKM theory from 1960s to 1980s; and various intersection theorems and equilibrium problems investigated by many authors. In 1983-2005, basic results in the theory was extended to convex spaces by Lassonde, to C-spaces by Horvath, and to G-convex spaces due to Park. In 2006, we introduced the concept of abstract convex spaces $(E, D; \Gamma)$ on which we can construct the KKM theory and study multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$. Moreover, abstract convex spaces satisfying an abstract form of the KKM theorem and its "open" version are called *KKM spaces*. We show that the class of KKM spaces are really adequate to establish the essential part of the KKM theory. Now the KKM theory becomes the study of the KKM spaces.

1. Introduction

One of the earliest equivalent formulations of the Brouwer fixed point theorem (1912) is a celebrated theorem of Knaster, Kuratowski, and Mazurkiewicz (1929) (simply, the KKM theorem), which is concerned with a particular type of multimaps called KKM maps later.

The KKM theory, first named by the author in 1992 [1], is the study of applications of various equivalent formulations of the KKM theorem and their generalizations. At the beginning, the basic theorems in the theory and their applications were established for convex subsets of topological vector spaces mainly by Ky Fan in 1961-84. A number of intersection theorems and applications to equilibrium problems followed. Then, the KKM theory has been extended to convex spaces by Lassonde in 1983, and to C-spaces

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(or H-spaces) by Horvath in 1984-93 and others. Since 1993, the theory is extended to generalized convex (G-convex) spaces in a sequence of papers of the author and others. Those basic theorems have many applications to various equilibrium problems in nonlinear analysis and other fields.

In the last decade, a number of authors have tried to imitate, modify, or generalize certain results on G-convex spaces $(X, D; \Gamma)$ and published a large number of papers. Many of them adopted artificial terminology and concepts without giving any proper examples or justifications. Some of them claimed to define new spaces more general than G-convex spaces. We found that all of such 'new' spaces are subsumed in the concept of ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ or spaces having a family $\{\phi_A\}_{A\in\langle D\rangle}$ of singular simplexes, where $\langle D\rangle$ denotes the set of all nonempty finite subsets of a set D. We noticed that this kind of spaces can be made into G-convex spaces.

In order to destroy such unnecessary concepts and to upgrade the KKM theory, in 2006-09, we proposed new concepts of abstract convex spaces and the KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [3-9]. Moreover, in the frame of such new spaces, certain broad classes $\mathfrak{KC}$ and $\mathfrak{KO}$ of multimaps (having the KKM property) are studied instead of traditional KKM maps. Now the KKM theory becomes the study of KKM spaces.

In this paper, we give a brief history of the KKM theory from the original KKM theorem to the birth of the new KKM spaces.

All references given by the form (year) can be found in [2] or references of [3-12].

2. Early works related to the KKM theory — From 1920s to 1980s

In 1910, the Brouwer fixed point theorem appeared:

Theorem (Brouwer, 1912). *A continuous map from an n-simplex to itself has a fixed point.*

In this theorem, the n-simplex can be replaced by the unit ball $\mathbf{B}^n$ or any compact convex subset of $\mathbf{R}^n$.

The "closed" version of the following is the origin of the KKM theory; see [2].

Theorem (KKM, 1929). *Let D be the set of vertices of an n-simplex Δ_n and $G : D \multimap \Delta_n$ be a KKM map (that is, $\operatorname{co} A \subset G(A)$ for each $A \subset D$) with closed [resp., open] values. Then $\bigcap_{z\in D} G(z) \neq \emptyset$.*

This is first applied to a direct proof of the Brouwer fixed point theorem by KKM (1929), and then to a von Neumann type minimax theorem for arbitrary topological vector spaces by Sion (1958).

