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Kyoto University
A survey on Shapovalov determinants of (generalized) quantum groups at roots of 1

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Abstract

This is an informal survey on a joint work [HY08b] with Istvan Heckenberger.

1 A quantum group $U(\chi)$ defined for any bi-character $\chi$

Recently study of Nichols algebras has been achieved very actively for the viewpoint of classification of Hopf algebras, see [AS98], [AS02], [Hec06]. One of their examples is the positive part $U^+(\chi)$ of a generalized quantum group $U(\chi)$ defined below.

Let $k$ be a field and $k^\times = k \setminus \{0\}$. For $n \in \mathbb{Z}_{\geq 0}$ and $x \in k$, let

\begin{equation}
[n]_x = \sum_{m=1}^{n} x^{m-1}, \quad [n]_x! = \prod_{m=1}^{n} [m]_x.
\end{equation}

For two elements $X_1$ and $X_2$ of a $k$-algebra we use the convention:

\begin{equation}
X_1 \leftrightarrow X_2 \quad \text{def} \quad \exists x \in k^\times \ X_1 = xX_2.
\end{equation}

Let $I$ be a finite index set. Let $\Pi = \{i \in I \mathbb{Z} \alpha_i \}$ be a rank $|I|$ free $\mathbb{Z}$-module with a basis $\Pi = \{\alpha_i | i \in I\}$. We say that a map $\chi : \Pi \times \Pi \to k^\times$ is a bi-character if $\chi(a + b, c) = \chi(a, c)\chi(b, c)$, and $\chi(a, b + c) = \chi(a, b)\chi(a, c)$ for all $a, b, c \in \Pi$.

Let $\chi$ be any bi-character. Then, as we explain more precisely in Section 2, Lusztig's definition [L, 3.1.1] of the quantum groups can be applied to define the Hopf $k$-algebra $U(\chi)$ with the generators

\begin{equation}
K, L \ (\lambda \in \Pi), \ E_i, F_i \ (i \in I),
\end{equation}
for which $K\ L$ $(\lambda, \mu \in \mathbb{Z}\Pi)$ are linearly independent and the following equations hold:

(4) $K_0 = L_0 = 1, \ K_+ = K K, \ L_+ = L L, \ K L = L K,$

(5) $K L E_j(K L)^{-1} = \frac{\chi(\lambda, \alpha_j)}{\chi(\alpha_j, \mu)} E_j, \ K L F_j(K L)^{-1} = \frac{\chi(\alpha_j, \mu)}{\chi(\lambda, \alpha_j)} F_j,$

(6) $E_i F_j - F_j E_i = \delta_{ij}(K\ L\).$

(7) $\Delta(K L) = K L, \ \Delta(K L)^{-1} = 1, \ S(K L) = (K L)^{-1},$

(8) $\Delta(E_i) = E_i + K, \ \Delta(F_i) = F_i + L, \ \varepsilon(E_i) = 1, \ \varepsilon(F_i) = 0,$

(9) $\varepsilon(E_i) = \varepsilon(F_i) = 0, \ S(E_i) = K^{-1} E_i, \ S(F_i) = F_i L^{-1}.$

Let $U^0(\chi) := k \cdot U \ L.$ Let $U^+(\chi)$ and $U^-(\chi)$ be the subalgebra of $U(\chi)$ generated by $E_i$ and $F_i$ with all $i \in I$ respectively. Then $U(\chi) = U^+(\chi) U^0(\chi) U^-(\chi),$ as a $k$-linear space. We have the $Z_{\geq 0}\Pi$-grading $U^\pm(\chi) = \varepsilon_{Z_{\geq 0}\Pi} U^\pm(\chi) \pm$ defined by $U^+(\chi) = \k E_i, \ U^-(\chi) = \k F_i,$ and $U^\pm(\chi) \subset U^\pm(\chi) \pm.$ We also have $\dim U^+(\chi) = \dim U^+(\chi)$ for all $\lambda \in Z_{\geq 0}\Pi.$

