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Families of characters of the imprimitive complex reflection groups

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ABSTRACT. The definition of Rouquier for the families of characters of Weyl groups in terms of blocks of the associated Iwahori-Hecke algebra has allowed the generalization of this notion to the case of complex reflection groups. In this paper, we will explain the combinatorics involved in the determination of the families of characters for the imprimitive complex reflection groups. We will also demonstrate the significant role played by the families of characters of the Weyl groups of type B.

1 Introduction

The work of G. Lusztig on the irreducible characters of reductive groups over finite fields (cf. [18]) has displayed the important role of the "families of characters" of the Weyl groups concerned. The Weyl groups are particular cases of complex reflection groups. For some complex reflection groups W, some data have been gathered which seem to indicate that behind the group W, there exists another mysterious object - the *Spets* (cf. [6], [21]) - that could play the role of the "series of finite reductive groups of Weyl group W". Therefore, it would be of great interest to generalize the notion of families of characters to the case of complex reflection groups.

Recent results of Gyoja [16] and Rouquier [24] have made possible the definition of a substitute for families of characters which can be applied to all complex reflection groups. In particular, Rouquier showed that the families of characters of a Weyl group W are exactly the blocks of characters of the Iwahori-Hecke algebra of W over a suitable coefficient ring, the "Rouquier ring". This definition generalizes without problem to all cyclotomic Hecke algebras of complex reflection groups. In [8], we showed that these "Rouquier blocks" of the cyclotomic Hecke algebras of a complex reflection group depend on a new numerical datum of the group, its "essential hyperplanes". Using this result, we were able to determine the families of characters for all exceptional irreducible complex reflection groups. Note that some particular cases had already been treated by Malle and Rouquier in [22].

In this paper, we will deal with the case of the groups of the infinite series, *i.e.*, the groups $G(de,e,r)$. In [4], Broué and Kim presented an algorithm for the determination of the Rouquier blocks of the cyclotomic Hecke algebras of the groups $G(d,1,r)$, *i.e.*, the *cyclotomic Ariki-Koike algebras*. However, it was recently realized that this algorithm works only when d is the power of a prime number. Using the theory of "essential hyperplanes", we will present here the correct algorithm for the determination of the Rouquier blocks of the cyclotomic Ariki-Koike algebras. The most important consequence of this algorithm is that we can obtain the Rouquier blocks of a cyclotomic Ariki-Koike algebra associated to $G(d, 1, r)$ from the families of characters of the Weyl groups of type B_n, $n \leq r$, already determined by Lusztig.

As far as the larger family $G(de, e, r)$ is concerned, we will explain how, in most of the cases (except for when $r = 2$ and e is even), we can obtain the Rouquier blocks of the associated cyclotomic Hecke algebras from the ones of $G(de, 1, r)$, as Kim did in [17]. The results of the determination of these blocks are thoroughly presented in [10].

2 Hecke algebras of complex reflection groups

2.1 Generic Hecke algebras

Let μ_∞ be the group of all the roots of unity in $\mathbb{C}$ and K a number field contained in $\mathbb{Q}(\mu_\infty)$. We denote by $\mu(K)$ the group of all the roots of unity of K. For every integer $d > 1$, we set $\zeta_d := \exp(2\pi i/d)$ and denote by μ_d the group of all the d-th roots of unity.

Let V be a K-vector space of finite dimension r. Let W be a finite subgroup of $\mathrm{GL}(V)$ generated by (pseudo-)reflections acting irreducibly on V. Let us denote by $\mathcal{A}$ the set of the reflecting hyperplanes of W and set $V^{\mathrm{reg}} := \mathbb{C} \otimes V - \bigcup_{H \in \mathcal{A}} \mathbb{C} \otimes H$. For $x_0 \in V^{\mathrm{reg}}$, let $B := \Pi_1(V^{\mathrm{reg}}/W, x_0)$ be the braid group associated to W (cf. [7], §2B).

For every orbit $\mathcal{C}$ of W on $\mathcal{A}$, we denote by $e_\mathcal{C}$ the common order of the subgroups W_H, where H is any element of $\mathcal{C}$ and W_H the subgroup of W formed by id_V and all the reflections fixing the hyperplane H.

We choose a set of indeterminates $\mathbf{u} = (u_{\mathcal{C},j})_{(\mathcal{C} \in \mathcal{A}/W)(0 \leq j \leq e_\mathcal{C}-1)}$ and we denote by $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ the Laurent polynomial ring in all the indeterminates $\mathbf{u}$. We define the *generic Hecke algebra* $\mathcal{H}(W)$ of W to be the quotient of the group algebra $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]B$ by the ideal generated by the elements of the form

$$(\mathbf{s} - u_{\mathcal{C},0})(\mathbf{s} - u_{\mathcal{C},1}) \ldots (\mathbf{s} - u_{\mathcal{C},e_\mathcal{C}-1}),$$

where $\mathcal{C}$ runs over the set $\mathcal{A}/W$ and $\mathbf{s}$ runs over the set of monodromy generators around the images in V^{reg}/W of the elements of the hyperplane orbit $\mathcal{C}$.

Example 2.1 Let $W := G_2 =< s, t \mid ststst = tststs, s^2 = t^2 = 1 >$ be the dihedral group of order 12. Then the generic Hecke algebra of W is defined over the Laurent polynomial ring in four indeterminates $\mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, w_0, w_0^{-1}, w_1, w_1^{-1}]$ and can be presented as follows:

$$\begin{aligned}\mathcal{H}(G_2) =< S, T \mid \quad STSTST = TSTSTS, \quad &(S - u_0)(S - u_1) = 0,\\ &(T - w_0)(T - w_1) = 0 > .\end{aligned}$$

From now on, we make the following assumptions for $\mathcal{H}(W)$, which have been verified for all but a finite number of irreducible complex reflection groups ([6], remarks before 1.17, § 2; [14]).

1. The algebra $\mathcal{H}(W)$ is a free $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$-module of rank $|W|$.
2. $\mathcal{H}(W)$ is endowed with a unique canonical symmetrizing form t.

Then we have the following result by G. Malle ([20], Theorem 5.2).

Theorem 2.2 *Let $\mathbf{v} = (v_{C,j})_{(C\in\mathcal{A}/W)(0\leq j\leq e_C-1)}$ be a set of indeterminates such that, for every C, j, we have*

$$v_{C,j}^{|\mu(K)|} = \zeta_{e_C}^{-j} u_{C,j}.$$

Then the $K(\mathbf{v})$-algebra $K(\mathbf{v})\mathcal{H}(W)$ is split semisimple.