Relatively early equivalent forms of the Brouwer theorem are as follows; see [2]:

1883 Poincaré's theorem.
1904 Bohl's non-retraction theorem.
1912 Brouwer's fixed point theorem.
1928 Sperner's combinatorial lemma.
1929 The Knaster-Kuratowski-Mazurkiewicz theorem.
1930 Caccioppoli's fixed point theorem.
1930 Schauder's fixed point theorem.
1935 Tychonoff's fixed point theorem.
1937 von Neumann's intersection lemma.
1941 Intermediate value theorem of Bolzano-Poincaré-Miranda.
1941 Kakutani's fixed point theorem.
1950 Bohnenblust-Karlin's fixed point theorem.
1950 Hukuhara's fixed point theorem.
1952 Fan-Glicksberg's fixed point theorem.
1955 Main theorem of mathematical economics on Walras equilibria of Gale (1955), Nikaido (1956), and Debreu (1959).
1957 Alexandroff-Pasynkoff's theorem.
1960 Kuhn's cubic Sperner lemma.
1961 Fan's KKM theorem.
1961 Fan's geometric or section property of convex sets.
1966 Fan's theorem on sets with convex sections.
1966 Hartman-Stampacchia's variational inequality.
1967 Browder's variational inequality.
1967 Scarf's intersection theorem.
1968 Browder's fixed point theorem.
1969 Fan's best approximation theorems.
1972 Fan's minimax inequality.
1972 Himmelberg's fixed point theorem.
1973 Shapley's generalization of the KKM theorem.
1976 Tuy's generalization of the Walras excess demand theorem.
1981 Gwinner's extension of the Walras theorem to infinite dimensions.
1983 Yannelis-Prabhakar's existence of maximal elements in mathematical economics.
1984 Fan's matching theorems.

Many other generalizations of these theorems are also known to be equivalent to the Brouwer theorem. For examples, Horvath and Lassonde (1997) obtained intersection theorems of the KKM-type, Klee-type, and Helly-type, which are all equivalent to the Brouwer theorem. Park and Jeong (2001) collected equivalent formulations closely related to Euclidean spaces or n-simplexes or n-balls.

3. Fan's works on the KKM theory — From 1960s to 1980s

From 1961, Ky Fan showed that the KKM theorem provides the foundations for many of the modern essential results in diverse areas of mathematical sciences. Actually, a milestone of the history of the KKM theory was erected by Fan (1961). He extended the KKM theorem to arbitrary topological vector spaces and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space.

Lemma (Fan, 1961). *Let X be an arbitrary set in a topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of a finite subset $\{x_1, \cdots, x_n\}$ of X is contained in $\bigcup_{i=1}^{n} F(x_i)$.*
(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

This is usually known as the KKMF theorem. Fan assumed the Hausdorffness of Y, which was known to be superfluous later.

Fan also obtained the following geometric or section property of convex sets, which is equivalent to the preceding Lemma.

Lemma (Fan, 1961). *Let X be a compact convex set in a topological vector space. Let A be a closed subset of $X \times X$ with the following properties:*

(i) *$(x, x) \in A$ for every $x \in X$.*
(ii) *For any fixed $y \in X$, the set $\{x \in X : (x, y) \notin A\}$ is convex (or empty).*

Then there exists a point $y_0 \in X$ such that $X \times \{y_0\} \subset A$.

Fan applied this Lemma to give a simple proof (1961) of the Tychonoff theorem and to prove two results (1963) generalizing the Pontrjagin-Iohvidov-Kreĭn theorem on existence of invariant subspaces of certain linear operators. Also, Fan (1964) applied his KKMF theorem to obtain an intersection theorem (concerning sets with convex sections) which implies the Sion minimax theorem and the Tychonoff theorem. The main results of Fan (1964) were extended by Ma (1969), who obtained a generalization of the Nash theorem for infinite case.

Moreover, "a theorem concerning sets with convex sections" was applied to prove the following results in Fan (1966):

An intersection theorem (which generalizes the von Neumann lemma (1937)).

An analytic formulation (which generalizes the equilibrium theorem of Nash (1951) and the minimax theorem of Sion (1958)).

A theorem on systems of convex inequalities of Fan (1957).

Extremum problems for matrices.

A theorem of Hardy-Littlewood-Pólya concerning doubly stochastic matrices.

A fixed point theorem generalizing Tychonoff (1935) and Iohvidov (1964).

Extensions of monotone sets.

Invariant vector subspaces.