2 Drinfeld pairing of $U(\chi)$

Here we will explain how to define $U(\chi)$ more precisely. By abuse of notation, we use the same symbols as above for the generators of the algebras introduced in this paragraph. Let $\tilde{U}^+(\chi)$ and $\tilde{U}^-(\chi)$ be the free $k$-algebras (with 1) with the generators $\{E_i|i \in I\}$ and $\{F_i|i \in I\}$ respectively. Let $\tilde{U}^0(\chi)$ be the $k$-linear space with the basis $\{K L | \lambda, \mu \in \mathbb{Z}\Pi\}.$ Let $\tilde{U}(\chi) = \tilde{U}^+(\chi) \k \tilde{U}^0(\chi) \k \tilde{U}^-(\chi).$ Identify $X \in \tilde{U}^+(\chi), \ Z \in \tilde{U}^0(\chi) \text{ and } Y \in \tilde{U}^-(\chi)$ with $X = 1, \ Y = 1.$ $\lambda \in \mathbb{Z}_{\geq 0}\Pi.$ and $\ell \ 1$ $1$ $Y$ respectively, and regard $\tilde{U}^+(\chi), \ \tilde{U}^0(\chi)$ and $\tilde{U}^-(\chi)$ as subspaces of $\tilde{U}(\chi)$ in this way. Then $\tilde{U}(\chi)$ can be regarded as the $k$-algebra (with 1) presented by the same generators as the ones for $U(\chi)$ and the relations (4), (5) and (6) (cf. [L, Prop. 3.2.4]). Further $\tilde{U}(\chi)$ can be regarded as the Hopf $k$-algebra with the same equalities as (7), (8) and (9). Let $\tilde{U}^{+,K}(\chi)$ be the subalgebra of $\tilde{U}(\chi)$ generated by $E_i$'s and $K$'s. Let $\tilde{U}^{L,-}(\chi)$ be the subalgebra of $\tilde{U}(\chi)$ generated by $F_i$'s and $L$'s. Then there exists a unique $k$-bilinear form

$$\langle , \rangle : \tilde{U}^{+,K}(\chi) \times \tilde{U}^{L,-}(\chi) \to k$$
with

\[(1) \quad \langle 1, Y \rangle = \epsilon(Y), \quad \langle X, 1 \rangle = \epsilon(X), \quad \langle S(X), Y \rangle = \langle X, S^{-1}(Y) \rangle,\]

\[(2) \quad \langle X_1 X_2, Y \rangle = \sum_g \langle X_2, Y^{(1)}_g \rangle \langle X_1, Y^{(2)}_g \rangle,\]

\[(3) \quad \langle X, Y_1 Y_2 \rangle = \sum_h \langle X_h^{(1)}, Y_1 \rangle \langle X_h^{(2)}, Y_2 \rangle,\]

\[(4) \quad \langle E_i, F_j \rangle = \delta_{ij}, \quad \langle K, L \rangle = \chi(\lambda, \mu), \quad \langle E_i, L \rangle = \langle K, F_j \rangle = 0\]

for \(X, X_1, X_2 \in \tilde{U}^{+,K}(\chi)\) with \(\Delta(X) = \sum_h X_h^{(1)} X_h^{(2)}\), and \(Y, Y_1, Y_2 \in \tilde{U}^{L,-}(\chi)\) with \(\Delta(Y) = \sum_g Y_g^{(1)} Y_g^{(2)}\) and for \(i, j \in I\) and \(\lambda, \mu \in \mathbb{Z}\Pi\). We see

\[(5) \quad \langle \tilde{E}K, \tilde{F}L \rangle = \langle \tilde{E}, \tilde{F} \rangle \langle K, L \rangle\]