By "Tits' deformation theorem" (cf., for example, [13], Theorem 7.4.6), it follows that the specialization $v_{C,j} \mapsto 1$ induces a bijection

$$\begin{array}{ccc} \mathrm{Irr}(K(\mathbf{v})\mathcal{H}(W)) & \leftrightarrow & \mathrm{Irr}(W) \\ \chi_{\mathbf{v}} & \mapsto & \chi. \end{array}$$

Moreover, we have that

$$t = \sum_{\chi\in\mathrm{Irr}(W)} \frac{1}{s_\chi}\chi_{\mathbf{v}},$$

where s_χ is the *Schur element* associated to $\chi_{\mathbf{v}} \in \mathrm{Irr}(K(\mathbf{v})\mathcal{H}(W))$. By [13], Proposition 7.3.9, we know that $s_\chi \in \mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$, where $\mathbb{Z}_K$ denotes the integral closure of $\mathbb{Z}$ in K.

The following result concerning the form of the Schur elements associated to the irreducible characters of $K(\mathbf{v})\mathcal{H}(W)$ is proved in [8], Theorem 3.2.5, using a case by case analysis.

Theorem 2.3 *The Schur element s_χ associated to the irreducible character $\chi_{\mathbf{v}}$ of $K(\mathbf{v})\mathcal{H}(W)$ is of the form*

$$s_\chi(\mathbf{v}) = \xi_\chi N_\chi \prod_{i\in I_\chi} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}}$$

where

- $\xi_\chi \in \mathbb{Z}_K$,
- $N_\chi = \prod_{C,j} v_{C,j}^{b_{C,j}}$ *is a monomial in* $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$ *such that* $\sum_{j=0}^{e_C-1} b_{C,j} = 0$ *for all* $C \in \mathcal{A}/W$,
- I_χ *is an index set,*

- $(\Psi_{\chi,i})_{i\in I_\chi}$ *is a family of K-cyclotomic polynomials in one variable (i.e., minimal polynomials of the roots of unity over K),*
- $(M_{\chi,i})_{i\in I_\chi}$ *is a family of monomials in* $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$ *such that if* $M_{\chi,i} = \prod_{C,j} v_{C,j}^{a_{C,j}}$, *then* $\gcd(a_{C,j}) = 1$ *and* $\sum_{j=0}^{e_C-1} a_{C,j} = 0$ *for all* $C \in \mathcal{A}/W$,
- $(n_{\chi,i})_{i\in I_\chi}$ *is a family of positive integers.*

The above factorization is unique in $K[\mathbf{v},\mathbf{v}^{-1}]$. *Moreover, the monomials* $(M_{\chi,i})_{i\in I_\chi}$ *are unique up to inversion.*

Example 2.4 Let $W := G_2$. We have seen that

$$\begin{aligned}\mathcal{H}(G_2) =< S,T \mid \quad STSTST = TSTSTS, \quad & (S-u_0)(S-u_1) = 0,\\ & (T-w_0)(T-w_1) = 0 > .\end{aligned}$$

Set $x_0^2 := u_0$, $x_1^2 := -u_1$, $y_0^2 := w_0$, $y_1^2 := -w_1$. By Theorem 2.2, the algebra $\mathbb{Q}(x_0,x_1,y_0,y_1)\mathcal{H}(G_2)$ is split semisimple and hence, there exists a bijection between its irreducible characters and the irreducible characters of G_2. The group G_2 has 4 irreducible characters of degree 1 and 2 irreducible characters of degree 2. Set

$$s_1(x_0,x_1,y_0,y_1) := \Phi_4(x_0x_1^{-1})\cdot\Phi_4(y_0y_1^{-1})\cdot\Phi_3(x_0x_1^{-1}y_0y_1^{-1})\cdot\Phi_6(x_0x_1^{-1}y_0y_1^{-1})$$

and

$$s_2(x_0,x_1,y_0,y_1) := 2x_0^{-2}x_1^2\cdot\Phi_3(x_0x_1^{-1}y_0y_1^{-1})\cdot\Phi_6(x_0x_1^{-1}y_0^{-1}y_1),$$

where

$$\Phi_3(x) = x^2+x+1, \Phi_4(x) = x^2+1, \Phi_6(x) = x^2-x+1.$$

The Schur elements of $\mathcal{H}(G_2)$ are

$$s_1(x_0,x_1,y_0,y_1), s_1(x_0,x_1,y_1,y_0), s_1(x_1,x_0,y_0,y_1), s_1(x_1,x_0,y_1,y_0),$$
$$s_2(x_0,x_1,y_0,y_1), s_2(x_0,x_1,y_1,y_0).$$

Due to the uniqueness (up to inversion) of the monomials appearing in the factorization of the Schur elements of $\mathcal{H}(W)$, we can define the *essential monomials* for W.

Definition 2.5 *Let* $\mathfrak{p}$ *be a prime ideal of* $\mathbb{Z}_K$ *and let* $M = \prod_{C,j} v_{C,j}^{a_{C,j}}$ *be a monomial in* $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$ *such that* $\gcd(a_{C,j}) = 1$. *We say that M is a $\mathfrak{p}$-essential monomial for W if there exists an irreducible character* $\chi \in \mathrm{Irr}(W)$ *and a K-cyclotomic polynomial Ψ such that*

- $\Psi(M)$ *is an irreducible factor of* $s_\chi(\mathbf{v})$.
- $\Psi(1) \in \mathfrak{p}$.

We say that M is an essential monomial for W, if there exists a prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ such that M is $\mathfrak{p}$-essential for W.

Example 2.6 Since $\Phi_3(1) = 3$, $\Phi_4(1) = 2$ and $\Phi_6(1) = 1$, the description of the Schur elements of $\mathcal{H}(G_2)$ in Example 2.4 implies that

- the $2\mathbb{Z}$-essential monomials for G_2 are $x_0x_1^{-1}$ and $y_0y_1^{-1}$ (and their inverses),
- the $3\mathbb{Z}$-essential monomials for G_2 are $x_0x_1^{-1}y_0y_1^{-1}$ and $x_0x_1^{-1}y_0^{-1}y_1$ (and their inverses).

2.2 Cyclotomic Hecke algebras

Let y be an indeterminate. We set $q := y^{|\mu(K)|}$.