An analogue of Helly's intersection theorem for convex sets.

On the other hand, Browder (1968) obtained an equivalent result to Fan's geometric lemma (1961) in the convenient form of a fixed point theorem by means of the Brouwer theorem and the partition of unity argument. Since then the following is known as the Fan-Browder fixed point theorem:

Theorem (Browder, 1968). *Let K be a nonempty compact convex subset of a topological vector space. Let T be a map of K into 2^K, where for each $x \in K$, $T(x)$ is a nonempty convex subset of K. Suppose further that for each y in K, $T^{-1}(y) = \{x \in K : y \in T(x)\}$ is open in K. Then there exists x_0 in K such that $x_0 \in T(x_0)$.*

Later this is also known to be equivalent to the Brouwer theorem. Browder (1968) applied his theorem to a systematic treatment of the interconnections between multivalued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems. This is also applied by Borglin and Keiding (1976) and Yannelis and Frabhakar (1983), to the existence of maximal elements in mathematical economics.

Motivated by Browder's works (1967, 1968) on fixed point theorems, Fan (1969) deduced the following from his geometric lemma:

Theorem (Fan, 1969). *Let X be a nonempty compact convex set in a normed vector space E. For any continuous map $f : X \to E$, there exists a point $y_0 \in X$ such that*

$$||\, y_0 - f(y_0)|| = \operatorname*{Min}_{x \in X} ||\, x - f(y_0)||.$$

(*In particular, if $f(X) \subset X$, then y_0 is a fixed point of f.*)

Fan also obtained a generalization of this theorem to locally convex Hausdorff topological vector spaces. Those are known as best approximation theorems and applied to generalizations of the Brouwer theorem and some nonseparation theorems on upper demicontinuous (u.d.c.) multimaps in Fan (1969).

Moreover, Fan established a minimax inequality from the KKMF theorem:

Theorem (Fan, 1972). *Let X be a compact convex set in a topological vector space. Let f be a real function defined on $X \times X$ such that:*

(a) *For each fixed $x \in X, f(x,y)$ is a lower semicontinuous function of y on X.*

(b) *For each fixed $y \in X, f(x,y)$ is a quasi-concave function of x on X.*

Then the minimax inequality

$$\min_{y\in X}\sup_{x\in X} f(x,y) \leq \sup_{x\in X} f(x,x)$$

holds.

Fan gave applications of this inequality as follows:

A variational inequality (extending Hartman-Stampacchia (1966) and Browder (1967)).

A geometric formulation of the inequality (equivalent to the Fan-Browder fixed point theorem).

Separation properties of u.d.c. multimaps, coincidence and fixed point theorems.

Properties of sets with convex sections (Fan, 1966).

A fundamental existence theorem in potential theory.

Furthermore, Fan (1979, 1984) introduced a KKM theorem with a coercivity (or compactness) condition for noncompact convex sets and, from this, extended many of known results to noncompact cases. We list some main results as follows:

Generalizations of the KKM theorem for noncompact cases.

Geometric formulations.

Fixed point and coincidence theorems.

Generalized minimax inequality (extends Allen's variational inequality (1977)).

A matching theorem for open (closed) covers of convex sets.

The 1978 model of the Sperner lemma.

Another matching theorem for closed covers of convex sets.

A generalization of Shapley's KKM theorem (Shapley, 1973).

Results on sets with convex sections.

A new proof of the Brouwer theorem.

While closing a sequence of lectures delivered at the NATO-ASI at Montreal in 1983, Fan listed various fields in mathematics which have applications of KKM maps, as follows:

Potential theory.

Pontrjagin spaces or Bochner spaces in inner product spaces.

Operator ideals.
Weak compactness of subsets of locally convex topological vector spaces.
Function algebras.
Harmonic analysis.
Variational inequalities.
Free boundary value problems.
Convex analysis.
Mathematical economics.
Game theory.
Mathematical statistics.

We may add the following fields to this list: nonlinear functional analysis, approximation theory, optimization theory, fixed point theory, and some others.