for \(\tilde{E} \in \tilde{U}^{+}(\chi)\) and \(\tilde{F} \in \tilde{U}^{-}(\chi)\). Further, letting \(\tilde{U}^{\pm}(\chi) = \mathbb{Z}_{\geq 0}\Pi^{\pm} \tilde{U}^{\pm}(\chi)_{\pm}\) be the \(\mathbb{Z}_{\geq 0}\Pi\)-grading on \(\tilde{U}^{\pm}(\chi)\) defined in a way similar to the one on \(U^{\pm}(\chi)\), we have \(\langle \tilde{U}^{+}(\chi), \tilde{U}^{-}(\chi) \rangle = \{0\}\) if \(\lambda \neq \mu\). Let

\[(6) \quad \tilde{J}^{+}(\chi) = \{\tilde{E} \in \tilde{U}^{+}(\chi) | \langle \tilde{E}, \tilde{U}^{-}(\chi) \rangle = \{0\}\},\]

\[(7) \quad \tilde{J}^{-}(\chi) = \{\tilde{F} \in \tilde{U}^{-}(\chi) | \langle \tilde{U}^{+}(\chi), \tilde{F} \rangle = \{0\}\},\]

\[(8) \quad \tilde{J}(\chi) = \text{Span}_k(\tilde{J}^{+}(\chi) \tilde{U}^{0}(\chi) \tilde{U}^{-}(\chi) + \tilde{U}^{+}(\chi) \tilde{U}^{0}(\chi) \tilde{J}^{-}(\chi))\]

Then \(\tilde{J}(\chi)\) is the kernel of the Hopf algebra epimorphism from \(\tilde{U}(\chi)\) to \(U(\chi)\) sending the generators to the ones denoted by the same symbols.

**Theorem 1.** (Kharchenko [Kha99]) There exist \(M \in \mathbb{N} \cup \{\infty\}\) and elements \(\hat{E}_i \in U^{+}(\chi)_{i}\), \((1 \quad i \quad M)\) for some \(\beta_i \in \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}\) such that we have the \(k\)-basis of \(U^{+}(\chi)\) formed by the elements

\[(9) \quad \left\{ \begin{array}{ll} \hat{E}_1^{m_1} \hat{E}_2^{m_2} & \text{if } M \text{ is finite, that is } M \in \mathbb{N}, \\ \hat{E}_1^{m_1} \hat{E}_2^{m_2} \hat{E}_M^{m_M} & \text{for some } M' \in \mathbb{N} \text{ if } M = \infty \end{array} \right.\]

with \(0 \quad m_i \quad h_i\), and \(h_i := \text{Max}\{n|\text{[|]}_n(\text{., .})! \neq 0\}\) \(\in \mathbb{N} \cup \{+\infty\}\).

Let

\[(10) \quad R_+ := \{\beta_i | 1 \quad i \quad M\}.\]

We say that \(\chi\) is finite-type if \(|R_+| < +\infty\). See [H09] for the classification. Note that if \(\dim U(\chi) < \infty\), then \(\chi\) is finite-type.
Theorem 2. (see [HY08b, Theorems 4.8, 4.9]) Assume that $\chi$ is finite-type. Then $|R_+|=M$ as for (20). We write $E_{i}=\hat{E}_{i}$ if $\hat{E}_{i}\in U(\chi)$. Then after re-choosing $E_{i}$ (as in (51)), we may assume that $E_{j}^{h+1}\equiv 0$ if $h<+\infty$ and that $E_{i}E_{j}\chi(\beta_{i},\beta_{j})E_{j}E_{i}\in (E|,i<r<j)$ for any $i<j$, so

\begin{equation}
\{E_{f(1)}^{m_{f(1)}}E_{f(2)}^{m_{f(2)}}E_{f(M)}^{m_{f(M)}}|0 \leq m_{i}\}
\end{equation}

is a $k$-basis of $U(\chi)$ for any bijective map $f:\{1,2,\ldots,M\}\rightarrow \{1,2,\ldots,M\}$.

