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A survey on Shapovalov determinants of (generalized) quantum groups at roots of 1

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Abstract

This is an informal survey on a joint work [HY08b] with Istvan Heckenberger.

1 A quantum group $U(\chi)$ defined for any bi-character $\chi$

Recently study of Nichols algebras has been achieved very actively for the viewpoint of classification of Hopf algebras, see [AS98], [AS02], [Hec06]. One of their examples is the positive part $U^+(\chi)$ of a generalized quantum group $U(\chi)$ defined below.

Let $k$ be a field and $k^\times = k \setminus \{0\}$. For $n \in \mathbb{Z}_{\geq 0}$ and $x \in k$, let

$$[n]_x = \sum_{m=1}^{n} x^{m-1}, \quad [n]_x! = \prod_{m=1}^{n} [m]_x.$$  

(1)

For two elements $X_1$ and $X_2$ of a $k$-algebra we use the convention:

$$X_1 X_2 \Longleftrightarrow \exists x \in k^\times \quad X_1 = xX_2.$$  

(2)

Let $I$ be a finite index set. Let $\mathbb{Z}I = \sum_{i \in I} \mathbb{Z} \alpha_i$ be a rank $|I|$ free $\mathbb{Z}$-module with a basis $\Pi = \{\alpha_i | i \in I\}$. We say that a map $\chi : \mathbb{Z}I \times \mathbb{Z}I \rightarrow k^\times$ is a bi-character if $\chi(a+b, c) = \chi(a, c)\chi(b, c)$, and $\chi(a, b + c) = \chi(a, b)\chi(a, c)$ for all $a, b, c \in \mathbb{Z}I$.

Let $\chi$ be any bi-character. Then, as we explain more precisely in Section 2, Lusztig’s definition [L, 3.1.1] of the quantum groups can be applied to define the Hopf $k$-algebra $U(\chi)$ with the generators

$$K, L (\lambda \in \mathbb{Z}I), E_i, F_i (i \in I),$$  

(3)
for which $K,L$ ($\lambda, \mu \in \mathbb{Z}\Pi$) are linearly independent and the following equations hold:

(4) $K_0 = L_0 = 1, \ K_+ = KK, \ L_+ = LL, \ KL = LK$,

(5) $KL E_j (KL)^{-1} = \frac{\chi(\lambda, \alpha_j)}{\chi(\alpha_j, \mu)} E_j, \ KL F_j (KL)^{-1} = \frac{\chi(\alpha_j, \mu)}{\chi(\lambda, \alpha_j)} F_j,$

(6) $E_i F_j \ F_j E_i = \delta_{ij} (K_i \ L_j).$

(7) $\Delta(KL) = KL \ K L, \ \epsilon(KL) = 1, S(KL) = (KL)^{-1},$

(8) $\Delta(E_i) = E_i + K, \ E_i, \ \Delta(F_i) = F_i + L, \ F_i,$

(9) $\epsilon(E_i) = \epsilon(F_i) = 0, \ S(E_i) = K^{-1} E_i, \ S(F_i) = F_i L^{-1}.$

Let $U^0(\chi) := k \mathbb{Z}_{\geq 0} \Pi k K L$. Let $U^+(\chi)$ and $U^-(\chi)$ be the subalgebra of $U(\chi)$ generated by $E_i$ and $F_i$ with all $i \in I$ respectively. Then $U(\chi) = U^+(\chi) \ U^0(\chi)$ $U^-(\chi)$, as a $\mathbb{k}$-linear space. We have the $\mathbb{Z}_{\geq 0} \Pi$-grading $U^\pm(\chi) = \mathbb{Z}_{\geq 0} \Pi U^\pm(\chi)$ defined by $U^+(\chi)i = k E_i, \ U^-(\chi)i = k F_i$, and $U^+(\chi) U^-(\chi) \subset U^\pm(\chi)$. We also have $\dim U^-(\chi) = \dim U^+(\chi)$ for all $\lambda \in \mathbb{Z}_{\geq 0} \Pi$.