Definition 2.7 *A cyclotomic specialization of $\mathcal{H}(W)$ is a $\mathbb{Z}_K$-algebra morphism $\phi : \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}] \to \mathbb{Z}_K[y, y^{-1}]$ with the following properties:*

- *$\phi : v_{C,j} \mapsto y^{n_{C,j}}$, where $n_{C,j} \in \mathbb{Z}$ for all C and j.*
- *For all $C \in \mathcal{A}/W$, if z is another indeterminate, then the element of $\mathbb{Z}_K[y, y^{-1}, z]$ defined by*

$$\Gamma_C(y, z) := \prod_{j=0}^{e_C - 1} (z - \zeta_{e_C}^j y^{n_{C,j}})$$

is invariant by the action of $\mathrm{Gal}(K(y)/K(q))$.

We can also write $\phi : u_{C,j} \mapsto \zeta_{e_C}^j q^{n_{C,j}}$.

If ϕ is a cyclotomic specialization of $\mathcal{H}(W)$, the corresponding *cyclotomic Hecke algebra* is the $\mathbb{Z}_K[y, y^{-1}]$-algebra, denoted by $\mathcal{H}_\phi$, which is obtained as the specialization of the $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$-algebra $\mathcal{H}(W)$ via the morphism ϕ. It also has a symmetrizing form t_ϕ defined as the specialization of the canonical form t.

Example 2.8 The *spetsial* Hecke algebra $\mathcal{H}_q^s(W)$ is the cyclotomic algebra obtained via the specialization

$$u_{C,0} \mapsto q, \; u_{C,j} \mapsto \zeta_{e_C}^j \text{ for } 1 \leq j \leq e_C - 1, \text{ for all } C \in \mathcal{A}/W.$$

For example, if $W := G_2$, then

$$\mathcal{H}_q^s(G_2) =< S, T \mid STSTST = TSTSTS, (S-q)(S+1) = (T-q)(T+1) = 0 > .$$

The following result is proved in [8] (remarks following Theorem 3.3.3):

Proposition 2.9 *The algebra $K(y)\mathcal{H}_\phi$ is split semisimple.*

When y specializes to 1, the algebra $K(y)\mathcal{H}_\phi$ specializes to the group algebra KW. Thus, by "Tits' deformation theorem", the specialization $v_{C,j} \mapsto 1$ defines the following bijections

$$\begin{array}{ccccc} \mathrm{Irr}(K(\mathbf{v})\mathcal{H}(W)) & \leftrightarrow & \mathrm{Irr}(K(y)\mathcal{H}_\phi) & \leftrightarrow & \mathrm{Irr}(W) \\ \chi_{\mathbf{v}} & \mapsto & \chi_\phi & \mapsto & \chi. \end{array}$$

The following result is an immediate consequence of Theorem 2.3.

Proposition 2.10 *The Schur element $s_{\chi_\phi}(y)$ associated to the irreducible character χ_ϕ of $K(y)\mathcal{H}_\phi$ is a Laurent polynomial in y of the form*

$$s_{\chi_\phi}(y) = \psi_{\chi,\phi} y^{a_{\chi,\phi}} \prod_{\Phi \in C_K} \Phi(y)^{n_{\chi,\phi,\Phi}}$$

where $\psi_{\chi,\phi} \in \mathbb{Z}_K$, $a_{\chi,\phi} \in \mathbb{Z}$, $n_{\chi,\phi,\Phi} \in \mathbb{N}$ and C_K is a set of K-cyclotomic polynomials.

2.3 Rouquier blocks of the cyclotomic Hecke algebras

Definition 2.11 *We call Rouquier ring of K and denote by $\mathcal{R}_K(y)$ the $\mathbb{Z}_K$-subalgebra of $K(y)$*

$$\mathcal{R}_K(y) := \mathbb{Z}_K[y, y^{-1}, (y^n - 1)^{-1}_{n\geq 1}]$$

Let $\phi : v_{C,j} \mapsto y^{n_{C,j}}$ be a cyclotomic specialization of $\mathcal{H}(W)$ and $\mathcal{H}_\phi$ the corresponding cyclotomic Hecke algebra. Set $\mathcal{O} := \mathbb{Z}_K[y, y^{-1}]$

Definition 2.12 *The Rouquier blocks of $\mathcal{H}_\phi$ are the blocks of the algebra $\mathcal{R}_K(y)\mathcal{H}_\phi := \mathcal{R}_K(y) \otimes_{\mathcal{O}} \mathcal{H}_\phi$, i.e., the partition $\mathcal{RB}(\mathcal{H}_\phi)$ of $\mathrm{Irr}(W)$ minimal for the property:*

$$\text{for all } B \in \mathcal{RB}(\mathcal{H}_\phi) \text{ and } h \in \mathcal{H}_\phi, \sum_{\chi \in B} \frac{\chi_\phi(h)}{s_{\chi_\phi}} \in \mathcal{R}_K(y).$$

It has been shown by Rouquier ([24]), that if W is a Weyl group and $\mathcal{H}_\phi$ is obtained via the "spetsial" cyclotomic specialization (see Example 2.8), then the Rouquier blocks of $\mathcal{H}_\phi$ coincide with the "families of characters" defined by Lusztig. This definition generalizes without problem to all cyclotomic Hecke algebras of complex reflection groups. Thus, the Rouquier blocks play an essential role in the program "Spets" (cf. [6]) whose ambition is to give to complex reflection groups the role of Weyl groups of as yet mysterious structures.

The Rouquier ring is a Dedekind ring (cf., for example, [8], Proposition 3.4.2). The following result is an immediate consequence of an elementary result on blocks and the form of the Schur elements of $\mathcal{H}_\phi$.

Proposition 2.13 *The characters $\chi, \psi \in \mathrm{Irr}(W)$ belong to the same Rouquier block of $\mathcal{H}_\phi$ if and only if there exist a finite sequence of irreducible characters $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of $\mathbb{Z}_K$ such that*

- $\chi_0 = \chi$ *and* $\chi_n = \psi$,
- $\forall i$ $(1 \leq i \leq n)$, χ_{i-1} *and* χ_i *belong to the same block of* $\mathcal{O}_{\mathfrak{p}_i\mathcal{O}}\mathcal{H}_\phi$.

Thanks to the above result, we have transferred the problem of the determination of the Rouquier blocks of $\mathcal{H}_\phi$ to that of the determination of the "$\mathfrak{p}$-blocks" of $\mathcal{H}_\phi$ (*i.e.*, the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$), where $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}_K$. Note that $\mathcal{O}_{\mathfrak{p}\mathcal{O}} \cong \mathcal{R}_K(y)_{\mathfrak{p}\mathcal{R}_K(y)}$ is a discrete valuation ring and thus, the $\mathfrak{p}$-blocks of $\mathcal{H}_\phi$ are in bijection with the blocks of $\mathbb{F}_\mathfrak{p}(y)\mathcal{H}_\phi$, where $\mathbb{F}_\mathfrak{p}$ denotes the finite field $\mathbb{Z}_K/\mathfrak{p}$.