Convention. Let $\chi_{1},\chi_{2}:\mathbb{Z}\Pi\times \mathbb{Z}\Pi\rightarrow k^{\times}$ be two bi-characters. Let $f_{1},f_{2}:U(\chi_{1})\rightarrow U(\chi_{2})$ be two $k$-algebra homomorphisms. Then we write

\begin{equation}
(22)\quad f_{1}=f_{2}
\end{equation}

if

\begin{equation}
(23)\quad f_{1}(K L )=f_{2}(K L ),\quad f_{1}(E_{i})=f_{2}(E_{i}),\quad f_{1}(F_{i})=f_{2}(F_{i})
\end{equation}

for all $\lambda,\mu\in \mathbb{Z}\Pi$ and $i\in I$.

3 Heckenberger’s Lusztig-type isomorphisms

Here we explain a generalization [H07] of Lusztig-type isomorphisms [L].

Assume $\chi$ to be any bi-character. Let

\begin{align*}
(24)\quad [X,Y]^{+} &= X Y \quad \chi(\lambda,\mu)Y X, \\
(25)\quad [X,Y]^{-} &= X Y \quad \chi(\lambda,\mu)^{-1}Y X, \\
(26)\quad [X,Y]^{\vee,+} &= X Y \quad \chi(\mu,\lambda)Y X, \\
(27)\quad [X,Y]^{\vee,-} &= X Y \quad \chi(\mu,\lambda)^{-1}Y X
\end{align*}

for $X\in U(\chi)$ and $Y\in U(\chi)$ with $\lambda,\mu\in \mathbb{Z}\Pi$. Let $i,j\in I$ be such that $i\neq j$. Let

\begin{align*}
E_{i}^{+} &= E_{i}, \quad E_{i}^{-}=E_{i}, \\
E_{j}^{+}+m_{i} &= [E_{i},E_{j}^{+}+(m-1)_{i}]_{+}, \quad E_{j}^{-}+m_{i} = [E_{i},E_{j}^{-}+(m-1)]_{-}, \\
F_{j}^{+}+m_{i} &= [F_{i},F_{j}^{+}+(m-1)]_{+}^{\vee}, \quad F_{j}^{-}+m_{i} = [F_{i},F_{j}^{-}+(m-1)]_{-}^{\vee},
\end{align*}

for $m\in \mathbb{N}$. For $m\in \mathbb{Z}_{\geq 0}$, we have

\begin{equation}
(29)\quad [m](\alpha,\beta)^{\dagger} \prod_{s=1}^{m}(1-\chi(\alpha_{i},\alpha_{i})^{s-1}\chi(\alpha_{i},\alpha_{j})\chi(\alpha_{j},\alpha_{i})) \neq 0
\end{equation}

$\iff E_{j}^{+}+m_{i} \neq 0 \iff E_{j}^{-}+m_{i} \neq 0 \iff F_{j}^{+}+m_{i} \neq 0 \iff F_{j}^{-}+m_{i} \neq 0$ 

$\iff \alpha_{j}+m\alpha_{i} \in R_{+}$. 

We also have
\[ [E_{j+m_{ij}}^{+}, F_{j+m_{ij}}^{+}] = (\chi(\alpha_{i}, \alpha_{j})^{m_{ij}-1}\chi(\alpha_{i}, \alpha_{j})\chi(\alpha_{j}, \alpha_{i}))^{m}[E_{j+m_{ij}}^{-}, F_{j+m_{ij}}^{-}] \]
\[ = (1)^{m}[m]! \prod_{s=1}^{m}(1 + \chi(\alpha_{i}, \alpha_{i})^{s-1}\chi(\alpha_{i}, \alpha_{j})\chi(\alpha_{j}, \alpha_{i}))^{m_{ij}}[E_{j+m_{ij}}^{-}, F_{j+m_{ij}}^{-}]. \]

**Theorem 3. ([H07])** Let \( i \in I \). Assume that for all \( j \in I \setminus \{i\} \), there exist \( m_{ij} \in \mathbb{Z}_{\geq 0} \) such that \( E_{j+m_{ij}}^{+} \neq 0 \) and \( E_{j+(m_{ij}+1)}^{+} = 0 \).

