Abstract
The short pulse model equation describes the propagation of ultra-short optical pulses in nonlinear media. We develop a systematic method for solving the short pulse equation and address the construction of the two-phase periodic solutions and their properties. The detail of the content of this paper is described in Ref. [11].

1.1 Maxwell equation
We start from the following Maxwell equation

\[ \text{div } \mathbf{D} = \rho, \quad \text{div } \mathbf{B} = 0, \quad \text{rot } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{rot } \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \] (1.1a)

\[ \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{B} = \mu_0 \mathbf{H} \] (1.1b)

where \( \mathbf{E} \) and \( \mathbf{H} \) are electric and magnetic field vectors, respectively, and \( \mathbf{D} \) and \( \mathbf{B} \) are corresponding electric and magnetic flux density.

We assume that \( \rho = 0 \) and \( \mathbf{j} = 0 \) and consider the one-dimensional propagation. Then Eq. (1.1) reduces to

\[ \mathbf{E} = E_3(x, t) \mathbf{e}_3, \quad \mathbf{H} = H_2(x, t) \mathbf{e}_2 \] (1.2)

\[ \frac{\partial H_2}{\partial x} = \frac{\partial D_3}{\partial t}, \quad \frac{\partial E_3}{\partial x} = \mu_0 \frac{\partial H_2}{\partial t}. \] (1.3)

Using (1.3) and the relation \( D_3 = \epsilon_0 E_3 + P_3 \), we eliminate \( H_2 \) from (1.3) to obtain

\[ E_{xx} - \frac{1}{c^2} E_{tt} = P_{tt} \] (1.4)

where we have put \( E = E_3, P = P_3/(\epsilon_0 c^2), c^2 = (\epsilon_0 \mu_0)^{-1} \). We further assume the relation

\[ P = P_{\text{lin}} + P_{\text{nl}} = \int_{-\infty}^{\infty} \chi(t - \tau) E(x, \tau) d\tau + \chi_3 E^3 \] (1.5a)

\[ \chi_{tt} = \chi_0 \delta(t). \] (1.5b)
Substituting (1.5) into (1.4), we obtain the nonlinear wave equation
\[ E_{xx} - \frac{1}{c^2} E_{tt} = \chi_0 E + \chi_3 (E^3)_t. \] (1.6)

1.2 Singular perturbation

In accordance with Schäfer and Wayne (2004), we apply the singular perturbation method to Eq. (1.6) to derive the short pulse (SP) equation. We expand \( E \) with respect to the small parameter \( \epsilon \)
\[ E(x, t) = \epsilon u_0(\phi, X) + \epsilon^2 u_1(\phi, X) + \cdots \] (1.7a)
where the new independent variables \( \phi \) and \( X \) are defined by
\[ \phi = \frac{t - \frac{x}{c}}{\epsilon}, \quad X = \epsilon x. \] (1.7b)

If we introduce (1.7) into (1.6), we obtain, at the lowest order \( O(\epsilon) \), the following PDE
\[ -\frac{2}{c} \frac{\partial^2 u_0}{\partial \phi \partial X} = \chi_0 u_0 + \chi_3 \frac{\partial^2 u_0^3}{\partial \phi^2}. \] (1.8)

After an appropriate change of the variables, we arrive at the normalized form of the SP equation:
\[ u_{xt} = u + \frac{1}{6} (u^3)_{xx}. \] (1.9)

1.3 Remarks

- The SP equation is a model equation describing the propagation of ultra-short optical pulses in nonlinear media.
- The SP equation has been derived in a mathematical context concerning the integrable PDE (Robelo (1989)).
- The following solutions are known for the SP equation:
- Analogous integrable equations (Matsuno (2006))
  \[ u_{xt} = \alpha u + \frac{1}{2} (1 - \beta) u_x^2 - uu_{xx} \]
  \( \beta = 2 \): Short-pulse model for Camassa-Holm equation
  \( \beta = 3 \): Short-pulse model for the Degasperis-Procesi equation, Vakhnenko equation
  \( \alpha = 0, \beta = 2 \): Hunter-Saxton equation

All the above equations have the solutions expressed by the parametric representation.
2. Exact method of solution

