On the geometric construction of symmetric crystals via quivers

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Expansion of Conibontorial Representation Theory

1. Introduction

1.1.

Recently in [5] and [6] with M. Kashiwara, the author presented an analogue of the LUST conjecture for the affine Hecke algebra of type B. In [6], we considered $U_v(g)$ and its Dynkin diagram involution $\theta$ and constructed an analogue $B_\theta(g)$ of the reduced $v$-analogue $B_v(g)$ (for the definition, see Definition 2.9 below). We gave a $B_\theta(g)$-module $V_\theta(\lambda)$ for a dominant integral weight $\lambda$ such that $\theta(\lambda) = \lambda$, which is an analogue of the $B_v(g)$-module $U_v^-(g)$ (for the definition, see Definition 2.10 below). We defined the notion of symmetric crystals and conjectured the existence of the global basis. In the case $g = g_{\infty}, I = Z_{odd}, \theta(i) = -i$ and $\lambda = 0$, we constructed the PBW type basis and the lower (and upper) global basis parametrized by the $\theta$-restricted multi-segments. We conjectured that irreducible modules of the affine Hecke algebras of type B are described by the global basis associated to the symmetric crystals.

In the paper [4], we construct the lower global basis for the symmetric crystals by using a geometry of quivers (with a Dynkin diagram involution). Hence for any symmetric quantized Kac-Moody algebra $U_v(g)$, we establish the existence of a crystal basis and a global basis for $V_\theta(0)$. This is analogous to Lusztig's geometric construction of $U_v^-(g)$ and its lower global basis.

1.2.

Lusztig's theory is summarized as follows.

Let $g$ be a symmetric Kac-Moody algebra and $I$ an index set of simple roots of $g$. For a fixed set of arrows $\Omega$, we consider $(I, \Omega)$ as a (finite) oriented graph. We call $(I, \Omega)$ a quiver. For an $I$-graded vector space $V$, we define the moduli space of representations of quiver $(I, \Omega)$ by

$$E_{V, \Omega} = \bigoplus_{i \in I} \text{Hom}(V_i, V_j).$$

The algebraic group $G_V = \prod_{i \in I} GL(V_i)$ acts on $E_{V, \Omega}$. Lusztig introduced a certain full subcategory $\mathcal{B}_{V, \Omega}$ of $\mathcal{D}(E_{V, \Omega})$ where $\mathcal{D}(E_{V, \Omega})$ is the bounded derived category of constructible complexes of sheaves on $E_{V, \Omega}$ (for the definition, see section 3). Let $\mathcal{B}(\mathcal{B}_{V, \Omega})$ be the Grothendieck group of $\mathcal{B}_{V, \Omega}$. He constructed the induction operators $f_i$ and the restriction operators $e_i$ on the Grothendieck group $K_\Omega := \oplus_{\Omega} K(\mathcal{B}_{V, \Omega})$, where $V$ runs over the isomorphism classes of $I$-graded vector spaces. He proved the following theorem.

**Theorem 1.1** (Lusztig).

(i) The operators $e_i$ and $f_i$ define the action of the reduced $v$-analogue $B_v(g)$ of $g$ on $K_\Omega \otimes \mathbb{Z}[v_1, v_1^{-1}] Q(v)$, and $K_\Omega \otimes \mathbb{Z}[v_1, v_1^{-1}] Q(v)$ is isomorphic to $U_v^-(g)$ as a $B_v(g)$-module. The involution induced by the Verdier duality functor coincides with the bar involution on $U_v^-(g)$.

(ii) The simple perverse sheaves in $\bigoplus_{\Omega} \mathcal{B}_{V, \Omega}$ give the lower global basis of $U_v^-(g)$.

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1.3.

This paper is a summarized version of [4].

We introduce the notion of $\theta$-quivers. This is a quiver $(I, \Omega)$ with an involution $\theta : I \to I$ (and $\theta : \Omega \to \Omega$) satisfying some conditions (see Definition 4.1). This notion is partially motivated by Yu Kato's construction [11] of the irreducible representations of the affine Hecke algebras of type $B$.

We also introduce the $\theta$-symmetric $I$-graded vector spaces. This is an $I$-graded vector space $V = (V_i)_{i \in I}$ endowed with a non-degenerate symmetric bilinear form such that $V_i$ and $V_j$ are orthogonal if $j \neq \theta(i)$. For a $\theta$-quiver $(I, \Omega)$ and a $\theta$-symmetric $I$-graded vector space $V$, we define the moduli space $\mathcal{E}_{V, \Omega}$ of representations of $(I, \Omega)$ adding a skew-symmetric condition on $E_{V, \Omega}$ with respect to the involution $\theta$.

Similar to Lusztig's arguments, we consider a certain full subcategory $\mathcal{O}_{V, \Omega}$ of $\mathcal{D}(\mathcal{E}_{V, \Omega})$ and its Grothendieck group $\mathcal{K}_{V, \Omega}$. We define the induction operators $F_i$ and the restriction operators $E_i$ on $\mathcal{K}_{V, \Omega} := \oplus V^\theta$ where $V$ runs over the isomorphism classes of the $\theta$-symmetric $I$-graded vector spaces. We prove the following main theorem which is an analogous result of Lusztig's geometric construction.