Now, set $m := \sum_{C \in \mathcal{A}/W} e_C$. If $M = \prod_{C,j} v_{C,j}^{a_{C,j}}$ is a $\mathfrak{p}$-essential monomial for W, then the hyperplane defined in $\mathbb{C}^m$ by the relation

$$\sum_{C,j} a_{C,j} t_{C,j} = 0,$$

where $(t_{C,j})_{C,j}$ is a set of m indeterminates, is called $\mathfrak{p}$-*essential hyperplane* for W. A hyperplane in $\mathbb{C}^m$ is called simply *essential* for W, if it is $\mathfrak{p}$-essential for some prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$.

- If the integers $n_{C,j}$ belong to no $\mathfrak{p}$-essential hyperplane (resp. no essential hyperplane) for W, then the $\mathfrak{p}$-blocks (resp. Rouquier blocks) of $\mathcal{H}_\phi$ are called $\mathfrak{p}$-*blocks associated with no essential hyperplane* (resp. *Rouquier blocks associated with no essential hyperplane*). They do not depend on the values of the $n_{C,j}$.
- If the integers $n_{C,j}$ belong to exactly one $\mathfrak{p}$-essential hyperplane H (resp. exactly one essential hyperplane H) for W, then the $\mathfrak{p}$-blocks (resp. Rouquier blocks) of $\mathcal{H}_\phi$ are called $\mathfrak{p}$-*blocks associated with the essential hyperplane H* (resp. *Rouquier blocks associated with the essential hyperplane H*). They do not depend on the values of the $n_{C,j}$.

The following result (cf. [8], Chapter 3) establishes the connection between the $\mathfrak{p}$-essential hyperplanes for W and the $\mathfrak{p}$-blocks of $\mathcal{H}_\phi$.

Theorem 2.14 *Let $\phi : v_{C,j} \mapsto y^{n_{C,j}}$ be a cyclotomic specialization and $\mathcal{H}_\phi$ the corresponding cyclotomic Hecke algebra. Let $\mathcal{E}_\mathfrak{p}$ be the set of all $\mathfrak{p}$-essential hyperplanes for W that the integers $n_{C,j}$ belong to. If $\mathcal{E}_\mathfrak{p} = \emptyset$, then the $\mathfrak{p}$-blocks of $\mathcal{H}_\phi$ are the $\mathfrak{p}$-blocks associated with no essential hyperplane. If $\mathcal{E}_\mathfrak{p} \neq \emptyset$, then two irreducible characters $\chi, \psi \in \mathrm{Irr}(W)$ belong to the same $\mathfrak{p}$-block of $\mathcal{H}_\phi$ if and only if there exist a finite sequence of irreducible characters $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence of $\mathfrak{p}$-essential hyperplanes $H_1, \ldots, H_n \in \mathcal{E}_\mathfrak{p}$ such that*

- *$\chi_0 = \chi$ and $\chi_n = \psi$,*
- *$\forall i$ $(1 \leq i \leq n)$, χ_{i-1} and χ_i belong to the same $\mathfrak{p}$-block associated with H_i.*

Thanks to Proposition 2.13 and Theorem 2.14, we obtain the connection between the essential hyperplanes for W and the Rouquier blocks of $\mathcal{H}_\phi$.

Corollary 2.15 *Let $\phi : v_{C,j} \mapsto y^{n_{C,j}}$ be a cyclotomic specialization and $\mathcal{H}_\phi$ the corresponding cyclotomic Hecke algebra. Let $\mathcal{E}$ be the set of all essential hyperplanes for W that the integers $n_{C,j}$ belong to. If $\mathcal{E} = \emptyset$, then the Rouquier blocks of $\mathcal{H}_\phi$ are the Rouquier blocks associated with no essential* *hyperplane. If $\mathcal{E} \neq \emptyset$, then two irreducible characters $\chi, \psi \in \mathrm{Irr}(W)$ belong to the same Rouquier block of $\mathcal{H}_\phi$ if and only if there exist a finite sequence of irreducible characters $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence of essential hyperplanes $H_1, \ldots, H_n \in \mathcal{E}$ such that*

- $\chi_0 = \chi$ *and* $\chi_n = \psi$,
- $\forall i$ $(1 \leq i \leq n)$, χ_{i-1} *and* χ_i *belong to the same Rouquier block associated with* H_i.

Thanks to the above results, in order to determine the Rouquier blocks of any cyclotomic Hecke algebra associated to a complex reflection group W, it suffices to determine the $\mathfrak{p}$-blocks, and thus the Rouquier blocks, associated with no and each essential hyperplane for W.

3 Families of characters of $G(d,1,r)$

The group $G(d,1,r)$ is the group of all $r \times r$ monomial matrices whose non-zero entries lie in μ_d. It is isomorphic to the wreath product $\mu_d \wr \mathfrak{S}_r$ and its field of definition (the field K of the previous section) is $\mathbb{Q}(\zeta_d)$. In particular, we have

- $G(1,1,r) \simeq A_{r-1}$ for $r \geq 2$,
- $G(2,1,r) \simeq B_r$ for $r \geq 2$ $\;(G(2,1,1) \simeq \mu_2)$.

We will start by introducing some combinatorial objects which will be necessary for the description of the Rouquier blocks of the cyclotomic Ariki-Koike algebras, *i.e.*, the cyclotomic Hecke algebras associated to the group $G(d,1,r)$.

3.1 Combinatorics

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ be a *partition*, *i.e.*, a finite decreasing sequence of positive integers:

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_h \geq 1.$$

The integer $|\lambda| := \lambda_1 + \lambda_2 + \ldots + \lambda_h$ is called *the size of* λ. We also say that λ *is a partition of* $|\lambda|$. The integer h is called *the height of* λ and we set $h_\lambda := h$. To each partition λ we associate its *β-number*, $\beta_\lambda = (\beta_1, \beta_2, \ldots, \beta_h)$, defined by

$$\beta_1 := h + \lambda_1 - 1, \beta_2 := h + \lambda_2 - 2, \ldots, \beta_h := h + \lambda_h - h.$$

Example 3.1 If $\lambda = (4,2,2,1)$, then $\beta_\lambda = (7,4,3,1)$.

Let $m \in \mathbb{N}$. The *m-shifted β-number* of λ is the sequence of numbers defined by

$$\beta_\lambda[m] = (\beta_1 + m, \beta_2 + m, \ldots, \beta_h + m, m-1, m-2, \ldots, 1, 0).$$

Example 3.2 If $\lambda = (4,2,2,1)$, then $\beta_\lambda[3] = (10,7,6,4,2,1,0)$.