2.1 Transformation to the sine-Gordon equation

Introduce the new variable $r$:

$$r^2 = 1 + u_x^2. \quad (2.1)$$

We rewrite the SP equation (1.9) into the form

$$r_t = \left( \frac{1}{2}u^2r \right)_x. \quad (2.2)$$

By means of the hodograph transformation $(x, t) \rightarrow (y, \tau)$

$$dy = rdx + \frac{1}{2}u^2rdt, \quad d\tau = dt \quad (2.3a)$$

or equivalently

$$\frac{\partial}{\partial x} = r\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \frac{1}{2}u^2r\frac{\partial}{\partial y} \quad (2.3b)$$

(2.1) and (2.2) are transformed to

$$r^2 = 1 + r^2u_y^2, \quad r_\tau = r^2uu_y. \quad (2.4)$$

Using the transformation

$$u_y = \sin \phi, \quad \phi = \phi(y, \tau) \quad (2.5)$$

(2.4) can be put into the form

$$\frac{1}{r} = \cos \phi. \quad (2.6)$$

It follows from (2.4)-(2.6) that $u = \phi_\tau$. Substituting this into (2.5), we obtain the sine-Gordon (sG) equation:

$$\phi_{y\tau} = \sin \phi. \quad (2.7)$$

We see from (2.3) that $x = x(y, \tau)$ satisfies the following linear PDE

$$x_y = \frac{1}{r}, \quad x_\tau = -\frac{1}{2}u^2. \quad (2.8)$$

2.2 Parametric representation of the solution

Since the integrability of Eq. (2.8), i.e. $x_{y\tau} = x_{\tau y}$ is assured by (2.4), we can integrate (2.8) to obtain

$$x(y, \tau) = \int^y \cos \phi \, dy + d \quad (2.9)$$

where $d$ is an integration constant. The expression of $u$ in terms $\phi$ is given by

$$u(y, \tau) = \phi_\tau. \quad (2.10)$$
2.3 A criterion for the single-valued functions

To derive a criterion for single-valued functions, we may simply require that \( u_x = \tan \phi \) exhibits no singularities. Thus, if

\[
-\frac{\pi}{2} < \phi < \frac{\pi}{2}, \quad (\text{mod } \pi), \quad (-\sqrt{2} + 1 < \tan \frac{\phi}{4} < \sqrt{2} - 1).
\] (2.11)

then the parametric solutions (2.9) and (2.10) will become single-valued functions for all values of \( x \) and \( t \).

3. Periodic solutions

Here, we are concerned with the construction of the periodic solutions of the SP equation, particularly focusing on the two-phase solutions.

3.1 Method of solution

We first introduce the two independent phase variables \( \xi \) and \( \eta \) according to

\[
\xi = ay + \frac{t}{a} + \xi_0, \quad \eta = ay - \frac{t}{a} + \eta_0
\] (3.1)

where \( a \neq 0 \), \( \xi_0 \) and \( \eta_0 \) are arbitrary constants. Then, the sG equation is transformed to

\[
\phi_{\xi\xi} - \phi_{\eta\eta} = \sin \phi, \quad \phi = \phi(\xi, \eta).
\] (3.2)

We seek solutions of the sG equation of the form

\[
\phi = 4 \tan^{-1} \left[ \frac{f(\xi)}{g(\eta)} \right].
\] (3.3)

This \( \phi \) satisfies the sG equation provided that

\[
f'^2 = -\kappa f^4 + \mu f^2 + \nu
\] (3.4a)

\[
g'^2 = \kappa g^4 + (\mu - 1)g^2 - \nu.
\] (3.4b)

Now, the parametric representation of \( u \) follows from (2.10) and (3.3)

\[
u = \frac{4}{a} \frac{f'g + fg'}{f^2 + g^2}.
\] (3.5)

To obtain the parametric form of \( x \), we note the relation

\[
\cos \phi = 1 - \frac{8f^2g^2}{(f^2 + g^2)^2}.
\] (3.6)