**Theorem 1.2** (Theorem 5.12). $\mathcal{K}_{V, \Omega}$ give a lower global basis of $V^\theta(0)$.

Though Lusztig proved Theorem 1.1 using some inner product on $K_0$, we prove Theorem 1.2 using a criterion of crystals (Theorem 2.14) and certain estimates for the actions of $E_i$ and $F_i$ on simple perverse sheaves (Theorem 5.3).

Theorem 5.3 and Lemma 5.5 are the most essential points of our proof of Theorem 1.2. But we omit the proof of them. But in the latter of section 5, we can know how to use them for our proof.

**Remark 1.3.** We give two remarks on a difference from the "folding" procedure and an overlap with perverse sheaves arising from graded Lie algebras by Lusztig.

(i) Our construction is different from Lusztig's construction, "Quiver with automorphisms", in his book [15, Chapter 12-14].

He considered actions $a : I \to I$ and $a : H \to H$ induced from a finite cyclic group $C$ generated by $a$. Put an orientation $\Omega$ such that $\text{out}(a(h)) = a(\text{out}(h))$ and $\text{in}(a(h)) = a(\text{in}(h))$. He said this orientation "compatible". Let $\mathcal{V}^a$ be the category of $I$-graded vector spaces $V$ such that $\dim V_i = \dim V_{a(i)}$ for any $i \in I$. For $V \in \mathcal{V}^a$, $a$ induces a natural automorphism on $\text{Ev}_{V, \Omega}$ and a functor $a^* : \mathcal{D}(\mathcal{E}_{V, \Omega}) \to \mathcal{D}(\mathcal{E}_{V, \Omega})$. He introduced "$C$-equivariant" simple perverse sheaves $(B, \phi)$, where $B$ is a $C$-equivariant sheaf and $\phi : a^*B \cong B$. Then he proved that the set $\cup_{V \in \mathcal{V}^a} \mathcal{B}_{V, \Omega}$ of $C$-equivariant perverse sheaves gives a lower global basis of $U^\theta_{\mathbb{C}}(g)$. Here $g$ has a non-symmetric Cartan matrix which is obtained by the "folding" procedure with respect to the $C$-action on $I$.

But in our construction, a $\theta$-orientation is not a compatible orientation. Moreover the essential difference is that his construction has no skew-symmetric condition in our sense. Hence the set of simple perverse sheaves $\mathcal{O}_{V, \Omega}$ and the space $\mathcal{K}_{V, \Omega} \otimes_{\mathbb{Z}[v, v^{-1}]} \mathbb{Q}(v) \cong V^\theta(0)$ are different from the $B_{V, \Omega}$ and $U^\theta_{\mathbb{C}}(g)$, respectively. The detailed crystal structure of $V^\theta(0)$ is unknown except for the case $g = g_{\infty}$, $I = \mathbb{Z}_{\text{odd}}$ and $\theta(i) = -i$ in [6].

(ii) In some special case, the lower global basis which constructs in this paper is obtained by Lusztig ([16] and [17]). Let us consider the case $G = SO(2n, \mathbb{C})$. Let $g$ be the Lie algebra of $G$ and $T$ a fixed maximal torus of $G$. Set $\varepsilon_{2i-1}$ $(1 \leq i \leq n)$ the fundamental characters of $T$. Assume $q \in C^*$ is not a root of unity. We choose a semisimple element $s \in T$ such that $\varepsilon_{2i-1}(s) \in q^{\pm 1/2}$ for any $i$ and put $d_{2i-1} = \{j|\varepsilon_{2j-1}(s) = q^{2i-1}\}$. Then the centralizer $G(s)$ of $s$ acts on:

\[ g_{2} := \{X \in g | sXs^{-1} = q^2X\} \]

which has finitely many $G(s)$-orbits. Lusztig considered the category $\mathcal{O}^a(g_2)$ of semisimple $G(s)$-equivariant complex on $g_{2}$ and constructed the canonical basis $B(g_{2})$ of $K(g_{2})$ which is the Grothendieck group of $\mathcal{O}^a(g_{2})$.

On the other hand, let us consider the $\theta$-symmetric vector space $V$ such that $\text{wt}(V) = \sum_{i=1}^{n} d_{2i-1}(\alpha_{2i-1}+$...}
$\alpha_{-2i+1}$ and the following $\theta$-quiver of type $A_{2n}$ and the $\theta$-orientation $\Omega$:

\[
\begin{array}{cccccccc}
-2n+1 & \cdots & -5 & -3 & -1 & 1 & 3 & 5 & \cdots & 2n-1 \\
\end{array}
\]

In this case, we have $G(s) = \prod_{i=1}^{n} GL(d_{2i-1}) = \mathfrak{g}_{V}$ and $g_{2} = \mathfrak{g}_{V,\Omega}$. Thus the set $\mathfrak{g}_{V,\Omega}$ of simple perverse sheaves coincide with $B(g_{2})$.

**Remark 1.4.** After writing the paper [4], the author found the notion of $\theta$-quivers has been already introduced by Derksen-Weyman in [3].