2 Drinfeld pairing of $U(\chi)$

Here we will explain how to define $U(\chi)$ more precisely. By abuse of notation, we use the same symbols as above for the generators of the algebras introduced in this paragraph. Let $\tilde{U}^+(\chi)$ and be $\tilde{U}^-(\chi)$ the free $\mathbb{k}$-algebras (with 1) with the generators $\{E_i| i \in I\}$ and $\{F_i| i \in I\}$ respectively. Let $\tilde{U}^0(\chi)$ be the $\mathbb{k}$-linear space with the basis $\{KL| \lambda, \mu \in \mathbb{Z}\Pi\}$. Let $\tilde{U}(\chi) = \tilde{U}^+(\chi) \ k \ \tilde{U}^0(\chi) \ k \ \tilde{U}^-(\chi)$. Identify $X \in \tilde{U}^+(\chi), \ Z \in \tilde{U}^0(\chi)$ and $Y \in \tilde{U}^-(\chi)$ with $X \ 1 \ 1, \ 1 \ Z \ 1$ and $1 \ 1 \ Y$ respectively, and regard $\tilde{U}^+(\chi), \ \tilde{U}^0(\chi)$ and $\tilde{U}^-(\chi)$ as subspaces of $\tilde{U}(\chi)$ in this way. Then $\tilde{U}(\chi)$ can be regarded as the $\mathbb{k}$-algebra (with 1) presented by the same generators as the ones for $U(\chi)$ and the relations (4), (5) and (6) (cf. [L, Prop. 3.2.4]). Further $\tilde{U}(\chi)$ can be regarded as the Hopf $\mathbb{k}$-algebra with the same equalities as (7), (8) and (9). Let $\tilde{U}^{+,K}(\chi)$ be the subalgebra of $\tilde{U}(\chi)$ generated by $E_i$'s and $K$'s. Let $\tilde{U}^{L,-}(\chi)$ be the subalgebra of $\tilde{U}(\chi)$ generated by $F_i$'s and $L$'s. Then there exists a unique $\mathbb{k}$-bilinear form

\begin{equation}
\langle , \rangle: \tilde{U}^{+,K}(\chi) \times \tilde{U}^{L,-}(\chi) \to \mathbb{k}
\end{equation}
with

(11) $\langle 1, Y \rangle = \epsilon(Y)$, $\langle X, 1 \rangle = \epsilon(X)$, $\langle S(X), Y \rangle = \langle X, S^{-1}(Y) \rangle$, 

(12) $\langle X_1X_2, Y \rangle = \sum_g \langle X_2, Y_g^{(1)} \rangle \langle X_1, Y_g^{(2)} \rangle$, 

(13) $\langle X, Y_1Y_2 \rangle = \sum_h \langle X_h^{(1)}, Y_1 \rangle \langle X_h^{(2)}, Y_2 \rangle$, 

(14) $\langle E_i, F_j \rangle = \delta_{ij}$, $\langle K, L \rangle = \chi(\lambda, \mu)$, $\langle E_i, L \rangle = \langle K, F_j \rangle = 0$ 

for $X, X_1, X_2 \in \tilde{U}^{+,K}(\chi)$ with $\Delta(X) = \sum_h X_h^{(1)}X_h^{(2)}$, and $Y, Y_1, Y_2 \in \tilde{U}^{L,-}(\chi)$ with $\Delta(Y) = \sum_g Y_g^{(1)}Y_g^{(2)}$ and for $i, j \in I$ and $\lambda, \mu \in \mathbb{Z}\Pi$. We see

(15) $\langle \tilde{E}K, \tilde{F}L \rangle = \langle \tilde{E}, \tilde{F} \rangle \langle K, L \rangle$ for $\tilde{E} \in \tilde{U}^{+}(\chi)$ and $\tilde{F} \in \tilde{U}^{-}(\chi)$.