1. There exist a bi-character \( r_{i}(\chi) : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \to k^{\times} \) and \( k \)-algebra isomorphisms \( \sigma_{i}^{r_{i}(\chi)} : \mathbb{Z}\Pi \to \mathbb{Z}\Pi \) by
\[
T_{i} = T_{i}^{+} : U(r_{i}(\chi)) \to U(\chi), \quad T_{i}^{-} : U(r_{i}(\chi)) \to U(\chi)
\]
such that
\[
T_{i}^{\pm}(K_{i}) = K_{-i}, \quad T_{i}^{\pm}(L_{i}) = L_{-i},
\]
\[
T_{i}^{\pm}(K_{j}) = K_{j + m_{ij}}^{\pm}, \quad T_{i}^{\pm}(L_{j}) = L_{j + m_{ij}}^{\pm},
\]
\[
T_{i}(E_{i}) = F_{i}^{\pm}L_{-i}, \quad T_{i}^{-}(E_{i}) = K_{-i}E_{i},
\]
\[
T_{i}^{\pm}(E_{j}) = E_{j}^{\pm} + m_{ij}^{\chi}, \quad T_{i}^{\pm}(F_{j}) = F_{j}^{\pm} + m_{ij}^{\chi},
\]
where \( j \in I \setminus \{i\} \).

2. \( r_{i}(r_{i}(\chi)) \) exists in the same way as above with \( r_{i}(\chi) \) in place of \( \chi \). Further \( r_{i}(r_{i}(\chi)) = \chi, \quad m_{ij}^{r_{i}(\chi)} = m_{ij} \) for all \( j \in I \setminus \{i\} \).

3. Let \( T_{i} : U(r_{i}(\chi)) \to U(\chi) \) be as in (31). Let \( T_{i}^{-} : U(\chi) \to U(r_{i}(\chi)) \) be the one as in (31) defined with \( r_{i}(\chi) \) in place of \( \chi \). Then \( T_{i}^{-}T_{i} = id_{U(\chi)} \) and \( T_{i}T_{i}^{-} = id_{U(r_{i}(\chi))} \).

4. Define the \( \mathbb{Z} \)-module isomorphism \( \sigma_{i}^{r_{i}(\chi)} : \mathbb{Z}\Pi \to \mathbb{Z}\Pi \) by \( T_{i}^{\pm}(U(r_{i}(\chi))) = U(\chi), \chi_{i} \) for all \( \lambda \in \mathbb{Z}\Pi \). Then
\[
\sigma_{i}^{r_{i}(\chi)}(\lambda) = \sigma_{i}(\sigma_{i}^{r_{i}(\chi)}(\lambda)) = id_{\mathbb{Z}\Pi}
\]
and
\[
\sigma_{i}^{r_{i}(\chi)}(R_{+}^{r_{i}(\chi)} \setminus \{\alpha_{i}\}) = R_{+} \setminus \{\alpha_{i}\}, \quad \sigma_{i}^{r_{i}(\chi)}(\alpha_{i}) = \alpha_{i}.
\]

**Theorem 4. ([H07])** Assume \( \chi \) to be finite-type. Let \( i, j \in I \) to be such that \( i \neq j \). Let \( M = |R_{+} \cap (\mathbb{Z}_{\geq 0}\alpha_{i} \cup \mathbb{Z}_{\geq 0}\alpha_{j})| \). For \( n \in \{1, 2, \ldots, M\} \), define two bi-characters \( \chi_{n}, \chi'_{n} \), two \( \mathbb{Z} \)-module automorphism \( \sigma_{n}, \sigma_{n}' \) of \( \mathbb{Z}\Pi \) and two \( k \)-algebra
isomorphisms \( \bar{T}_n : U(\chi_n) \rightarrow U(\chi), \bar{T}'_n : U(\chi'_n) \rightarrow U(\chi) \) in the way that \( \chi_1 = \chi'_1 = \chi, \bar{\sigma}_1 = \bar{\sigma}'_1 = \text{id}_{\mathbb{Z}I}, \bar{T}_1 = \bar{T}'_1 = \text{id}_{U(\chi)} \), and