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## 2. Preliminaries

### 2.1. Quantum enveloping algebras

#### 2.1.1 Quantum enveloping algebras and reduced $\upsilon$-analogue

We shall recall the quantized universal enveloping algebra $U_{\upsilon}(g)$. In this paper, we treat only the symmetric Cartan matrix case. Let $I$ be an index set (for simple roots), and $Q$ the free $\mathbb{Z}$-module with a basis $\{\alpha_{i}\}_{i\in I}$. Let $(\cdot, \cdot) : Q \times Q \to \mathbb{Z}$ be a symmetric bilinear form such that $(\alpha_{i}, \alpha_{i}) = 2$ and $(\alpha_{i}, \alpha_{j}) \in \mathbb{Z}_{\geq 0}$ for $i \neq j$.

Let $\upsilon$ be an indeterminate and set $K := \mathbb{Q}(\upsilon)$. We define its subrings $A_{0}$, $A_{\infty}$ and $A$ as follows.

\[
A_{0} = \{ f \in K \mid f \text{ is regular at } \upsilon = 0 \}, \\
A_{\infty} = \{ f \in K \mid f \text{ is regular at } \upsilon = \infty \}, \\
A = \mathbb{Q}[\upsilon, \upsilon^{-1}].
\]

**Definition 2.1.** The quantized universal enveloping algebra $U_{\upsilon}(g)$ is the $K$-algebra generated by elements $e_{i}, f_{i}$ and invertible elements $t_{i}$ ($i \in I$) with the following defining relations.

1. The $t_{i}$'s commute with each other.
2. $t_{i}e_{i}t_{i}^{-1} = e_{i}^{(\alpha_{i}, \alpha_{i})}e_{i}$ and $t_{j}f_{j}f_{j}^{-1} = f_{j}^{-(\alpha_{j}, \alpha_{j})}f_{j}$ for any $i, j \in I$.
3. $[e_{i}, f_{j}] = \delta_{ij} \frac{t_{i} - t_{i}^{-1}}{\upsilon - \upsilon^{-1}}$ for $i, j \in I$.
4. ($\upsilon$-Serre relation) For $i \neq j$,

\[
\sum_{k=0}^{b}(-1)^{k}e_{i}^{(k)}e_{j}e_{i}^{(b-k)} = 0, \\
\sum_{k=0}^{b}(-1)^{k}f_{i}^{(k)}f_{j}f_{i}^{(b-k)} = 0.
\]

Here $b = 1 - (\alpha_{i}, \alpha_{j})$ and

\[
e_{i}^{(k)} = e_{i}^{k}/[k]_{\upsilon}!, \\
f_{i}^{(k)} = f_{i}^{k}/[k]_{\upsilon}!, \\
[k]_{\upsilon} = (\upsilon^{k} - \upsilon^{-k})/(\upsilon - \upsilon^{-1}), \\
[k]_{\upsilon}! = [1]_{\upsilon} \cdots [k]_{\upsilon}.
\]

Let us denote by $U_{\upsilon}^{-}(g)$ the subalgebra of $U_{\upsilon}(g)$ generated by the $f_{i}$'s.

Let $e_{i}'$ and $e_{i}^{\dagger}$ be the operators on $U_{\upsilon}^{-}(g)$ defined by

\[
[e_{i}'a] = \frac{(e_{i}'a)t_{i} - t_{i}^{-1}e_{i}^{\dagger}a}{\upsilon - \upsilon^{-1}} (a \in U_{\upsilon}^{-}(g)).
\]
These operators satisfy the following formulas similar to derivations:

\[ e'_i(ab) = (e'_i(a)b + (\text{Ad}(t_i)a)e'_i b. \]

The algebra \( U^-_v(\mathfrak{g}) \) has a unique symmetric bilinear form \((\cdot, \cdot)\) such that \((1,1) = 1\) and

\[ (e'_i a, b) = (a, f_i b) \text{ for any } a, b \in U^-_v(\mathfrak{g}). \]

It is non-degenerate. The left multiplication operator \( f_j \) and \( e'_i \) satisfy the commutation relations

\[ e'_i f_j = v^{-(\alpha_i, \alpha_j)} f_j e'_i + \delta_{ij} \tag{1} \]

and the \( e'_i \)’s satisfy the \( v \)-Serre relations (Definition 2.1(4)).

**Definition 2.2.** The reduced \( v \)-analogue \( B_v(\mathfrak{g}) \) of \( \mathfrak{g} \) is the \( \mathbb{Q}(v) \)-algebra generated by \( e'_i \) and \( f_i \) which satisfy (1) and the \( v \)-Serre relations for \( e'_i \) and \( f_i \) \((i, j \in \Gamma)\) as the defining relations.