We modify the right-hand side of (3.6) by introducing the function \( Y = Y(\xi, \eta) \)

\[
Y = \frac{c_1(f^2)' + c_2(g^2)'}{f^2 + g^2}.
\] (3.7)
We calculate $Y_y$. Using (3.4), we can modify this in the form

$$
Y_y = \frac{a}{(f^2 + g^2)^2} \left[ -2\kappa(c_1f^6 + 3c_1f^4g^2 - 3c_2f^2g^4 - c_2g^6) - 4c_2f^2g^2 \\
+2(c_1 + c_2) \{-2fgf'g' + 2\mu f^2g^2 - \nu(f^2 - g^2)\} \right].
$$

If we put $c_1 + c_2 = 0$ and $c_1 = -2/a$, then (3.8) simplifies to

$$
Y_y = 4\kappa(f^2 + g^2) - \frac{8f^2g^2}{(f^2 + g^2)^2}.
$$

If we compare (3.6) and (3.9), we obtain

$$
\cos \phi = 1 + Y_y - 4\kappa(f^2 + g^2).
$$

Finally, substituting (3.10) into (2.9) and integrating, we obtain the parametric representation of $x$:

$$
x = y - \frac{4}{a} \frac{ff' - gg'}{f^2 + g^2} - \frac{4\kappa}{a} \left( \int f^2(\xi)d\xi + \int g^2(\eta)d\eta \right) + d.
$$

### 3.2 Examples

Here, we present the three examples of the periodic solutions:

**a. Example 1**

$$
f(\xi) = A \operatorname{cn}(\beta\xi, k_f), \ g(\eta) = \frac{1}{\operatorname{cn}(\Omega\eta, k_g)}
$$

$$
k_f^2 = \frac{A^2}{1 + A^2} \left( 1 + \frac{1}{\beta^2(1 + A^2)} \right)
$$

$$
k_g^2 = \frac{A^2}{1 + A^2} \left( 1 - \frac{1}{\Omega^2(1 + A^2)} \right)
$$

$$
\Omega^2 = \beta^2 + \frac{1 - A^2}{1 + A^2}.
$$

The inequality $0 \leq k_f \leq 1$ implies that the parameter $\beta$ must be restricted by the condition

$$
\frac{A}{\sqrt{1 + A^2}} \leq \beta.
$$

The parametric solution takes the form

$$
u = \frac{4A - \beta \operatorname{sn}(\beta\xi, k_f)\operatorname{dn}(\beta\xi, k_f)\operatorname{cn}(\Omega\eta, k_g) + \Omega \operatorname{cn}(\beta\xi, k_f)\operatorname{sn}(\Omega\eta, k_g)\operatorname{dn}(\Omega\eta, k_g)}{A^2\operatorname{cn}^2(\beta\xi, k_f)\operatorname{cn}^2(\Omega\eta, k_g) + 1}
$$

$$
x = y + \frac{4\beta}{a} \frac{\operatorname{cn}(\beta\xi, k_f)\operatorname{cn}(\Omega\eta, k_g)}{A^2\operatorname{cn}^2(\beta\xi, k_f)\operatorname{cn}^2(\Omega\eta, k_g) + 1} \left\{ A^2\operatorname{sn}(\beta\xi, k_f)\operatorname{dn}(\beta\xi, k_f)\operatorname{cn}(\Omega\eta, k_g) \right\}
$$
$$-\frac{\beta k_f^2}{\Omega k_g'2} \{\text{cn}(\beta \xi, k_f) \text{sn}(\Omega \eta, k_g) \text{dn}(\Omega \eta, k_g)\}$$

$$-\frac{4\beta}{a} \left[ E(\beta \xi, k_f) - k_f'^2 \beta \xi - \frac{\beta k_f^2}{A^2 \Omega k_g'^2} \left\{E(\Omega \eta, k_g) - k_g'^2 \Omega \eta\right\} \right] + d.$$  (3.15b)

### Properties of the solution

- The solution is a multiply periodic function. It becomes a single-valued function if $0 < A < \sqrt{2} - 1$.
- Under the condition $L = m_\xi L_\xi/a = m_\eta L_\eta/a, (m_\xi, m_\eta) = 1$ where $L_\xi \equiv 4K(k_f)/\beta$ and $L_\eta \equiv 4K(k_g)/\Omega$, the solution has a period $\Lambda$

$$\Lambda = L \left[ 1 - 4\beta^2 \left\{ \frac{E(k_f)}{K(k_f)} - \frac{k_f^2}{A^2(1-k_g^2)} \frac{E(k_g)}{K(k_g)} + \frac{1}{\beta^2(1+A^2)} \right\} \right]$$  (3.16)

where $K(k_f)$ and $E(k_f)$ are the complete elliptic integral of the first and second kinds, respectively. Figure 1 shows a profile of $u$ at $t = 0$ for Example 1.