(39) \( \chi_{2n} = r_i(\chi_{2n-1}), \chi_{2n+1} = r_j(\chi_{2n}), \chi'_{2n} = r_j(\chi'_{2n-1}), \chi'_{2n+1} = r_i(\chi'_{2n}) \),

(40) \( \bar{\sigma}_2 = \bar{\sigma}_{2n-1} \sigma_i^{2n}, \bar{\sigma}_{2n+1} = \bar{\sigma}_{2n} \sigma_j^{2n}, \bar{\sigma}'_2 = \bar{\sigma}'_{2n-1} \sigma_j^{2n}, \bar{\sigma}'_{2n+1} = \bar{\sigma}'_{2n} \sigma_i^{2n} \),

(41) \( \bar{T}_2 = \bar{T}_{2n-1} T_i, \bar{T}_{2n+1} = \bar{T}_{2n} T_j, \bar{T}'_2 = \bar{T}'_{2n-1} T_j, \bar{T}'_{2n+1} = \bar{T}'_{2n} T_i \).

Then we have

(42) \( \chi_M = \chi'_M \),

(43) \( \bar{\sigma}_M = \bar{\sigma}'_M \)

and

(44) \( \bar{T}_M = \bar{T}'_M \).

## 4 Longest elements of Weyl groupoids

In this section we always assume \( \chi \) to be finite-type, and refer to [CH08] for categorical definitions of Weyl groupoids.

**Convention.** For a category \( C \), we denote the product of the morphisms by \( \cdot \). That is, for two morphism \( f_1 \in \text{Mor}(a_1, b_1) \) and \( f_2 \in \text{Mor}(a_2, b_2) \) with \( a_1, b_1, a_2 \) and \( b_2 \in \text{Ob}(C) \), we denote their product by

(45) \( f_1 f_2 \) if \( b_2 = a_1 \).

Set

(46) \( C(\chi) = \{\chi\} \cup \bigcup_{n=1}^{\infty} \{ r_{i_1} \ldots r_{i_n}(\chi) | i_1, \ldots, i_n \in I \} \).

Let \( W = W(\chi) \) be the category with \( \text{Ob}(W) = C(\chi) \) and generated by the maps \( \sigma_i \in \text{Mor}_W(\chi', r_i(\chi')) \) with \( \chi' \in \text{Ob}(W) \) and \( i \in I \). Let \( \mathcal{W} = \mathcal{W}(\chi) \) be the (abstract) category with \( \text{Ob}(\mathcal{W}) = C(\chi) \) defined by generators \( s_i' \in \text{Mor}_W(\chi', r_i(\chi')) \) with \( \chi' \in \text{Ob}(W) \) and \( i \in I \) and relations

(47) \( s_i' s_i'^{-1} = 1_{r_i(\chi')} \),
We call $\mathcal{W}$ the Weyl groupoid. Define the morphism $\phi : \mathcal{W} \rightarrow W$ by $\phi(s_i^\ell) = s_i$. Then $\phi$ is bijective, see [HY08a, Theorem 1]. Let $\ell(1^0) = 0$ for $\chi' \in C(\chi)$. Let $\ell(s_i^\ell) = 1$. For $w \in \text{Mor}_\mathcal{W}(\chi_1, \chi_2)$, let $\ell(w)$ be the least number $\ell(w') + \ell(w'')$ with $w = w' w''$ for some $\chi_3 \in C(\chi)$, and some $w' \in \text{Mor}_\mathcal{W}(\chi_3, \chi_2)$, some $w'' \in \text{Mor}_\mathcal{W}(\chi_1, \chi_3)$. By [HY08a, Lemma 8(iii)], we have