Then $\tilde{J}(\chi)$ is the kernel of the Hopf algebra epimorphism from $\tilde{U}(\chi)$ to $U(\chi)$ sending the generators to the ones denoted by the same symbols.

Theorem 1. (Kharchenko [Kha99]) There exist $M \in \mathbb{N} \cup \{\infty\}$ and elements

$\hat{E}_i \in U^{+}(\chi)$, $\hat{E}_i \in U^{+(\chi)} \ i \ M$ for some $\beta_i \in \mathbb{Z}_{\geq 0}\Pi \backslash \{0\}$ such that we have the $k$-basis of $U^{+}(\chi)$ formed by the elements

(19) $\begin{cases} 
\hat{E}_1^{m_1} \hat{E}_2^{m_2} \hat{E}_M^{m_M} & \text{if } M \text{ is finite, that is } M \in \mathbb{N}, \\
\hat{E}_1^{m_1} \hat{E}_2^{m_2} \hat{E}_{M'}^{m_{M'}} & \text{for some } M' \in \mathbb{N} \text{ if } M = \infty
\end{cases}$

with $0 \ m_i \ \text{and } h_i := \text{Max}\{n||n|\ (\ , \ )! \neq 0\} \in \mathbb{N} \cup \{+\infty\}$.

Let

(20) $R_+ := \{\beta_i | 1 \ i \ M\}$. 

Note that $|R_+| = M$, that is, $\beta_i$ and $\beta_j$ may be the same for some $i \neq j$.

We say that $\chi$ is finite-type if $|R_+| < +\infty$. See [H09] for the classification.

Note that if $\dim U(\chi) < \infty$, then $\chi$ is finite-type.
Theorem 2. (see [HY08b, Theorems 4.8, 4.9]) Assume that $\chi$ is finite-type. Then $|R_+| = M$ as for (20). We write $E_i = \hat{E}_i$ if $\hat{E}_i \in U(\chi)$. Then after re-choosing $E_i$ (as in (51)), we may assume that $E^{h_{\beta}+1}_j = 0$ if $h < +\infty$ and that $E_iE_j, \chi(\beta_i, \beta_j)E_jE_i \in <E_i| i < r < j>$ for any $i < j$, so

(21) \[ \{E^{m_{f(1)}}_1 E^{m_{f(2)}}_2 E^{m_{f(M)}}_M | 0 \leq m_i \leq h_i \} \]

is a $k$-basis of $U(\chi)$ for any bijective map $f : \{1, 2, \ldots, M\} \rightarrow \{1, 2, \ldots, M\}$.

Convention. Let $\chi_1, \chi_2 : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \rightarrow k^x$ be two bi-characters. Let $f_1, f_2 : U(\chi_1) \rightarrow U(\chi_2)$ be two $k$-algebra homomorphisms. Then we write

(22) \[ f_1 = f_2 \]

if

(23) \[ f_1(KL) = f_2(KL), f_1(E_i) = f_2(E_i), f_1(F_i) = f_2(F_i) \]

for all $\lambda, \mu \in \mathbb{Z}\Pi$ and $i \in I$.

3 Heckenberger's Lusztig-type isomorphisms

Here we explain a generalization [H07] of Lusztig-type isomorphisms [L].

Assume $\chi$ to be any bi-character. Let

(24) \[ [X, Y]^+ = XY, \chi(\lambda, \mu) YX \]

(25) \[ [X, Y]^+ = XY, \chi(\lambda, \mu)^{-1} YX \]

(26) \[ [X, Y]^\vee, + = XY, \chi(\mu, \lambda) YX \]