![Figure 1](image_url)

**Figure 1**: $A = 0.2, m_\xi = 1, m_\eta = 2, a = 1.0, \beta = 0.5832, \Omega = 1.124, k_f = 0.3837, k_g = 0.0958, \Lambda = 10.37$.

**Long-wave limit** $\Lambda \rightarrow \infty$

In the long-wave limit, the parametric solution reduces to

$$u \sim \frac{4A\Omega}{a} - \frac{2A \sinh \beta \xi \cos \Omega \eta + \cosh \beta \xi \sin \Omega \eta}{\cosh^2 \beta \xi + A^2 \cos^2 \Omega \eta}$$  (3.17a)

$$x \sim y - \frac{2\Omega \sinh 2\beta \xi + A \sin 2\Omega \eta}{\cosh^2 \beta \xi + A^2 \cos^2 \Omega \eta} + d.$$  (3.17b)
Figure 2: Long-wave limit of the solution depicted in Figure 1.

b. Example 2

\[ f(\xi) = A \frac{\text{sn}(\beta \xi, k_f)}{\text{cn}(\beta \xi, k_f)}, \quad g(\eta) = \frac{1}{\text{dn}(\Omega \eta, k_g)} \]  
(3.18)

\[ k_f^2 = 1 - A^2 + \frac{A^2}{\beta^2(1 - A^2)} \]  
(3.19a)

\[ k_g^2 = 1 - \frac{1}{A^2} + \frac{1}{\Omega^2(1 - A^2)} \]  
(3.19b)

\[ \Omega = \beta A \]  
(3.19c)

\[ \frac{1}{\sqrt{1 - A^2}} \leq \beta \leq \frac{1}{1 - A^2} \]  
(3.20)

\[ u = \frac{4A}{a} \beta \frac{\text{dn}(\beta \xi, k_f)\text{dn}(\Omega \eta, k_g) + k_g^2\Omega \text{sn}(\beta \xi, k_f)\text{cn}(\beta \xi, k_f)\text{sn}(\Omega \eta, k_g)\text{cn}(\Omega \eta, k_g)}{A^2\text{sn}^2(\beta \xi, k_f)\text{dn}^2(\Omega \eta, k_g) + \text{cn}^2(\beta \xi, k_f)} \]  
(3.21a)

\[ x = y - \frac{4\beta}{a} \frac{1}{A^2\text{sn}^2(\beta \xi, k_f)\text{dn}^2(\Omega \eta, k_g) + \text{cn}^2(\beta \xi, k_f)} \times \]  
\[ \times \left[ (A^2\text{dn}^2(\Omega \eta, k_g) - 1)\text{sn}(\beta \xi, k_f)\text{cn}(\beta \xi, k_f)\text{dn}(\beta \xi, k_f) \right] + \frac{4\beta}{a} (-E(\beta \xi, k_f) + AE(\Omega \eta, k_g)) + d \]  
(3.21b)

\[ \Lambda = L \left[ 1 - 4\beta^2 \left\{ \frac{E(k_f)}{K(k_f)} - A^2 \frac{E(k_g)}{K(k_g)} \right\} \right] . \]  
(3.22)

Figure 3 shows a profile of \( u \) at \( t = 5 \) for Example 2.
Figure 3: $A = 0.2, m_\xi = 2, m_\eta = 1, a = 1.0, \beta = 1.027, \Omega = 0.2053, k_f = 0.9998, k_g = 0.8421, \Lambda = 5.938.$