$$\ell(w) = |\{\alpha \in R_+^1| \phi(w)(\alpha) \in R_+^2\}|.$$  

Moreover for each $\chi_1 \in C(\chi)$, there exists unique $\chi_2 \in C(\chi)$ and $1w_0 \in \text{Mor}_\mathcal{W}(\chi_2, \chi_1)$ such that $\phi(1w_0)|_{R_+^2} = R_+^2$. We call $1w_0$ the longest element since $\ell(1w_0)$ $\ell(w')$ for any $w' \in \text{Mor}_\mathcal{W}(\chi_3, \chi_4)$ for any $\chi_3, \chi_4 \in C(\chi)$.

Let $\tilde{\mathcal{W}} = \mathcal{W}(\chi)$ be the (abstract) category with $\text{Ob}(\tilde{\mathcal{W}}) = C(\chi)$ defined by generators $\tilde{s}_i \in \text{Mor}_{\tilde{\mathcal{W}}}(\chi', r_i(\chi'))$ with $\chi' \in \text{Ob}(W)$ and $i \in I$ and relations

$$\tilde{s}_i^3 \tilde{s}_i^j \tilde{s}_i^k = \tilde{s}_i^j \tilde{s}_i^k \tilde{s}_i^j \tilde{s}_i^k$$

(both sides are composed of $|R_+ \cap (\mathbb{Z}\alpha_i \mathbb{Z}\alpha_j)|$-factors).

Let $\tilde{\mathcal{W}} = \mathcal{W}(\chi', \chi')$ denote the identity morphism. Define the morphism $\tilde{\phi} : \tilde{\mathcal{W}} \rightarrow \mathcal{W}$ by $\tilde{\phi}(\tilde{s}_i^\ell) = s_i^\ell$.

Let $\ell(1^0) = 0$ for $\chi' \in C(\chi)$. Let $\ell(\tilde{s}_i^\ell) = 1$. For $\tilde{w} \in \text{Mor}_{\tilde{\mathcal{W}}}(\chi_1, \chi_2)$, let $\tilde{\ell}(\tilde{w})$ be the least number $\tilde{\ell}(\tilde{w}') + \tilde{\ell}(\tilde{w}'')$ with $\tilde{w} = \tilde{w}' \tilde{w}''$ for some $\chi_3 \in C(\chi)$, and some $\tilde{w}' \in \text{Mor}_{\tilde{\mathcal{W}}}(\chi_3, \chi_2)$, some $\tilde{w}'' \in \text{Mor}_{\tilde{\mathcal{W}}}(\chi_1, \chi_3)$.

**Theorem 5.** ([HY08a, Theorem 5, Corollary 6]) Let $\chi_1, \chi_2 \in C(\chi)$. For $w \in \text{Mor}_\mathcal{W}(\chi_1, \chi_2)$ and $\tilde{w}_1, \tilde{w}_2 \in \tilde{w} \in \text{Mor}_{\tilde{\mathcal{W}}}(\chi_1, \chi_2)$ with $\tilde{\phi}(\tilde{w}_1) = \tilde{\phi}(\tilde{w}_2) = w$ and $\ell(w) = \ell(\tilde{\phi}(\tilde{w}_1)) = \ell(\tilde{\phi}(\tilde{w}_2))$, we have $\tilde{w}_1 = \tilde{w}_2$. Further, if $\tilde{w} \in \text{Mor}_{\tilde{\mathcal{W}}}(\chi_1, \chi_2)$ is such that $\tilde{\ell}(\tilde{w}) > \ell(\tilde{\phi}(\tilde{w}))$, then $\tilde{w} = \tilde{w}' \tilde{s}_i^3 \tilde{s}_i^3 \tilde{w}''$ for some $i \in I$, $\tilde{w}' \in \text{Mor}_{\tilde{\mathcal{W}}}(\chi_1, \chi_3)$ and $\tilde{w}'' \in \text{Mor}_{\tilde{\mathcal{W}}}(\chi_3, \chi_2)$ with $\ell(\tilde{w}') + \ell(\tilde{w}'') = \ell(\tilde{w})$.