4. Intersection theorems and equilibrium problems

Intersection theorems are concerned with conditions under which members of a certain subcover of a cover of a given set have a nonempty intersection. Such intersection theorems on the standard simplex or other convex sets were given by the covering property of Sperner (1928), the KKM theorem (1929), Alexandroff-Pasynkoff's theorem (1957), the KKMF theorem due to Fan (1961), Peleg's generalization (1967) of the KKM theorem, Scarf's theorem (1967), the KKMS theorem due to Shapley (1973), Gale's theorem (1984), Ichiishi's theorem (1988), the intersection theorems of Horvath and Lassonde (1997), and others. These theorems are applied to the existence of solutions of mathematical programming problems, to economic equilibrium theory, and to game theoretic problems.

In 1967, motivated by the search for equilibrium points in non-cooperative games, Peleg established the following extension of the KKM theorem:

Lemma (Peleg, 1967). *For each $i \in I = \{1, \cdots, n\}$, let C_i^j, $j = 1, \cdots, m_i+1$, be closed subsets of $\Pi_{i\in I}\Delta_{m_i}$ such that for each $A_i \subset \{1, \cdots, m_i+1\}$ and $i \in I$,*

$$\Delta_{m_1} \times \cdots \times \Delta_{A_i} \times \cdots \times \Delta_{m_n} \subset \bigcup_{j\in A_i} C_i^j,$$

where Δ_{A_i} denotes the face of Δ_{m_i} corresponding to A_i. Then

$$\bigcap_{i\in I} \bigcap_{j=1}^{m_i+1} C_i^j \neq \emptyset.$$

Since then Peleg's lemma has been widely used in the framework of game theory in order to prove existence results concerning different solution concepts, like the bargaining set and the kernel.

The KKMS theorem is a very useful tool to show that the core of any balanced non-transferable utility game is nonempty, a result first shown in Scarf (1967) by means of a constructive method being related to the methods introduced in Scarf (1967, 1973). In fact, Shapley (1973) extended the KKM theorem on closed covers of a simplex to the case of more general closed covers of a simplex incorporating the notion of balancedness, and obtained a theorem now called the KKMS theorem. Shapley proved the theorem constructively using an analogous generalization of the Sperner lemma (1928).

Let $N = \{1, \cdots, n\}$ and let $\langle N \rangle$ be the family of all nonempty subsets of N. Let $\{e^i : i \in N\}$ be the standard basis of $\mathbf{R}^n$, that is, e^i is an n-vector whose i-th coordinate is 1 and 0 otherwise. Let Δ be the simplex $\mathrm{co}\{e^i : i \in N\}$ and, for an $S \in \langle N \rangle$, let Δ^S be the face of Δ spanned by $\{e^i : i \in S\}$; that is, $\Delta^S = \mathrm{co}\{e^i : i \in S\}$. A subfamily $\langle B \rangle$ of $\langle N \rangle$ is said to be *balanced* if there are nonnegative weights λ^S, $S \in \langle B \rangle$, such that $\sum_{S \in \langle B \rangle} \lambda^S e^S = e^N$, where e^S denotes the n-vector whose i-th coordinate is 1 if $i \in S$ and 0 otherwise. It is easily seen that $\langle B \rangle$ is balanced if and only if $m^N \in \mathrm{co}\{m^S : S \in \langle B \rangle\}$, where m^S denotes the center of gravity of the face Δ^S; that is, $m^S = \sum_{i \in S} e^i/|S|$.

Theorem (Shapley, 1973). *Let $\{C_S : S \in \langle N \rangle\}$ be a family of closed subsets of Δ such that for each $T \in \langle N \rangle$*