**2.1.2 Review on crystal bases and global bases of \( U^-_v \)**

Since \( e'_i \) and \( f_i \) satisfy the \( v \)-boson relation, any element \( a \in U^-_v(\mathfrak{g}) \) can be uniquely written as

\[ a = \sum_{n \geq 0} f_i^{(n)} a_n \text{ with } e'_i a_n = 0. \]

Here \( f_i^{(n)} = \frac{f_i^n}{n!}. \)

**Definition 2.3.** We define the modified root operators \( \tilde{e}_i \) and \( \tilde{f}_i \) on \( U^-_v(\mathfrak{g}) \) by

\[ \tilde{e}_ia = \sum_{n \geq 1} f_i^{(n-1)} a_n, \quad \tilde{f}_ia = \sum_{n \geq 0} f_i^{(n+1)} a_n. \]

**Theorem 2.4** ([8]). We define

\[ L(\infty) = \sum_{\ell \geq 0, i_1, \ldots, i_{\ell} \in I} A_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_{\ell}} \cdot 1 \subset U^-_v(\mathfrak{g}), \]

\[ B(\infty) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_{\ell}} \cdot 1 \mod vL(\infty) \mid \ell \geq 0, i_1, \ldots, i_{\ell} \in I \right\} \subset L(\infty)/vL(\infty). \]

Then we have

1. \( \tilde{e}_i L(\infty) \subset L(\infty) \) and \( \tilde{f}_i L(\infty) \subset L(\infty), \)
2. \( B(\infty) \) is a basis of \( L(\infty)/vL(\infty), \)
3. \( \tilde{f}_i B(\infty) \subset B(\infty) \) and \( \tilde{e}_i B(\infty) \subset B(\infty) \cup \{0\}. \)

We call \((L(\infty), B(\infty))\) the crystal basis of \( U^-_v(\mathfrak{g}) \).

**Definition 2.5.** We define \( e_i(b) := \max\{m \in \mathbb{Z}_{\geq 0} \mid b^m \neq 0\} \) for \( i \in \Gamma \) and \( b \in B(\infty). \)

Let \( \theta \) be the automorphism of \( K \) sending \( v \) to \( v^{-1} \). Then \( \theta \) coincides with \( A_{\infty} \).

Let \( V \) be a vector space over \( K \), \( L_0 \) an \( A \)-submodule of \( V \), \( L_\infty \) an \( A_{\infty} \)-submodule, and \( V_A \) an \( A \)-submodule. Set \( E := L_0 \cap L_\infty \cap V_A. \)

**Definition 2.6** ([8]). We say that \((L_0, L_\infty, V_A)\) is balanced if each of \( L_0, L_\infty \) and \( V_A \) generates \( V \) as a \( K \)-vector space, and if one of the following equivalent conditions is satisfied.

1. \( E \to L_0/vL_0 \) is an isomorphism,
2. \( E \to L_\infty/v^{-1}L_\infty \) is an isomorphism,
3. \((L_0 \cap V_A) \oplus (v^{-1}L_\infty \cap V_A) \to V_A \) is an isomorphism.
4. \( A_0 \otimes Q E \to L_0, A_\infty \otimes Q E \to L_\infty, A \otimes Q E \to V_A \) and \( K \otimes Q E \to V \) are isomorphisms.
Let $-\theta$ be the ring automorphism of $U_v(g)$ sending $v$, $t_i$, $e_i$, $f_i$ to $v^{-1}$, $t_i^{-1}$, $e_i$, $f_i$.

Let $U_v(g)_A$ be the $A$-subalgebra of $U_v(g)$ generated by $e_i^{(n)}$, $f_i^{(n)}$ and $t_i$. Similarly we define $U_v^{-}(g)_A$.

**Theorem 2.7.** $(L(\infty), L(\infty)^{-}, U_v^{-}(g)_A)$ is balanced.

Let

$$G^{low}: L(\infty)/vL(\infty) \rightarrow E := L(\infty) \cap L(\infty)^{-} \cap U_v^{-}(g)_A$$

be the inverse of $E \rightarrow L(\infty)/vL(\infty)$. Then $\{G^{low}(b) \mid b \in B(\infty)\}$ forms a basis of $U_v^{-}(g)$. We call it a (lower) global basis. It is first introduced by G. Lusztig ([13]) under the name of "canonical basis" for the $A$, $D$, $E$ cases.

**Definition 2.8.** Let $\{G^{up}(b) \mid b \in B(\infty)\}$ be the dual basis of $\{G^{low}(b) \mid b \in B(\infty)\}$ with respect to the inner product $(\cdot, \cdot)$. We call it the upper global basis of $U_v^{-}(g)$.

### 2.2. Symmetric Crystals

Let $\theta$ be an automorphism of $I$ such that $\theta^2 = id$ and $(\alpha_{\theta(i)}, \alpha_{\theta(j)}) = (\alpha_i, \alpha_j)$. Hence it extends to an automorphism of the root lattice $Q$ by $\theta(\alpha_i) = \alpha_{\theta(i)}$, and induces an automorphism of $U_v(g)$.

**Definition 2.9.** Let $B_v(g)$ be the $K$-algebra generated by $E_i$, $F_i$ and invertible elements $T_i(i \in I)$ satisfying the following defining relations:

(i) the $T_i$'s commute with each other,

(ii) $T_i = T_i$ for any $i$,

(iii) $T_i E_j T_i^{-1} = v^{\alpha_j + \alpha_{\theta(i)} - \alpha_i} E_j$ and $T_i F_j T_i^{-1} = v^{\alpha_j + \alpha_{\theta(i)} - \alpha_i} F_j$ for $i, j \in I$,

(iv) $E_i F_j = v^{-1} F_j E_i + (\delta_{i,j} + \delta_{\theta(i),j} T_i)$ for $i, j \in I$,

(v) the $E_i$'s and the $F_i$'s satisfy the $v$-Serre relations.