Assume $w_0$ to be $s_{j_1}^1 s_{j_2}^2 s_{j_M}^M$, where $M = |R_+|$, $r_1(\chi_1) = \chi$, and $r_j(\chi_j) = \chi_{j-1}$. Let $\overline{T}_1 = \text{id}_U$. For $2 \leq n \leq M$, define the k-algebra isomorphism $T_n : U(\chi_{n-1}) \rightarrow U(\chi)$ by $T_n = \overline{T}_{n-1} T_{j_{n-1}}$. Then as for $E_i$ of Theorem 2, we may put

$$E_i = \overline{T}_i(E_i)$$

for $1 \leq j \leq M$.  

\[E_i = \overline{T}_i(E_i)\] 

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5 Shapovalov determinants

Let $\chi$ be a bi-character. We define the Shapovalov matrix $\text{Sh}$ in the natural way for each $\alpha \in \mathbb{Z}_{\geq 0}\Pi$. More precisely, $\text{Sh}$ is a $\dim U^{+}(\chi) \times \dim U^{+}(\chi)$-matrix whose components are elements of $U^{0}(\chi)$. Let $\rho : \mathbb{Z}\Pi \rightarrow k^{\times}$ be the (abelian) group homomorphism defined by $\rho(\alpha_{i}) = \chi(\alpha_{i}, \alpha_{i})$. We use the Kostant partition function $P(\alpha, \beta, t) := \dim E^{t}U^{+}(\chi) - t$, where we define $P(\alpha, \beta, t) = 0$ in case $\alpha \neq \beta \notin \mathbb{Z}_{\geq 0}\Pi$.

**Theorem 6.** ([HY08b, Theorem 7.3]) Let $\chi$ be finite-type. Assume that $\chi(\beta, \beta) \neq 1$ for all $\beta \in R_{+}$. Then for $\alpha \in \mathbb{Z}_{\geq 0}\Pi$, we have

$$\text{det} \text{Sh} = c \prod_{\in R^{+}_{\chi}} \prod_{t=1}^{h_{\beta}^{\chi}} (\rho(\beta)K \chi(\beta, \beta)^{t}L)^{P^{\chi}(\alpha, \beta, t)}$$

for some $c \in k^{\times}$.

As stated below, for $U(\chi)$ which is the (ordinary or small) quantum group of a finite dimensional Lie algebra $\mathfrak{g}$, we have the generalization of (1) the one [dDK90] for $q \in \mathbb{C}^{\times}$ which is not a root of unity, and (2) the one [KL97] for $q \in \mathbb{C}^{\times}$ which is a primitive $p$-th root of unity for some prime number $p$.

**Corollary 7.** Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of type $A$-$G$ or a finite dimensional simple Lie superalgebra of type $A$-$G$. Then the Shapovalov determinant of the quantum group $U_{q}(\mathfrak{g})$ when $q$ is not root of unity or the small quantum group $u_{q}(\mathfrak{g})$ when $q$ is a primitive $r$-th root of unity for some positive integer $r \neq 2$ is given by

$$c \prod_{\in R^{+}_{\chi}} \prod_{t=1}^{\infty} (q^{2(t)}K \chi(\beta, \beta)^{t}L)^{P^{\chi}(\alpha, \beta, t)}$$

for some $c \in \mathbb{C}^{\times}$.

We even recover the original ones due to Shapovalov [Sha72], and Kac [Kac77] (super cases):

**Corollary 8.** Let $\mathfrak{g}$ be as above. Then the Shapovalov determinant of the enveloping algebra $U(\mathfrak{g})$ is given by

$$c \prod_{\in R^{+}_{\chi}} (H + (\rho, \beta) \frac{(\beta, \beta)t}{2})^{P^{\chi}(\alpha, \beta, t)}$$

for some $c \in \mathbb{C}^{\times}$. 
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References


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