$$\Delta^T \subset \bigcup_{S \subset T} C_S.$$

Then there is a balanced family $\langle B \rangle$ such that

$$\bigcap_{S \in \langle B \rangle} C_S \neq \emptyset.$$

Since Scarf's core theorem is very important in mathematical economics and since Shapley's proof of the KKMS theorem was rather complicated, several authors explored the logical connection between Scarf's theorem and fixed point theory, either by proving the KKMS theorem from a standard fixed point theorem or by going directly to Scarf's theorem by a different route. Kannai (1970) showed that Scarf's theorem (1967) is equivalent to the Brouwer theorem. Todd (1978) applied the Kakutani fixed point theorem (1941) to prove a special case of the KKMS theorem, sufficient to prove the core theorem. An easy non-constructive proof of the KKMS theorem due to Ichiishi (1981) based on a coincidence theorem of Fan (1969). Keiding and Thorlund-Peterson (1985) proved the core theorem through the KKM theorem. And Ichiishi (1981) initiated a cooperative extension of the noncooperative game and, more systematically (1993); in particular, his theorem includes as special cases the Nash equilibrium theorem in noncooperative game theory and Scarf's core theorem in cooperative game theory. Moreover, Ichiichi (1988) obtained a dual version of the KKMS theorem, again using Fan's coincidence theorem, and then applied it to the core theorem.

Shapley and Vohra (1991) gave proofs of both Scarf's core theorem and the KKMS theorem involving either Kakutani's fixed point theorem or Fan's coincidence theorem. Komiya (1994) gave a proof of the KKMS theorem based on the Kakutani theorem, the separating hyperplane theorem, and the Berge maximum theorem. Krasa and Yannelis (1994) gave a proof of the KKMS theorem by means of the Brouwer theorem, the separating hyperplane theorem, and a continuous selection theorem. Zhou (1994) considered intersection theorems close to the Ichiishi theorem and the KKMS theorem. Moreover, Herings (1997) gave a very elementary and simple proof of the KKMS theorem using only the Brouwer theorem and some elementary calculus. This shows that the KKMS theorem and the Brouwer theorem should be regarded as "equivalent" since it is elementary to show the Brouwer theorem using the KKMS theorem.

5. Convex spaces of Lassonde

The concept of convex sets in a topological vector space is extended to convex spaces by Lassonde (1983), and further to C-spaces by Horvath in (1983-91). A number of other authors also extended the concept of convexity for various purposes.

Let X be a subset of a vector space and D a nonempty subset of X. We call (X, D) a *convex space* if $\operatorname{co} D \subset X$ and X has a topology that induces the Euclidean topology on the convex hulls of any $N \in \langle D \rangle$; see Park (1994). Note that (X, D) can be represented by $(X, D; \Gamma)$ where $\Gamma : \langle D \rangle \to X$ is the convex hull operator. If $X = D$ is convex, then $X = (X, X)$ becomes a convex space in the sense of Lassonde (1983). Every nonempty convex subset X of a topological vector space is a convex space with respect to any nonempty subset D of X, and the converse is known to be not true.

The subject matter of Lassonde (1983) belongs to nonlinear analysis, and its aim is to present a simple and unified treatment of a large variety of minimax and fixed point problems. More specifically, he gave several KKM type theorems for convex spaces (X, D) and proposed a systematic development of the method based on the KKM theorem; the principal topics treated by him may be listed as follows:

Fixed point theory for multifunctions.
Minimax equalities.
Extensions of monotone sets.
Variational inequalities.
Special best approximation problems.

Applying Lassonde's conception, from coincidence theorems on compositions of the admissible maps, Park (1994) deduced generalizations of the KKM theorem, the Fan-Browder theorem, a matching theorem, an analytic alternative, the Fan minimax inequalities, section properties of convex spaces, and other fundamental theorems in the theory. These new results extend, improve, and unify main theorems in more than one hundred published works.

One of the most important applications of Lassonde's convex spaces is the following:

Existence of maximizable quasiconcave functions on convex spaces; see Park and Bae (1991).

In fact, the author (1992, 2002) applied the existence theorem to obtain coincidence, fixed point, and surjectivity theorems, and existence theorems on critical points for a class of convex-valued multimaps larger than that of upper hemicontinuous ones. One of the main fixed point theorems (1992) is concerned with generalized upper hemicontinuous maps whose domains and ranges may have different topologies. Furthermore, the existence theorem or the fixed point theorems were applied to

Condensing inward multimaps.
Matching theorems for closed coverings.
The Fan type nonseparation theorems.
Existence of maximizable linear functionals with preassigned particular properties.
Generalized extremal principles originated from Mazur and Schauder.