We set $F_i^{(n)} = F_i^n/n!$.

**Proposition 2.10** ([6, Proposition2.11]). Let

$$\lambda \in P_+ := \{\lambda \in \text{Hom}(Q, \mathbb{Q}) \mid \lambda(\alpha_i) \in \mathbb{Z}_{\geq 0} \text{ for any } i \in I\}$$

be a dominant integral weight such that $\theta(\lambda) = \lambda$.

(i) There exists a $B_v(g)$-module $V_v(\lambda)$ generated by a non-zero vector $\phi_\lambda$ such that

(a) $E_i \phi_\lambda = 0$ for any $i \in I$,

(b) $T_i \phi_\lambda = v^{\alpha_i} \phi_\lambda$ for any $i \in I$,

(c) $\{u \in V_v(\lambda) \mid E_i u = 0 \text{ for any } i \in I\} = K \phi_\lambda$.

Moreover such a $V_v(\lambda)$ is irreducible and unique up to an isomorphism.

(ii) There exists a unique non-degenerate symmetric bilinear form $(\cdot, \cdot)$ on $V_v(\lambda)$ such that $(\phi_\lambda, \phi_\lambda) = 1$ and $(E_i u, v) = (u, F_i v)$ for any $i \in I$ and $u, v \in V_v(\lambda)$.

(iii) There exists an endomorphism $-\theta$ of $V_v(\lambda)$ such that $-\theta \phi_\lambda = \phi_\lambda$ and $\overline{\theta v} = \overline{a v} F_i \overline{v} = F_i \overline{v}$ for any $a \in K$ and $v \in V_v(\lambda)$.

Hereafter we assume further that

there is no $i \in I$ such that $\theta(i) = i$.

In [6] we conjectured that $V_v(\lambda)$ has a crystal basis. This means the following. Since $E_i$ and $F_i$ satisfy the $v$-boson relation $E_i F_i = v^{-1} F_i E_i + 1$, we define the modified root operators:

$$\tilde{E}_i(u) = \sum_{n \geq 1} F_i^{(n-1)}u_n \text{ and } \tilde{F}_i(u) = \sum_{n \geq 0} F_i^{(n+1)}u_n.$$


when writing $u = \sum_{n \geq 0} f_i^{(n)}u_n$ with $E_iu_n = 0$. Let $L_\theta(\lambda)$ be the $A_\theta$-submodule of $V_{\theta}(\lambda)$ generated by $\tilde{F}_{i_1}\cdots\tilde{F}_{i_\ell}\phi_\lambda$ ($\ell \geq 0$ and $i_1, \ldots, i_\ell \in I$), and let $B_\theta(\lambda)$ be the subset
\[ \{\tilde{F}_{i_1}\cdots\tilde{F}_{i_\ell}\phi_\lambda \mod vL_\theta(\lambda) \mid \ell \geq 0, i_1, \ldots, i_\ell \in I\} \]
of $L_\theta(\lambda)/vL_\theta(\lambda)$.

**Conjecture 2.11.** Let $\lambda$ be a dominant integral weight such that $\theta(\lambda) = \lambda$.

1. $\tilde{F}_iL_\theta(\lambda) \subset L_\theta(\lambda)$ and $\tilde{E}_iL_\theta(\lambda) \subset L_\theta(\lambda)$,
2. $B_\theta(\lambda)$ is a basis of $L_\theta(\lambda)/vL_\theta(\lambda)$,
3. $\tilde{F}_iB_\theta(\lambda) \subset B_\theta(\lambda)$ and $\tilde{E}_iB_\theta(\lambda) \subset B_\theta(\lambda) \cup \{0\}$,
4. $\tilde{F}_i\tilde{E}_i(b) = b$ for any $b \in B_\theta(\lambda)$ such that $\tilde{E}_i b \neq 0$, and $\tilde{E}_i\tilde{F}_i(b) = b$ for any $b \in B_\theta(\lambda)$.

Moreover we conjectured that $V_{\theta}(\lambda)$ has a global crystal basis. Namely we have

**Conjecture 2.12.** $(L_\theta(\lambda), \overline{L_\theta(\lambda)}, V_{\theta}(\lambda)_{\Lambda}^{low})$ is balanced. Here $V_{\theta}(\lambda)_{\Lambda}^{low} := U^{-}_{v}(g)_{\Lambda}\phi_{\lambda}$.

**Example 2.13.** Suppose $g = g_{\infty}$, the Dynkin diagram involution $\theta$ of $I$ defined by $\theta(i) = -i$ for $i \in I = \mathbb{Z}_{odd}$.

And assume $\lambda = 0$. In this case, we can prove
\[ V_{\theta}(0) \cong U_{\lambda}^{-} / \sum_{i \in I} U_{\lambda}^{-}(f_i - f_{\theta(i)}). \]

Moreover we can construct a PBW type basis, a crystal basis and an upper and lower global basis on $V_{\theta}(0)$ parametrized by "the $\theta$-restricted multisegments". For more details, see [6].