(27) \[ [X, Y]^\vee, + = XY, \chi(\mu, \lambda)^{-1} YX \]

for $X \in U(\chi)$ and $Y \in U(\chi)$ with $\lambda, \mu \in \mathbb{Z}\Pi$. Let $i, j \in I$ be such that $i \neq j$. Let

\[ E^+_i = E_i, \quad E^-_i = E_i, \]

\[ E^+_{j+m} = [E_i, E^{+}_{j+(m-1)}, E^-_{j+m} = [E_i, E^{-}_{j+(m-1)}, ]^{\vee, -}, F^+_{j+m} = [F_i, F^{+}_{j+(m-1)}, F^-_{j+m} = [F_i, F^{-}_{j+(m-1)}, ]^{-} \]

for $m \in \mathbb{N}$. For $m \in \mathbb{Z}_{\geq 0}$, we have

\[ [m] (\cdot, \cdot)! \prod_{s=1}^m (1 - \chi(\alpha_i, \alpha_i)^{s-1} \chi(\alpha_i, \alpha_j) \chi(\alpha_j, \alpha_i)) \neq 0 \]

(29) \[ \Leftrightarrow E^+_{j+m} \neq 0 \Leftrightarrow E^-_{j+m} \neq 0 \Leftrightarrow F^+_{j+m} \neq 0 \Leftrightarrow F^-_{j+m} \neq 0 \Leftrightarrow \alpha_j + m\alpha_i \in R_+ \].
We also have

\[(30)\]

\[[E_{j+m_{ij}}^{+}, F_{j+m_{ij}}^{+}] = (\chi(\alpha_{i}, \alpha_{i})^{m-1}\chi(\alpha_{i}, \alpha_{j})\chi(\alpha_{j}, \alpha_{i}))^{m}[E_{j-m_{ij}}^{-}, F_{j-m_{ij}}^{-}] = (1)^{m}([m]_{ii}! \prod_{s=1}^{m}(1 \chi(\alpha_{i}, \alpha_{i})^{s-1}\chi(\alpha_{i}, \alpha_{j})\chi(\alpha_{j}, \alpha_{i}))(K_{j-m_{ij}}+m_{ij}L_{j-m_{ij}}).\]

**Theorem 3.** ([H07]) Let \(i \in I\). Assume that for all \(j \in I \setminus \{i\}\), there exist \(m_{ij} \in \mathbb{Z}_{\geq 0}\) such that \(E_{j+m_{ij}}^{+} \neq 0\) and \(E_{j+(m_{ij}+1)}^{+} = 0\).

(1) There exist a bi-character \(r_{i}(\chi) : \mathbb{Z}_{\Pi} \times \mathbb{Z}_{\Pi} \rightarrow k^{\times}\) and \(k\)-algebra isomorphisms

\[(31)\]

\(T_{i} = T_{i}^{+} : U(r_{i}(\chi)) \rightarrow U(\chi), \quad T_{i}^{-} : U(r_{i}(\chi)) \rightarrow U(\chi)\)

such that

\[(32)\]

\(T_{i}^{\pm}(K_{i}) = K_{-i}, T_{i}^{\pm}(L_{i}) = L_{-i},\)

\[(33)\]

\(T_{i}^{\pm}(K_{j}) = K_{j-m_{ij}^{\chi}}, T_{i}^{\pm}(L_{j}) = L_{j-m_{ij}^{\chi}},\)

\[(34)\]

\(T_{i}(E_{i}) = F_{i}L_{-i}, T_{i}(F_{i}) = K_{-i}E_{i},\)

\[(35)\]

\(T_{i}^{-}(E_{i}) = K_{-i}F_{i}, T_{i}^{-}(F_{i}) = E_{i}L_{-i},\)

\[(36)\]

\(T_{i}^{\pm}(E_{j}) = E_{j-m_{ij}^{\chi}}, T_{i}^{\pm}(F_{j}) = F_{j-m_{ij}^{\chi}},\)

where \(j \in I \setminus \{i\}\).

(2) \(r_{i}(r_{i}(\chi))\) exists in the same way as above with \(r_{i}(\chi)\) in place of \(\chi\). Further \(r_{i}(r_{i}(\chi)) = \chi, m_{ij}^{r_{i}(\chi)} = m_{ij}\) for all \(j \in I \setminus \{i\}\).