Let d be a positive integer. A family of d partitions $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)})$ is called a *d-partition*. We set

$$h^{(a)} := h_{\lambda^{(a)}}, \quad \beta^{(a)} := \beta_{\lambda^{(a)}}$$

and we have

$$\lambda^{(a)} = (\lambda_1^{(a)}, \lambda_2^{(a)}, \ldots, \lambda_{h^{(a)}}^{(a)}).$$

The integer $|\lambda| := |\lambda^{(0)}| + |\lambda^{(1)}| + \ldots + |\lambda^{(d-1)}|$ is called *the size of* λ. We also say that λ *is a d-partition of* $|\lambda|$.

Now, let us suppose that we have a given "weight system", *i.e.*, a family of integers

$$m := (m^{(0)}, m^{(1)}, \ldots, m^{(d-1)}).$$

We call (d,m)*-charged height of* λ the family $(hc^{(0)}, hc^{(1)}, \ldots, hc^{(d-1)})$, where

$$hc^{(0)} := h^{(0)} - m^{(0)}, hc^{(1)} := h^{(1)} - m^{(1)}, \ldots, hc^{(d-1)} := h^{(d-1)} - m^{(d-1)}.$$

We define the *m-charged height of* λ to be the integer

$$hc_\lambda := \max\{hc^{(a)} \mid 0 \le a \le d-1\}.$$

Definition 3.3 *The m-charged standard symbol of* λ *is the family of numbers defined by*

$$Bc_\lambda = (Bc_\lambda^{(0)}, Bc_\lambda^{(1)}, \ldots, Bc_\lambda^{(d-1)}),$$

where, for all a $(0 \le a \le d-1)$, *we have*

$$Bc_\lambda^{(a)} := \beta^{(a)}[hc_\lambda - hc^{(a)}].$$

The m-charged content of λ *is the multiset*

$$\mathrm{Contc}_\lambda = Bc_\lambda^{(0)} \cup Bc_\lambda^{(1)} \cup \ldots \cup Bc_\lambda^{(d-1)}.$$

Example 3.4 Let us take $d = 2$, $\lambda = ((2,1),(3))$ and $m = (-1,2)$. Then

$$Bc_\lambda = \begin{pmatrix} 3 & 1 & & & \\ 7 & 3 & 2 & 1 & 0 \end{pmatrix}$$

We have $\mathrm{Contc}_\lambda = \{0,1,1,2,3,3,7\}$.

Remark: If $m_0 = m_1 = \ldots = m_{d-1} = 0$, then hc_λ is called the *height of* λ and $B_\lambda := Bc_\lambda$ is the *ordinary standard symbol of* λ.

3.2 Ariki-Koike algebras

The *generic Ariki-Koike algebra* associated to $G(d,1,r)$ (cf. [3], [5]) is the algebra $\mathcal{H}_{d,r}$ generated over the Laurent ring of polynomials in $d+1$ indeterminates

$$\mathcal{O}_d := \mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, \ldots, u_{d-1}, u_{d-1}^{-1}, x, x^{-1}]$$

by the elements $\mathbf{s}, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{r-1}$ satisfying the relations

- $\mathbf{s}\mathbf{t}_1\mathbf{s}\mathbf{t}_1 = \mathbf{t}_1\mathbf{s}\mathbf{t}_1\mathbf{s}$, $\mathbf{s}\mathbf{t}_j = \mathbf{t}_j\mathbf{s}$ for $j \neq 1$,
- $\mathbf{t}_j\mathbf{t}_{j+1}\mathbf{t}_j = \mathbf{t}_{j+1}\mathbf{t}_j\mathbf{t}_{j+1}$, $\mathbf{t}_i\mathbf{t}_j = \mathbf{t}_j\mathbf{t}_i$ for $|i-j|>1$,
- $(\mathbf{s}-u_0)(\mathbf{s}-u_1)\ldots(\mathbf{s}-u_{d-1}) = (\mathbf{t}_j - x)(\mathbf{t}_j+1) = 0$.

For every d-partition $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)})$ of r, we consider the free $\mathcal{O}_d$-module which has as basis the family of standard tableaux of λ. We can give to this module the structure of a $\mathcal{H}_{d,r}$-module (cf. [3], [1], [15]) and thus obtain the *Specht module* $\mathbf{Sp}^\lambda$ associated to λ.

Set $\mathcal{K}_d := \mathbb{Q}(u_0, u_1, \ldots, u_{d-1}, x)$ the field of fractions of $\mathcal{O}_d$. The $\mathcal{K}_d\mathcal{H}_{d,r}$-module $\mathcal{K}_d\mathbf{Sp}^\lambda$, obtained by extension of scalars, is absolutely irreducible and every irreducible $\mathcal{K}_d\mathcal{H}_{d,r}$-module is isomorphic to a module of this type. Thus $\mathcal{K}_d$ is a splitting field for $\mathcal{H}_{d,r}$. We denote by χ_λ the (absolutely) irreducible character of the $\mathcal{K}_d\mathcal{H}_{d,r}$-module $\mathbf{Sp}^\lambda$.

Since the algebra $\mathcal{K}_d\mathcal{H}_{d,r}$ is split semisimple, the Schur elements of its irreducible characters belong to $\mathcal{O}_d$. They have been calculated independently by Geck, Iancu, Malle in [14] and by Mathas in [23].

Theorem 3.5 *Let λ be a d-partition of r with ordinary standard symbol $B_\lambda = (B_\lambda^{(0)}, B_\lambda^{(1)}, \ldots, B_\lambda^{(d-1)})$. We set $B_\lambda^{(s)} = (b_1^{(s)}, b_2^{(s)}, \ldots, b_h^{(s)})$, where h is the height of λ. Let $a := r(d-1) + \binom{d}{2}\binom{h}{2}$ and $b := dh(h-1)(2dh-d-3)/12$. Then the Schur element of the irreducible character χ_λ is given by the formulae $s_\lambda = (-1)^a x^b (x-1)^{-r}(u_0u_1\ldots u_{d-1})^{-r}\nu_\lambda/\delta_\lambda$, where*

$$\nu_\lambda = \prod_{0\le s<t<d}(u_s-u_t)^h \prod_{0\le s,t<d}\ \prod_{b_s\in B_\lambda^{(s)}}\ \prod_{1\le k\le b_s}(x^k u_s - u_t),$$

$$\delta_\lambda = \prod_{0\le s<t<d}\ \prod_{(b_s,b_t)\in B_\lambda^{(s)}\times B_\lambda^{(t)}}(x^{b_s}u_s - x^{b_t}u_t)\prod_{0\le s<d}\ \prod_{1\le i<j\le h}(x^{b_i^{(s)}}u_s - x^{b_j^{(s)}}u_s).$$

Now let

$$\phi : \begin{cases} u_j \mapsto \zeta_d^j q^{m_j}, (0\le j<d), \\ x \mapsto q^n \end{cases}$$

be a cyclotomic specialization of $\mathcal{H}_{d,r}$. Following the description of the Schur elements of $\mathcal{H}_{d,r}$, we deduce (cf. [9]) that the essential hyperplanes for $G(d,1,r)$ are:

- $N = 0$,
- $kN + M_s - M_t = 0$ for all $-r < k < r$ and $0 \leq s < t < d$ such that $\zeta_d^s - \zeta_d^t$ is not a unit in $\mathbb{Z}[\zeta_d]$.