Long-wave limit $\Lambda \to \infty$

\[ u \sim \frac{4\beta A \cosh \beta \xi \cosh \Omega \eta + A \sinh \beta \xi \sinh \Omega \eta}{a \left( A^2 \sinh^2 \beta \xi + \cosh^2 \Omega \eta \right)} \]  \hspace{1cm} (3.23a)  

\[ x \sim y - \frac{2\beta A^2 \sinh 2\beta \xi - A \sinh 2\Omega \eta}{a \left( A^2 \sinh^2 \beta \xi + \cosh^2 \Omega \eta \right)} + d. \]  \hspace{1cm} (3.23b)  

Figure 4: Long-wave limit of the solution depicted in Figure 3.

c. Example 3

\[ f(\xi) = A \dn(\beta \xi, k_f), \ g(\eta) = \frac{\cn(\Omega \eta, k_g)}{\sn(\Omega \eta, k_g)} \]  \hspace{1cm} (3.24)  

\[ k_f^2 = 1 - \frac{1}{A^2} + \frac{1}{\beta^2(A^2 - 1)} \]  \hspace{1cm} (3.25a)
\[
k_g^2 = 1 - A^2 + \frac{A^2}{\Omega^2(A^2 - 1)}
\]

(3.25b)

\[
\Omega = \frac{\beta}{A}
\]

(3.25c)

\[
\frac{A}{\sqrt{A^2 - 1}} \leq \beta \leq \frac{A^2}{A^2 - 1}, \quad A > 1
\]

(3.26)

\[
u = -\frac{4A}{a} \frac{\Omega \text{dn}(\beta \xi, k_f) \text{dn}(\Omega \eta, k_g) + \beta k_f^2 \text{sn}(\beta \xi, k_f) \text{cn}(\beta \xi, k_f) \text{sn}(\Omega \eta, k_g) \text{cn}(\Omega \eta, k_g)}{A^2 \text{dn}^2(\beta \xi, k_f) \text{sn}^2(\Omega \eta, k_g) + \text{cn}^2(\Omega \eta, k_g)}
\]

(3.27a)

\[
x = y - \frac{4\beta}{a} \frac{1}{A^2 \text{dn}^2(\beta \xi, k_f) \text{sn}^2(\Omega \eta, k_g) + \text{cn}^2(\Omega \eta, k_g)} \times \left[ \frac{1}{A} (1 - A^2 \text{dn}^2(\beta \xi, k_f)) \text{sn}(\Omega \eta, k_g) \text{cn}(\Omega \eta, k_g) \text{dn}(\Omega \eta, k_g) \right] - \frac{4\beta}{a} \left( E(\beta \xi, k_f) - \frac{1}{A} E(\Omega \eta, k_g) \right) + d
\]

(3.27b)

\[
\Lambda = L \left[ 1 - 4\beta^2 \left\{ \frac{E(k_f)}{K(k_f)} - \frac{1}{A^2} \frac{E(k_g)}{K(k_g)} \right\} \right]
\]

(3.28)

Figure 5 shows a profile of \(u\) at \(t = 5\) for Example 3.

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{profile.png}
\caption{Figure 5: \(A = 5, m_\xi = 2, m_\eta = 1, a = 1.0, \beta = 1.027, \Omega = 0.2053, k_f = 0.9998, k_g = 0.8421, \Lambda = 5.938\).}
\end{figure}

Long-wave limit \(\Lambda \to \infty\)

\[
u \sim -\frac{4\beta \cosh \beta \xi \cosh \Omega \eta + A \sinh \beta \xi \sinh \Omega \eta}{a \cosh^2 \beta \xi + A^2 \sinh^2 \Omega \eta}
\]

(3.29a)

\[
x \sim y - \frac{2\beta \sinh 2\beta \xi - A \sinh 2\Omega \eta}{a \cosh^2 \beta \xi + A^2 \sinh^2 \Omega \eta} + d.
\]

(3.29b)
Figure 6: Long-wave limit of the solution depicted in Figure 5.

4. Conclusion

- By means of a novel method of exact solution, we obtained periodic solutions of the SP equation and investigated their properties.

- Of particular interest is the nonsingular periodic solution which reduces to the breather solution in the long-wave limit.

- The construction of a more general class of periodic solutions is under study. It is produced by the multiphase solutions of the sG equation expressed by Riemann's theta functions.

5. References