(3) Let \(T_{i} : U(r_{i}(\chi)) \rightarrow U(\chi)\) be as in (31). Let \(T_{i}^{-} : U(\chi) \rightarrow U(r_{i}(\chi))\) be the one as in (31) defined with \(r_{i}(\chi)\) in place of \(\chi\). Then \(T_{i}^{-}T_{i} = id_{U(r_{i}(\chi))}\) and \(T_{i}T_{i}^{-} = id_{U(\chi)}\).

(4) Define the \(\mathbb{Z}\)-module isomorphism \(\sigma_{i}^{r_{i}(\chi)} = \sigma_{i} : \mathbb{Z}_{\Pi} \rightarrow \mathbb{Z}_{\Pi}\) by \(T_{i}^{\pm}(U(r_{i}(\chi))) = U(\chi)_{i}(\chi)\) for all \(\lambda \in \mathbb{Z}_{\Pi}\). Then

\[(37)\]

\(\sigma_{i}^{r_{i}(\chi)} = \sigma_{i}, \quad \sigma_{i} \sigma_{i}^{r_{i}(\chi)} = id_{\mathbb{Z}_{\Pi}}\)

and

\[(38)\]

\(\sigma_{i}^{r_{i}(\chi)}(R_{i}^{r_{i}(\chi)} \\cap \\{\alpha_{i}\}) = R_{i}^{\chi} \\cap \\{\alpha_{i}\}, \quad \sigma_{i}^{r_{i}(\chi)}(\alpha_{i}) = \alpha_{i}.\)

**Theorem 4.** ([H07]) Assume \(\chi\) to be finite-type. Let \(i, j \in I\) to be such that \(i \neq j\). Let \(M = |R_{i} \cap (\mathbb{Z}_{\geq 0}\alpha_{i} - \mathbb{Z}_{\geq 0}\alpha_{j})|\). For \(n \in \{1, 2, \ldots, M\}\), define two bi-characters \(\chi_{n}, \chi_{n}',\) two \(\mathbb{Z}\)-module automorphisms \(\tilde{\sigma}_{n}, \tilde{\sigma}_{n}'\) of \(\mathbb{Z}_{\Pi}\) and two \(k\)-algebra
isomorphisms $\tilde{T}_n : U(\chi_n) \rightarrow U(\chi)$, $\bar{T}_n' : U(\chi'_n) \rightarrow U(\chi)$ in the way that $\chi_1 = \chi'_1 = \chi$, $\bar{\sigma}_1 = \bar{\sigma}'_1 = \text{id}_{\mathbb{Z}\Pi}$, $\bar{T}_1 = \bar{T}'_1 = \text{id}_{U(\chi)}$, and

(39) $\chi_{2n} = r_i(\chi_{2n-1})$, $\chi_{2n+1} = r_j(\chi_{2n})$, $\chi'_{2n} = r_j(\chi'_{2n-1})$, $\chi'_{2n+1} = r_i(\chi'_{2n})$,
(40) $\bar{\sigma}_{2n} = \bar{\sigma}_{2n-1}\sigma_i^{2n}$, $\bar{\sigma}_{2n+1} = \bar{\sigma}_{2n}\sigma_j^{2n+1}$, $\bar{\sigma}'_{2n} = \bar{\sigma}'_{2n-1}\sigma_j^{2n}$, $\bar{\sigma}'_{2n+1} = \bar{\sigma}'_{2n}\sigma_i^{2n+1}$,
(41) $\bar{T}_{2n} = \bar{T}_{2n-1}T_i$, $\bar{T}_{2n+1} = \bar{T}_{2n}T_j$, $\bar{T}'_{2n} = \bar{T}'_{2n-1}T_j$, $\bar{T}'_{2n+1} = \bar{T}'_{2n}T_i$.

Then we have

(42) $\chi_M = \chi'_M$,  
(43) $\bar{\sigma}_M = \bar{\sigma}'_M$

and

(44) $\bar{T}_M = \bar{T}'_M$.