3.3 Residues of multipartitions

Due to Proposition 2.13, the Rouquier blocks of a cyclotomic Hecke algebra can be determined by its $\mathfrak{p}$-blocks, where $\mathfrak{p}$ runs over the set of prime ideals of $\mathbb{Z}_K$. The algorithm of Lyle and Mathas for the blocks of the Ariki-Koike algebras over any field ([19]) provides us with a characterization of the $\mathfrak{p}$-blocks of $\mathcal{H}_{d,r}$, which will be used for the determination of the Rouquier blocks associated with the essential hyperplanes for $G(d,1,r)$.

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ lying over a prime number p. We set

$$[\lambda] := \{(i,j,a) \,|\, (0 \leq a \leq d-1)(1 \leq i \leq h^{(a)})(1 \leq j \leq \lambda_i^{(a)})\}.$$

A *node* is any ordered triple $(i,j,a) \in [\lambda]$. If

$$\phi : \begin{cases} u_j \mapsto \zeta_d^j q^{m_j}, (0 \leq j < d), \\ x \mapsto q^n \end{cases}$$

is a cyclotomic specialization of $\mathcal{H}_{d,r}$, then the $\mathfrak{p}$-*residue* of the node $x = (i,j,a)$ with respect to ϕ is

$$\mathrm{res}_{\mathfrak{p},\phi}(x) = \begin{cases} \phi(u_a x^{(j-i)}) \bmod \mathfrak{p} & \text{if } n \neq 0, \\ ((j-i) \bmod p, \phi(u_a) \bmod \mathfrak{p}) & \text{if } n = 0 \text{ and } \phi(u_b) \not\equiv \phi(u_a) \bmod \mathfrak{p} \text{ for } b \neq a, \\ \phi(u_a) \bmod \mathfrak{p} & \text{otherwise.} \end{cases}$$

Let $\mathrm{Res}_{\mathfrak{p},\phi} := \{\mathrm{res}_{\mathfrak{p},\phi}(x) \,|\, x \in [\lambda] \text{ for some } d\text{-partition } \lambda \text{ of } r\}$ be the set of all possible residues. For any d-partition λ of r and $f \in \mathrm{Res}_{\mathfrak{p},\phi}$, we define

$$C_f(\lambda) = \#\{x \in [\lambda] \,|\, \mathrm{res}(x) = f\}.$$

We say that the d-partitions λ and μ of r are $\mathfrak{p}$-*residue equivalent* with respect to ϕ if $C_f(\lambda) = C_f(\mu)$ for all $f \in \mathrm{Res}_{\mathfrak{p},\phi}$. The following result is an immediate consequence of [19], Theorem 2.11.

Proposition 3.6 *Let λ and μ be two d-partitions of r. The irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same $\mathfrak{p}$-block of $(\mathcal{H}_{d,r})_\phi$ if and only if λ and μ are $\mathfrak{p}$-residue equivalent.*

3.4 Rouquier blocks of the cyclotomic Ariki-Koike algebras

Theorem 3.13 of [4] gives a description of the Rouquier blocks of the cyclotomic Ariki-Koike algebras in terms of charged contents of multipartitions. However, in its proof, it is supposed that $1 - \zeta_d$ always belongs to a prime ideal of $\mathbb{Z}[\zeta_d]$. This is not correct, unless d is the power of a prime number. Therefore, we will state here the part of the theorem that is correct.

Theorem 3.7 *Let ϕ be a cyclotomic specialization such that $\phi(x) = q$. If two irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$, then* $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$ *with respect to the weight system $m = (m_0, m_1, \ldots, m_{d-1})$. The converse holds when d is the power of a prime number.*

Thanks to Corollary 2.15, in order to obtain the Rouquier blocks of any cyclotomic Ariki-Koike algebra, it suffices to calculate the Rouquier blocks associated with no and each essential hyperplane for $G(d,1,r)$. If

$$\phi : \begin{cases} u_j \mapsto \zeta_d^j q^{m_j}, (0 \leq j < d), \\ x \mapsto q^n \end{cases}$$

is a cyclotomic specialization such that the $(m_0, m_1, \ldots, m_{d-1}, n)$ do not belong to any essential hyperplane for $\mathcal{H}_{d,r}$, then all the Schur elements of $(\mathcal{H}_{d,r})_\phi$ are invertible in the Rouquier ring. Thus, we obtain that:

Proposition 3.8 *The Rouquier blocks associated with no essential hyperplane for $G(d,1,r)$ are trivial.*

The two results that follow are proved in detail in [9]. Here we will only give some idea of their proofs.

Proposition 3.9 *Let λ, μ be two d-partitions of r. The characters χ_λ and χ_μ are in the same Rouquier block associated with the essential hyperplane $N = 0$ if and only if $|\lambda^{(a)}| = |\mu^{(a)}|$ for all $a = 0, 1, \ldots, d-1$.*

Proof: Let

$$\phi : \begin{cases} u_j \mapsto \zeta_d^j q^{m_j}, (0 \leq j < d), \\ x \mapsto 1 \end{cases}$$

be a cyclotomic specialization such that $m_s \neq m_t$ for all $0 \leq s < t < d$. The Rouquier blocks of $(\mathcal{H}_{d,r})_\phi$ are the Rouquier blocks associated with the essential hyperplane $N = 0$.

Suppose first that $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. Due to Proposition 2.13, we may assume that there exists a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_d]$ such that $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same $\mathfrak{p}$-block of $(\mathcal{H}_{d,r})_\phi$. Since the m_a $(0 \leq a < d)$ can take any value, Proposition 3.6 yields

$$\begin{aligned} |\lambda^{(a)}| = \; & \#\{(i,j,a) \mid (1 \leq i \leq h_\lambda^{(a)})(1 \leq j \leq \lambda_i^{(a)})\} & = \\ = \; & \#\{(i,j,a) \mid (1 \leq i \leq h_\mu^{(a)})(1 \leq j \leq \mu_i^{(a)})\} & = |\mu^{(a)}| \end{aligned}$$

for all $a = 0, 1, \ldots, d-1$.