### 2.3. Criterion for crystals

Let $K[e, f]$ be the ring generated by $e$ and $f$ with the defining relation $ef = v^{-2}fe + 1$. We call this algebra the $v$-boson algebra. Let $P$ be a free $Z$-module, and let $\alpha$ be a non-zero element of $P$. Let $M$ be a $K[e, f]$-module. Assume that $M$ has a weight decomposition $M = \bigoplus_{\xi \in P} M_{\xi}$ and $eM_{\lambda} \subset M_{\lambda + \alpha}$ and $fM_{\lambda} \subset M_{\lambda - \alpha}$ . Assume the following finiteness conditions:

for any $\lambda \in P$, dim $M_{\lambda} < \infty$ and $M_{\lambda + n\alpha} = 0$ for $n \gg 0$.

Hence for $u \in M$, we can write $u = \sum_{n \geq 0} f^{(n)}u_n$ with $eu_n = 0$. We define endomorphisms $\tilde{e}$ and $\tilde{f}$ of $M$ by
\[ \tilde{e}u = \sum_{n \geq 1} f^{(n-1)}u_n, \quad \tilde{f}u = \sum_{n \geq 0} f^{(n+1)}u_n. \]

Let $B$ be a crystal with weight decomposition by $P$ in the following sense. We have wt: $B \to P$, $f$: $B \to B$, $\tilde{e}$: $B \to B \cup \{0\}$ and $\epsilon$: $B \to \mathbb{Z}_{\geq 0}$ satisfying the following properties, where $B_{\lambda} = \text{wt}^{-1}(\lambda)$:

1. $\tilde{f}B_{\lambda} \subset B_{\lambda - \alpha}$ and $\tilde{e}B_{\lambda} \subset B_{\lambda + \alpha} \cup \{0\}$ for any $\lambda \in P$,
2. $\tilde{f}\tilde{e}b = b$ if $\tilde{e}b \neq 0$, and $\tilde{e} \circ \tilde{f} = \text{id}_B$,
3. for any $\lambda \in P$, $B_{\lambda}$ is a finite set and $B_{\lambda + n\alpha} = \emptyset$ for $n \gg 0$,
4. $\epsilon(b) = \max\{n \geq 0 \mid \tilde{e}^n b \neq 0\}$ for any $b \in B$. 


Set \( \text{ord}(a) = \sup \{ n \in \mathbb{Z} | a \in v^nA_0 \} \) for \( a \in K \). We understand \( \text{ord}(0) = \infty \).

Let \( \{ G(b) \}_{b \in B} \) be a system of generators of \( M \) with \( G(b) \in M_{\text{wt}(b)} \). Assume that we have expressions:

\[
eq G(b) = \sum_{b' \in B} E_{b,b'} G(b), \quad fG(b) = \sum_{b' \in B} F_{b,b'} G(b).
\]

Now consider the following conditions for these data, where \( \ell = \varepsilon(b) \) and \( \ell' = \varepsilon(b') \):

\[
\begin{align*}
\text{ord}(F_{b,b'}) & \geq 1 - \ell', \\
\text{ord}(E_{b,b'}) & \geq -\ell', \\
F_{b,\overline{f}b} & \in v^{-\ell}(1 + vA_0), \\
E_{b,\overline{f}b} & \in v^{1-\ell}(1 + vA_0), \\
\text{ord}(F_{b,b'}) & > 1 - \ell' \text{ if } \ell < \ell' \text{ and } b \neq \overline{f}b, \\
\text{ord}(E_{b,b'}) & > -\ell' \text{ if } \ell < \ell' + 1 \text{ and } b \neq \overline{f}b.
\end{align*}
\]

Theorem 2.14 ([6, Theorem 4.1, Corollary 4.4]). Assume the conditions (2)-(7). Let \( L \) be the \( A_0 \)-submodule \( \sum_{b \in B} A_0 G(b) \) of \( M \). Then we have \( \overline{\alpha} L \subset L \) and \( fL \subset L \). Moreover we have

\[
\overline{\alpha} G(b) \equiv G(\overline{a}b) \mod vL, \quad \overline{f} G(b) \equiv G(\overline{f}b) \mod vL
\]

for any \( b \in B \). Here we understand \( G(0) = 0 \).

In [6], this theorem is proved under weaker assumptions.

### 2.4. Perverse Sheaves

#### 2.4.1 Perverse Sheaves

In this paper, we consider algebraic varieties over \( \mathbb{C} \). Let \( \mathcal{D}(X) \) be the bounded derived category of constructible complexes of sheaves on an algebraic variety \( X \). For a morphism \( f: X \to Y \) of algebraic varieties \( X \) and \( Y \), let \( f^\ast \) be the inverse image, \( f \) the direct image with proper support and \( D: \mathcal{D}(X) \to \mathcal{D}(X) \) the Verdier duality functor. Let \( \mathcal{P}(\mathcal{D}^{\geq 0}(X), \mathcal{P}\mathcal{D}^{\leq 0}(X)) \) be the perverse \( t \)-structure and \( \mathcal{P}(\mathcal{D}(X)) \) be the perverse \( t \)-structure for \( H^k(\cdot) \) be the \( k \)-th perverse cohomology sheaf. We say that an object \( L \) in \( \mathcal{D}(X) \) is semisimple if \( L \) is isomorphic to the direct sum \( \oplus_a \mathcal{P}H^k(L)[-k] \) and if each \( \mathcal{P}H^k(L) \) is a semisimple perverse sheaf. Assume that we are given an action of a connected algebraic group \( G \) on \( X \). A semisimple object \( L \) in \( \mathcal{D}(X) \) is said to be \( G \)-equivariant if each \( \mathcal{P}H^k(L) \) is a \( G \)-equivariant perverse sheaf. We denote by \( 1_X \) the constant sheaf on \( X \).