4 Longest elements of Weyl groupoids

In this section we always assume $\chi$ to be finite-type, and refer to [CH08] for categorical definitions of Weyl groupoids.

Convention. For a category $C$, we denote the product of the morphisms by $\cdot$. That is, for two morphism $f_1 \in \text{Mor}(a_1, b_1)$ and $f_2 \in \text{Mor}(a_2, b_2)$ with $a_1, b_1, a_2$ and $b_2 \in \text{Ob}(C)$, we denote their product by

(45) $f_1 \cdot f_2$ if $b_2 = a_1$.

Set

(46) $C(\chi) = \{\chi\} \cup \bigcup_{n=1}^{\infty}\{r_{i_1}\cdots r_{i_n}(\chi) \mid i_1, \ldots, i_n \in I\}$.

Let $W = W(\chi)$ be the category with $\text{Ob}(W) = C(\chi)$ and generated by the maps $\sigma_i \in \text{Mor}_W(\chi', r_i(\chi'))$ with $\chi' \in \text{Ob}(W)$ and $i \in I$. Let $\mathcal{W} = \mathcal{W}(\chi)$ be the (abstract) category with $\text{Ob}(\mathcal{W}) = C(\chi)$ defined by generators $s_i \in \text{Mor}_\mathcal{W}(\chi', r_i(\chi'))$ with $\chi' \in \text{Ob}(W)$ and $i \in I$ and relations

(47) $s_i \cdots s_i = 1_{r_i(\chi')}$. 

if

For some morphism isomorphism relations by k-algebra Theorem for number Define some the have least for Ob define we with some longe $\mathcal{W}$ as unique

Let $\tilde{\mathcal{W}}=	ilde{\mathcal{W}}(\chi)$ be the (abstract) category with $\text{Ob}(\tilde{\mathcal{W}})=\mathcal{C}(\chi)$ defined by generators $\tilde{s}_i'\in Mor_{\tilde{\mathcal{W}}}(\chi', r_i(\chi'))$ with $\chi'\in \text{Ob}(W)$ and $i\in I$ and relations

$$
\tilde{s}_i' \tilde{s}_j^{r_i(\chi')}\tilde{s}_i^{r_j(\chi')} = \tilde{s}_j' \tilde{s}_i^{r_j(\chi')}\tilde{s}_j^{r_i(\chi')}
$$

(both sides are composed of $|R_+ \cap (Z\alpha_i \quad Z\alpha_j)|$-factors).

Moreover for each $\chi_1 \in \mathcal{C}(\chi)$, there exists unique $\chi_2 \in \mathcal{C}(\chi)$ and $1w_0 \in \text{Mor}_\mathcal{W}(\chi_2, \chi_1)$ such that $\phi(1w_0)(R_+^2) = R_+^2$. We call $1w_0$ the longest element since $\ell(1w_0)$ $\ell(w')$ for any $w' \in \text{Mor}_\mathcal{W}(\chi_3, \chi_4)$ for any $\chi_3, \chi_4 \in \mathcal{C}(\chi)$.

Theorem 5. ([HY08a, Theorem 5, Corollary 6]) Let $\chi_1, \chi_2 \in \mathcal{C}(\chi)$. For $w \in \text{Mor}_\mathcal{W}(\chi_1, \chi_2)$ and $\tilde{\mathcal{W}}=\tilde{\mathcal{W}}(\chi_1, \chi_2)$ with $\ell(w) = \ell(\tilde{\phi}(w))$, we have $w_1 = w_2 = w$. Further, if $\tilde{w} \in \text{Mor}_{\tilde{\mathcal{W}}}(\chi_1, \chi_2)$ is such that $\ell(\tilde{w}) > \ell(\tilde{\phi}(\tilde{w}))$, then $\tilde{w} = \tilde{w}' \tilde{s}_i^{r_i(\chi')} \tilde{s}_i^{r_i(\chi')}$ for some $i \in I, \chi' \in \mathcal{C}(\chi_1, \chi_2)$.