Now, let $a \in \{0, 1, \ldots, d-1\}$. It is enough to show that if λ and μ are two d-partitions of r such that

$$|\lambda^{(a)}| = |\mu^{(a)}| \text{ and } \lambda^{(b)} = \mu^{(b)} \text{ for all } b \neq a,$$

then $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. Set $l := |\lambda^{(a)}| = |\mu^{(a)}|$. The generic Ariki-Koike algebra of the symmetric group $\mathfrak{S}_l$ specializes to the group algebra $\mathbb{Z}[\mathfrak{S}_l]$ when x specializes to 1 . It is well-known that all irreducible characters of $\mathfrak{S}_l$ belong to the same Rouquier block of $\mathbb{Z}[\mathfrak{S}_l]$ (see also [24], §3, Rem.1). Due to Proposition 2.13, we may assume, without loss of generality, that $\chi_{\lambda^{(a)}}$ and $\chi_{\mu^{(a)}}$ belong to the same p-block of $\mathfrak{S}_l$ for some prime number p. Hence, by Proposition 3.6, $\lambda^{(a)}$ and $\mu^{(a)}$ are $p\mathbb{Z}$-residue equivalent. If $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}[\zeta_d]$ lying over p, then, by definition of the $\mathfrak{p}$-residue, λ and μ are $\mathfrak{p}$-residue equivalent, and thus, $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. ■

Finally, let H be an essential hyperplane for $G(d,1,r)$ of the form $kN + M_s - M_t = 0$ and let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}[\zeta_d]$ such that $\zeta_d^s - \zeta_d^t \in \mathfrak{p}$. Then H is a $\mathfrak{p}$-essential hyperplane for $G(d,1,r)$. Let

$$\phi_H : \left\{ \begin{array}{l} u_j \mapsto \zeta_d^j q^{m_j}, (0 \leq j < d), \\ x \mapsto q^n \end{array} \right.$$

be a cyclotomic specialization such that $kn + m_s - m_t = 0$ and the integers $(m_0, m_1, \ldots, m_{d-1}, n)$ belong to no other essential hyperplane for $G(d,1,r)$. The Rouquier blocks of $(\mathcal{H}_{d,r})_{\phi_H}$ are the Rouquier blocks associated with the hyperplane H. Our following result gives their description.

Proposition 3.10 *Let λ, μ be two distinct d-partitions of r. The irreducible characters $(\chi_\lambda)_{\phi_H}$ and $(\chi_\mu)_{\phi_H}$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_{\phi_H}$ if and only if the following conditions are satisfied:*

1. *We have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s,t\}$.*
2. *If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$ and $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$, then $\mathrm{Contc}_{\lambda^{st}} = \mathrm{Contc}_{\mu^{st}}$ with respect to the weight system $(0,k)$.*

Proof: We can assume, without loss of generality, that $n = 1$. We can also assume that $m_s = 0$ and $m_t = k$.

Suppose that $(\chi_\lambda)_{\phi_H}$ and $(\chi_\mu)_{\phi_H}$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_{\phi_H}$. Due to Theorem 3.7, we have $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$ with respect to the weight system $m = (m_0, m_1, \ldots, m_{d-1})$. Since the m_a, $a \notin \{s,t\}$ can take any value (as long as they don't belong to another essential hyperplane), the equality $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$ yields conditions 1 and 2.

Now let us suppose that the conditions 1 and 2 are satisfied. Set $l := |\lambda^{st}|$. Due to the first condition, we must have $|\mu^{st}| = l$. Let $\mathcal{H}_{2,l}$ be the generic Ariki-Koike algebra associated to the group $G(2,1,l)$ defined over the Laurent polynomial ring

$$\mathbb{Z}[U_0, U_0^{-1}, U_1, U_1^{-1}, X, X^{-1}].$$

Let us consider the cyclotomic specialization

$$\vartheta : U_0 \mapsto 1, U_1 \mapsto -q^k, X \mapsto q.$$

Due to Theorem 3.7, the condition 2 implies that the characters $(\chi_{\lambda^{st}})_\vartheta$ and $(\chi_{\mu^{st}})_\vartheta$ belong to the same Rouquier block of $(\mathcal{H}_{2,l})_\vartheta$. Therefore, we must have that $kN + M_0 - M_1 = 0$ is a $2\mathbb{Z}$-essential hyperplane for $G(2,1,l)$ and that $(\chi_{\mu^{st}})_\vartheta$ and $(\chi_{\lambda^{st}})_\vartheta$ belong to the same $2\mathbb{Z}$-block of $(\mathcal{H}_{2,l})_\vartheta$. By Proposition 3.6, λ^{st} and μ^{st} are $2\mathbb{Z}$-residue equivalent. Following the definition of the $\mathfrak{p}$-residue, we deduce that $(\chi_\lambda)_{\phi_H}$ and $(\chi_\mu)_{\phi_H}$ belong to the same $\mathfrak{p}$-block and hence to the same Rouquier block of $(\mathcal{H}_{d,r})_{\phi_H}$. ■

The following result is a corollary of the above proposition. However, it can also be obtained independently using the Morita equivalences established by Theorem 1.1 of [12], according to which the algebra $(\mathcal{H}_{d,r})_{\phi_H}$ defined over the Rouquier ring is Morita equivalent to the algebra

$$\bigoplus_{\substack{n_1,\dots,n_{d-1}\geq 0 \\ n_1+\dots+n_{d-1}=r}} (\mathcal{H}_{2,n_1})_{\phi_H} \otimes \mathcal{H}(\mathfrak{S}_{n_2})_{\phi_H} \otimes \dots \otimes \mathcal{H}(\mathfrak{S}_{n_{d-1}})_{\phi_H}.$$

Corollary 3.11 *Let λ, μ be two distinct d-partitions of r. The irreducible characters $(\chi_\lambda)_{\phi_H}$ and $(\chi_\mu)_{\phi_H}$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_{\phi_H}$ if and only if the following conditions are satisfied:*

1. *We have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s,t\}$.*
2. *If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$, $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$ and $l := |\lambda^{st}| = |\mu^{st}|$, then the characters $(\chi_{\lambda^{st}})_\vartheta$ and $(\chi_{\mu^{st}})_\vartheta$ belong to the same Rouquier block of the cyclotomic Ariki-Koike algebra of $G(2,1,l)$ obtained via the specialization*
$$\vartheta : U_0 \mapsto q^{m_s}, U_1 \mapsto -q^{m_t}, X \mapsto q^n.$$