Moreover, in the frame of the convex space theory, we obtained the following remarkable consequences:

The KKM principle implies many fixed point theorems; Park (2004).
Generalized equilibrium, generalized complementarity, and eigenvector problems; Park (1997) and Li and Park (2006).

6. C-spaces of Horvath

The KKM theorem was further extended to pseudo-convex spaces, contractible spaces, and spaces with certain contractible subsets or c-spaces by Horvath (1983, 1984, 1987, 1990, 1991). In these papers, replacing convexity by contractibility, most of Fan's results in the KKM theory are extended to c-spaces; and a large number of new deep examples of c-spaces were given. Horvath also added some applications of his results to various types of new spaces. This line of generalizations was followed by Bardaro and Ceppitelli (1988, 1989, 1990) and many others.

A triple $(X, D; \Gamma)$ is called an *H-space* by Park (1992) if X is a topological space, D a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $D = X$, we denote $(X; \Gamma)$ instead of $(X, X; \Gamma)$, which is called a c-space by Horvath (1991) or an H-space by Bardaro and Ceppitelli (1988).

Any convex space X is an H-space $(X;\Gamma)$ by putting $\Gamma_A = \operatorname{co} A$, the convex hull of $A \in \langle D \rangle$. Other examples of $(X;\Gamma)$ are any pseudo-convex space (Horvath, 1983), any homeomorphic image of a convex space, any contractible space, and so on; see Bardaro and Ceppitelli (1988) and Horvath (1991). Every n-simplex Δ_n is an H-space $(\Delta_n, D;\Gamma)$, where D is the set of vertices and $\Gamma_A = \operatorname{co} A$ for $A \in \langle D \rangle$.

With these terminology, Park (1992) established new versions of KKM theorems, matching theorems, Fan-Browder type coincidence theorems, minimax inequalities, and others on H-spaces. These results were stated in forms sufficiently general enough to include the basic KKM theorems due to Lassonde (1983).

A number of other authors also extended the concept of convexity on topological spaces for various purposes.

7. G-convex spaces

In the last decade of the 20th century, Park and Kim (1993, 1996-98) unified various general convexities to generalized convex spaces or G-convex spaces. For these spaces, the foundations of the KKM theory with respect to admissible maps were established by Park and Kim (1997), and some general fixed point theorems were obtained by Kim (1998) and Park (1999).

Definition. A *generalized convex space* or a *G-convex space* $(X, D;\Gamma)$ consists of a topological space X, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\Delta_n = \operatorname{co}\{e_i\}_{i=0}^n$ is the standard n-simplex, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \operatorname{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$ and $(X, \Gamma) = (X, X;\Gamma)$.

There are lots of examples of G-convex spaces; see [2] and references therein. For details on G-convex spaces, see Park and Kim (1996-98) and Park (2000), where basic theory was extensively developed.

For a G-convex space $(X, D;\Gamma)$, a map $F : D \multimap X$ is called a *KKM map* if $\Gamma_N \subset F(N)$ for each $N \in \langle D \rangle$. So, the KKM theory was extended to the study of KKM maps on G-convex spaces. The following is basic in this theory:

Theorem. *Let $(X, D;\Gamma)$ be a G-convex space, Y a Hausdorff space, $S : D \multimap Y$, $T : X \multimap Y$ maps, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that*

(1) *for each $x \in D$, Sx is open in Y;*
(2) *for each $y \in F(X)$, $M \in \langle S^- y \rangle$ implies $\Gamma_M \subset T^- y$;*
(3) *there exists a nonempty compact subset K of Y such that $\overline{F(X)} \cap K \subset S(D)$; and*

(4) *either*

(i) $Y \setminus K \subset S(M)$ *for some* $M \in \langle D \rangle$*; or*

(ii) $X \supset D$ *and, for each* $N \in \langle D \rangle$*, there exists a compact* Γ*-convex subset* L_N *of* X *containing* N *such that* $F(L_N) \setminus K \subset S(L_N \cap D)$.