#### 2.4.2 Fourier-Sato-Deligne transforms

Let \( E \to S \) be a vector bundle and \( E^* \to S \) the dual vector bundle. Hence \( C^* \) acts on \( E \) and \( E^* \). We say that \( L \in \mathcal{D}(E) \) is monodromic if \( H^j(L) \) is locally constant on every \( C^* \)-orbit of \( E \). Let \( \mathcal{D}_{\text{mono}}(E) \) be the full subcategory of \( \mathcal{D}(E) \) consisting of monodromic objects. Then we can define the Fourier transform

\[
\Phi_{E/S}: \mathcal{D}_{\text{mono}}(E) \to \mathcal{D}_{\text{mono}}(E^*).
\]

#### 2.5. Quivers

Let \( I \) and \( \alpha \)'s be as in 2.1.

**Definition 2.15.** A double quiver \((I, H)\) associated with the symmetric Cartan matrix is a following data:

(i) a set \( H \),

(ii) two maps \( \text{out, in}: H \to I \) such that \( \text{out}(h) \neq \text{in}(h) \) for any \( h \in H \),

(iii) an involution \( h \mapsto \overline{h} \) on \( H \) satisfying \( \text{out}(\overline{h}) = \text{in}(h) \) and \( \text{in}(\overline{h}) = \text{out}(h) \),
(iv) $h \in H \setminus \Omega$ out$(h) = i$, in$(h) = j = -\alpha_i, \alpha_j$ for $i \neq j$.

An orientation of a double quiver $(I,H)$ is a subset $\Omega$ of $H$ such that $\Omega \cap \Omega^\prime = \emptyset$ and $\Omega \cup \Omega^\prime = H$. For an orientation $\Omega$, we call $(I, \Omega)$ a quiver.

For a fixed orientation $\Omega$, we call a vertex $i \in I$ a sink if out$(h) \neq i$ for any $h \in \Omega$.

**Definition 2.16.** Let $\mathcal{V}$ be the category of $I$-graded vector spaces $V = (V_i)$, with morphisms being linear maps respecting the grading. Put $\operatorname{wt}(V) = \sum_{i \in I} (\dim V_i) \alpha_i$.

Let $S_i$ be an $I$-graded vector space such that $\operatorname{wt}(S_i) = \alpha_i$.

**Definition 2.17.** For $V \in \mathcal{V}$ and a subset $\Omega$ of $H$, we define

$$E_{V,\Pi}: = \bigoplus_{h \in \Omega} \operatorname{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)}).$$

The algebraic group $G_V = \prod_{i \in I} \operatorname{GL}(V_i)$ acts on $E_{V,\Pi}$ by $(g, x) \mapsto gx$ where $(gx)_h = g_{\text{in}(h)}x_h g_{\text{out}(h)}^{-1}$.

The group $(\mathbb{C}^\times)^{\Omega}$ also acts on $E_{V,\Pi}$ by $x_h \mapsto c_h x_h (h \in \Omega, c_h \in \mathbb{C}^\times)$.

For $x \in E_{V,\Pi}$, an $I$-graded subspace $W \subset V$ is $x$-stable if $x_h(W_{\text{out}(h)}) \subset W_{\text{in}(h)}$ for any $h \in \Omega$.

Note that $E_{S_i,\Pi} \cong \{pt\}$.

### 3. A Review of Lusztig’s geometric construction

We give a quick review on Lusztig’s theory in [13] and [14] (cf. [15]). For a sequence $i = (i_1, \ldots, i_m) \in I^m$ and a sequence $a = (a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^m$, a flag of type $(i, a)$ is by definition a finite decreasing sequence $F = (V = F_0 \supset F_1 \supset \cdots \supset F_m = \{0\})$ of $I$-graded subspaces of $V$ such that the $I$-graded vector space $F^{-1}/F^t$ vanishes in degrees $\neq i_t$ and has dimension $a_t$ in degree $i_t$. We denote by $\tilde{F}_{i,a,\Pi}$ the set of pairs $(x, F)$ such that $x \in E_{V,\Pi}$ and $F$ is an $x$-stable flag of type $(i, a)$. The group $G_V$ acts on $\tilde{F}_{i,a,\Pi}$. The first projection $\pi_{i,a} : \tilde{F}_{i,a,\Pi} \to E_{V,\Pi}$ is a $G_V$-equivariant projective morphism.