Example 3.12 Let $W := G(3,1,2)$. The irreducible characters of W are parametrized by the 3-partitions of 2. These are:

$$\begin{array}{lll}
\lambda_{(2),0} = ((2),\emptyset,\emptyset), & \lambda_{(2),1} = (\emptyset,(2),\emptyset), & \lambda_{(2),2} = (\emptyset,\emptyset,(2)), \\
\lambda_{(1,1),0} = ((1,1),\emptyset,\emptyset), & \lambda_{(1,1),1} = (\emptyset,(1,1),\emptyset), & \lambda_{(1,1),2} = (\emptyset,\emptyset,(1,1)), \\
\lambda_{\emptyset,0} = (\emptyset,(1),(1)), & \lambda_{\emptyset,1} = ((1),\emptyset,(1)), & \lambda_{\emptyset,2} = ((1),(1),\emptyset).
\end{array}$$

The generic Ariki-Koike algebra associated to W is the algebra $\mathcal{H}_{3,2}$ generated over the Laurent polynomial ring in 4 indeterminates

$$\mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, u_2, u_2^{-1}, x, x^{-1}]$$

by the elements $\mathbf{s}$ and $\mathbf{t}$ satisfying the relations

- $\mathbf{stst} = \mathbf{tsts}$,
- $(\mathbf{s} - u_0)(\mathbf{s} - u_1)(\mathbf{s} - u_2) = (\mathbf{t} - x)(\mathbf{t} + 1) = 0.$

Let

$$\phi : \begin{cases} u_j \mapsto \zeta_3^j q^{m_j}, (0 \le j \le 2), \\ x \mapsto q^n \end{cases}$$

be a cyclotomic specialization for $\mathcal{H}_{3,2}$. The essential hyperplanes for W are:

- $N = 0$.
- $kN + M_0 - M_1 = 0$ for $k \in \{-1, 0, 1\}$.
- $kN + M_0 - M_2 = 0$ for $k \in \{-1, 0, 1\}$.
- $kN + M_1 - M_2 = 0$ for $k \in \{-1, 0, 1\}$.

Let us take $m_0 := 0$, $m_1 := 0$, $m_2 := 5$ and $n := 1$. These integers belong only to the essential hyperplane $M_0 - M_1 = 0$. Following Proposition 3.10, two irreducible characters $(\chi_\lambda)_\phi$, $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{2,3})_\phi$ if and only if

1. We have $\lambda^{(2)} = \mu^{(2)}$.
2. If $\lambda^{01} := (\lambda^{(0)}, \lambda^{(1)})$ and $\mu^{01} := (\mu^{(0)}, \mu^{(1)})$, then $\mathrm{Contc}_{\lambda^{01}} = \mathrm{Contc}_{\mu^{01}}$ with respect to the weight system $(0, 0)$.

The first condition yields that the irreducible characters corresponding to the partitions $\lambda_{(2),2}$ and $\lambda_{(1,1),2}$ are singletons. Moreover, we have

$$B_{\lambda^{01}_{(2),0}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \; B_{\lambda^{01}_{(2),1}} = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

$$B_{\lambda^{01}_{(1,1),0}} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \; B_{\lambda^{01}_{(1,1),1}} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

$$B_{\lambda^{01}_{\emptyset,0}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; B_{\lambda^{01}_{\emptyset,1}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; B_{\lambda^{01}_{\emptyset,2}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, the Rouquier blocks of $(\mathcal{H}_{3,2})_\phi$ are:

$$\{\lambda_{(2),0}, \lambda_{(2),1}\}, \; \{\lambda_{(2),2}\}, \{\lambda_{(1,1),0}, \lambda_{(1,1),1}\}, \; \{\lambda_{(1,1),2}\}, \{\lambda_{\emptyset,0}, \lambda_{\emptyset,1}\}, \; \{\lambda_{\emptyset,2}\}.$$

4 Families of characters of $G(de, e, r)$

Let d, e, r be three positive integers. The group $G(de, e, r)$ is the group of all $r \times r$ monomial matrices with non-zero entries in μ_{de} such that the product of all non-zero entries lies in μ_d. In particular, we have

- $G(2, 2, r) \simeq D_r$ for $r \ge 4$,
- $G(e, e, 2) \simeq I(e)$, where $I(e)$ denotes the dihedral group of order $2e$.

The algorithm of Kim for the determination of the Rouquier blocks for the group $G(de, e, r)$ (cf.[17]) is not entirely correct. In [10] we give the correct algorithm and we study separately the case when $r = 2$ and e is even, which had never been studied before.

4.1 Clifford theory and the Hecke algebras of $G(de, e, r)$

Let W be a complex reflection group and let us denote by $\mathcal{H}(W)$ its generic Hecke algebra. Let W' be another complex reflection group such that, for a certain choice parameters, $\mathcal{H}(W)$ becomes the *twisted symmetric algebra of a finite cyclic group G over the subalgebra* $\mathcal{H}(W')$ (for the definition, see [8], Definition 2.3.6). Then, if we know the blocks of $\mathcal{H}(W)$, we can obtain the blocks of $\mathcal{H}(W')$ with the use of a generalization of some classic results, known as "Clifford theory", to the case of twisted symmetric algebras of finite groups (cf., for example, [11], [8] §2.3). Thanks to a result by Ariki ([2], Proposition 1.16), we obtain that

1. the generic Hecke algebra of $G(de, 1, r)$ specializes to the twisted symmetric algebra of the cyclic group μ_e over the generic Hecke algebra of $G(de, e, r)$ in the case where $r > 2$ or $r = 2$ and e is odd.

2. the generic Hecke algebra of $G(de, 2, 2)$ specializes to the twisted symmetric algebra of the cyclic group $\mu_{e/2}$ over the generic Hecke algebra of $G(de, e, 2)$ in the case where e is even.

In the first case, we can obtain the Rouquier blocks of the cyclotomic Hecke algebras associated to $G(de, e, r)$ from the Rouquier blocks of the cyclotomic Ariki-Koike algebras, already determined in the previous section. In the second case, we need to know the Rouquier blocks of the cyclotomic Hecke algebras of $G(de, 2, 2)$. These have been explicitly calculated in [10] §4.1, using again the theory of essential hyperplanes. For the results of the application of Clifford Theory in both cases, the reader should refer to Theorems 3.10 and 4.8 of [10].

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