Then there exists an $\overline{x} \in X$ *such that* $F\overline{x} \cap T\overline{x} \neq \emptyset$.

This was due to Park and Kim (1996, 1997), where this had been reformulated to more than a dozen foundational results in the KKM theory. The *admissible* class $\mathfrak{A}_c^\kappa$ in the above theorem can be replaced by the *better* admissible class $\mathfrak{B}$ for G-convex spaces. Moreover, there have appeared some fixed point theorems for the class $\mathfrak{B}$ on G-convex spaces; see, for example, Park [2,10,12].

Moreover, Park and Kim (1999) gave a Peleg type KKM theorem (1967) on G-convex spaces and applied this to a coincidence theorem, a whole intersection property, a geometric lemma, an analytic alternative for multimaps, and existence theorems of equilibrium points in qualitative games and in n-person games.

Contrary to the preceding progress, many authors have tried to imitate, modify, or generalize G-convex spaces and published a large number of papers. In fact, in the last decade, there have appeared authors who introduced spaces of the form $(X, \{\varphi_A\})$ having a family $\{\varphi_A\}$ of continuous functions defined on simplexes. Such example are L-spaces due to Ben-El-Mechaiekh et al., spaces having property (H) due to Huang, FC-spaces due to Ding, convexity structures satisfying the H-condition by Xiang et al., M-spaces and another L-spaces due to González et al., and others. Some authors claimed that such spaces generalize G-convex spaces without giving any justifications or proper examples. Some authors also tried to generalize the KKM theorem for their own settings. They introduced various types of generalized KKM maps; for example, generalized KKM maps on L-spaces, generalized R-KKM maps, and many other artificial terminology. We found that most of such spaces are subsumed in the concept of ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, which can be made into G-convex spaces; see [5,6,10,12].

8. Theory of the KKM spaces

In order to destroy such unnecessary concepts and to upgrade the KKM theory, recently in 2006-09, we proposed new concepts of abstract convex spaces and the KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [3,5-9].

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of nonempty sets E, D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. Then $(X, D'; \Gamma|_{\langle D' \rangle})$ is called a Γ-*convex subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example. In [5-9], we gave plenty of examples of abstract convex spaces as follows:

1. The original KKM theorem (1929) is for the triple $(\Delta_n \supset V; \mathrm{co})$, where V is the set of vertices of Δ_n and $\mathrm{co} : \langle V \rangle \multimap \Delta_n$ the convex hull operation.

2. A triple $(X \supset D; \Gamma)$, where X and D are subsets of a t.v.s. E such that $\mathrm{co}\, D \subset X$ and $\Gamma :=$co. Fan's celebrated KKM lemma (1961) is for $(E \supset D; \mathrm{co})$, where D is a nonempty subset of E.

3. A convex space $(X, D; \Gamma)$ of the Lassonde type.

4. An H-space.

5. A generalized convex space or a G-convex space. This class contains all of the above classes in 1-4.

6. A ϕ_A-*space* $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with $|A| = n + 1$. Every ϕ_A-space can be made into a G-convex space; see [5,10-12].

7. A convexity space $(E, \mathcal{C})$ in the classical sense is an abstrct convex space. For details, see Sortan (1984), where the bibliography lists 283 papers.

8. According to Horvath (2008), a convexity on a set X is an algebraic closure operator $A \mapsto [[A]]$ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that $[[\{x\}]] = \{x\}$ for all $x \in X$, or equivalently, a family $\mathcal{C}$ of subsets of X, the convex sets, which contains the whole space and the empty set as well as singletons and which is closed under arbitrary intersections and updirected unions.

Note that each of these examples has a large number of concrete examples.

From now on, in an abstract convex space $(E, D; \Gamma)$, E is assumed to be a topological space.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all } A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp, $F \in \mathfrak{KO}(E, Z)$].

Definition. For an abstract convex topological space $(E, D; \Gamma)$, the *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$ and the *partial KKM principle* is $1_E \in \mathfrak{KC}(E, E)$.

A *KKM space* is an abstract convex space satisfying the KKM principle.