Assume $w_0$ to be $s_{j_1}^1 s_{j_2}^2 s_{j_M}^M$, where $M = |R_+|$, $r_1(\chi_1) = \chi$, and $r_j(\chi_3) = \chi_{j-1}$. Let $T_1 = \text{id}_{U(\chi_i)}$. For $2 \leq n \leq M$, define the k-algebra isomorphism $T_n : U(\chi_{n-1}) \rightarrow U(\chi)$ by $T_n = T_{n-1}T_{j_{n-1}}$. Then as for $E_i$ of Theorem 2, we may put

$$
E_i = \tilde{T}_i(E_{j_i})
$$

for $1 \leq j \leq M$. 

159
5 Shapovalov determinants

Let $\chi$ be a bi-character. We define the Shapovalov matrix $\text{Sh}$ in the natural way for each $\alpha \in \mathbb{Z}_{\geq 0}\Pi$. More precisely, $\text{Sh}$ is a $\dim U^{+}(\chi) \times \dim U^{+}(\chi)$ matrix whose components are elements of $U^{0}(\chi)$. Let $\rho : \mathbb{Z}\Pi \to k^{\times}$ be the (abelian) group homomorphism defined by $\rho(\alpha_{i}) = \chi(\alpha_{i}, \alpha_{i})$. We use the Kostant partition function $P(\alpha, \beta, t) := \dim E^{t}U^{+}(\chi)_{-t}$, where we define $P(\alpha, \beta, t) = 0$ in case $\alpha \neq t\beta \not\in \mathbb{Z}_{\geq 0}\Pi$.

Theorem 6. ([HY08b, Theorem 7.3]) Let $\chi$ be finite-type. Assume that $\chi(\beta, \beta) \neq 1$ for all $\beta \in R_{+}$. Then for $\alpha \in \mathbb{Z}_{\geq 0}\Pi$, we have

$$\det \text{Sh} = c \prod_{\alpha \in R^{+}} \prod_{t=1}^{h_{\beta}^{\chi}} (\rho(\beta)K(\beta, \beta)^{t}L)^{P^{\chi}(\alpha, \beta, t)}$$

for some $c \in k^{\times}$.

As stated below, for $U(\chi)$ which is the (ordinary or small) quantum group of a finite dimensional Lie algebra $\mathfrak{g}$, we have the generalization of (1) the one [dDK90] for $q \in \mathbb{C}^{\times}$ which is not a root of unity, and (2) the one [KL97] for $q \in \mathbb{C}^{\times}$ which is a primitive $p$-th root of unity for some prime number $p$.

Corollary 7. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of type $A$-$G$ or a finite dimensional simple Lie superalgebra of type $A$-$G$. Then the Shapovalov determinant of the quantum group $U_{q}(\mathfrak{g})$ when $q$ is not root of unity or the small quantum group $u_{q}(\mathfrak{g})$ when $q$ is a primitive $r$-th root of unity for some positive integer $r = 2$ is given by

$$c \prod_{\alpha \in R^{+}} \prod_{t=1}^{h_{\beta}^{\chi}} (q^{2(\beta, \beta)^{t}}K(\beta, \beta)^{t}K^{-1})^{P^{\chi}(\alpha, \beta, t)}$$

for some $c \in \mathbb{C}^{\times}$.

We even recover the original ones due to Shapovalov [Sha72], and Kac [Kac77] (super cases):

Corollary 8. Let $\mathfrak{g}$ be as above. Then the Shapovalov determinant of the enveloping algebra $U(\mathfrak{g})$ is given by

$$c \prod_{\alpha \in R^{+}} \prod_{t=1}^{\infty} (H + (\rho, \beta) \frac{(\beta, \beta)^{t}}{2})^{P^{\chi}(\alpha, \beta, t)}$$

for some $c \in \mathbb{C}^{\times}$. 
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References


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