In our recent work [7,8], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to KKM spaces and abstract convex spaces satisfying the partial KKM principle.

Example. We give examples of KKM spaces:

1. Every G-convex space is a KKM space.
2. A connected linearly ordered space $(X, \leq)$ can be made into a KKM space.
3. The extended long line L^* is a KKM space $(L^* \supset D; \Gamma)$ with the ordinal space $D := [0, \Omega]$. But L^* is not a G-convex space.
4. For Horvath's convex space $(X, \mathcal{C})$ (2008) with the weak Van de Vel property, the corresponding abstract convex space $(X; \Gamma)$ is a KKM space, where $\Gamma_A := [[A]] = \bigcap\{C \in \mathcal{C} \mid A \subset C\}$ is metrizable for each $A \in \langle X \rangle$.

Example. We give examples of abstract convex spaces satisfying the partial KKM principle:

1. All KKM spaces.
2. For Horvath's convex space $(X, \mathcal{C})$ (2008) with the weak Van de Vel property, the $(X; \Gamma)$ is a partial KKM space, where $\Gamma_A := [[A]]$ for each $A \in \langle X \rangle$.

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Space satisfying the partial KKM principle $\Longrightarrow$ Abstract convex space.

In the KKM theory, it is routine to reformulate the (partial) KKM principle to the following equivalent forms:

Fan type matching property
Another intersection property
Geometric or section properties
The Fan-Browder type fixed point theorem
Existence theorem of maximal elements, and others

Any of such statements can be used to characterize the KKM spaces. For example, the Fan-Browder type theorem is used for the following:

Theorem. *An abstract convex space* $(X, D; \Gamma)$ *is a KKM space iff for any maps* $S : D \multimap X$, $T : X \multimap X$ *satisfying*

(1) $S(z)$ *is open [resp., closed] for each* $z \in D$*;*
(2) *for each* $y \in X$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$*; and*
(3) $X = \bigcup_{z \in M} S(z)$ *for some* $M \in \langle D \rangle$*,*

T *has a fixed point* $x_0 \in X$*; that is* $x_0 \in T(x_0)$*.*

Moreover, from the partial KKM principle we have a whole intersection property of the Fan type. From this, we can deduce the following:

Theorem. *Let* $(X, D; \Gamma)$ *satisfy the partial KKM principle,* K *be a nonempty compact subset of* X*, and* $G : D \multimap X$ *a map such that*

(1) $\bigcap_{z \in D} G(z) = \bigcap_{z \in D} \overline{G(z)}$ *[that is,* G *is transfer closed-valued];*
(2) $\overline{G}$ *is a KKM map; and*
(3) *either*
  (i) $\bigcap\{\overline{G(z)} \mid z \in M\} \subset K$ *for some* $M \in \langle D \rangle$*; or*
  (ii) *for each* $N \in \langle D \rangle$*, there exists a compact* Γ*-convex subset* L_N *of* X *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$L_N \cap \bigcap\{\overline{G(z)} \mid z \in D'\} \subset K.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$*.*

From this theorem we can deduce its equivalent formulations of the following forms for abstract convex spaces satisfying the partial KKM principle:

Analytic alternatives (a basis of various equilibrium problems)
Fan type minimax inequalities
Variational inequalities, and others.

Consequently, for a compact abstract convex spaces $(X; \Gamma)$ satisfying the partial KKM principle, we deduced 15 theorems from any of the characterizations of such spaces. Moreover, we noticed there that, for a compact G-convex space $(X; \Gamma)$, each of these 15 theorems and their corollaries is equivalent to the original KKM theorem.

Further applications of our theory on abstract convex spaces satisfying the partial KKM principle are given in [7,8] as follows:

Best approximations
The von Neumann type minimax theorem

The von Neumann type intersection theorem
The Nash type equilibrium theorem
The Himmelberg fixed point theorem for KKM spaces
Weakly KKM maps [11]

Finally, recall that there are several hundred published works on the KKM theory and we can cover only an essential part of it. For the more historical background for the related fixed point theory, the reader can consult with [2] and references therein. For more involved or generalized versions of the results in this paper, see the references below and the literature therein.

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