By the decomposition theorem [2], $L_{i,a,\Pi} = (\pi_{i,a})_* (\tilde{F}_{i,a,\Pi}) \in \mathcal{D}(E_{V,\Pi})$ is a semisimple complex. We define $\mathcal{P}_{V,\Pi}$ as the set of the isomorphism classes of simple perverse sheaves $L \in \mathcal{D}(E_{V,\Pi})$ satisfying the following property: $L$ appears as a direct summand of $L_{i,a,\Pi}[d]$ for some $d$ and $(i, a)$. We denote by $\mathcal{P}_{V,\Pi}$ the full subcategory of $\mathcal{D}(E_{V,\Pi})$ consisting of all objects which are isomorphic to finite direct sums of the form $L[d]$ for various $L \in \mathcal{P}_{V,\Pi}$ and various integers $d$. Any complex in $\mathcal{P}_{V,\Pi}$ is $G_V \times (\mathbb{C}^\times)^{\Pi}$-equivariant.

Let $T, W, V$ be $I$-graded vector spaces such that $\operatorname{wt}(V) = \operatorname{wt}(W) + \operatorname{wt}(T)$. We consider the following diagram

$$E_{T,\Pi} \times E_{W,\Pi} \xrightarrow{\mathbf{p}_1} E_{\Pi} \xrightarrow{\mathbf{p}_2} E_{V,\Pi} \xrightarrow{\mathbf{p}_3} E_{V,\Pi}.$$  

Here $E_{T,\Pi}$ is the variety of $x, W$, $E_{W,\Pi}$ and $E_{V,\Pi}$ is an $x$-stable $I$-graded subspace of $V$ such that $\operatorname{wt}(W) = \operatorname{wt}(W)$. The variety $E_{\Pi}$ consists of $(x, W, \varphi^W, \varphi^T)$ where $(x, W) \in E^T, \varphi^W : W \cong W$, and $\varphi^T : T \cong V/W$. The morphisms $p_1, p_2$ and $p_3$ are given by $p_1(x, W, \varphi^W, \varphi^T) = (x, W, \varphi^W, \varphi^T)$ and $p_2(x, W) = x$ and $p_3(x, W) = x$. Then $p_1$ is smooth with connected fibers, $p_2$ is a principal $G_T \times G_W$-bundle, and $p_3$ is projective. For a $G_T$-equivariant semisimple complex $K_T$ and a $G_W$-equivariant semisimple complex $K_W$, there exists a unique semisimple complex $K''$ satisfying $p_1'(K_T \boxtimes K_W) = p_2(K'')$. We define $K_T \ast K_W : = (p_3)_*(K'') \in \mathcal{D}(E_{V,\Pi})$.

For an $I$-graded subspace $U$ of $V$ such that $V/U \cong T$, we also consider the following diagram

$$E_{T,\Pi} \times E_{U,\Pi} \xrightarrow{\mathbf{p}_4} E(U, V) \xrightarrow{\mathbf{p}_5} E_{V,\Pi}. $$

Here $E(U, V)_{\Pi}$ is the variety of $x \in E_{V,\Pi}$ such that $U$ is $x$-stable. For $K \in \mathcal{D}(E_{V,\Pi})$, we define $\operatorname{Res}_{U/V}(K) : = p_4^*(K')$.

We define $K_{V,\Pi}$ as the Grothendieck group of $\mathcal{P}_{V,\Pi}$. It is the additive group generated by the isomorphism classes $(L)$ of objects $L \in \mathcal{P}_{V,\Pi}$ with the relation $(L) = (L') + (L'')$ when $L \cong L' \oplus L''$. The group $K_{V,\Pi}$ has a $\mathbb{Z}[v, v^{-1}]$-module structure by $v(L) = (L[1])$ and $v^{-1}(L) = (L[-1])$ for $L \in \mathcal{P}_{V,\Pi}$. 


Hence, $K_{V, \Omega}$ is a free $\mathbb{Z}[v, v^{-1}]$-module with a basis \{(L)|L \in \mathcal{P}_{V, \Omega}\}. We define $K_{\Omega} := \bigoplus_{V} K_{V, \Omega}$ where $V$ runs over the isomorphism classes of $I$-graded vector spaces. Recall that $S_i$ is an $I$-graded vector space such that $\text{wt}(S_i) = \alpha_i$. Then we can define the induction $f_i : K_{W, \Omega} \to K_{V, \Omega}$ and the restriction $e_i : K_{V, \Omega} \to K_{W, \Omega}$ by

$$f_i(K) := v^{\dim W_i + \sum_{i \neq j} \dim W_j} (1_{S_i} * K), \quad e_i(K) := v^{-\dim W_i + \sum_{i \neq j} \dim W_j} \text{Res}_{S_i, V}(K).$$

Then Lusztig’s main theorem is stated as follows.

**Theorem 3.1** (Lusztig).

(i) The operators $e_i$ and $f_i$ define the action of the reduced $v$-analogue $B_v(g)$ of $g$ on $K_{\Omega} \otimes_{\mathbb{Z}[v, v^{-1}]} \mathbb{Q}(v)$. The $B_v(\mathfrak{g})$-module $K_{\Omega} \otimes_{\mathbb{Z}[v, v^{-1}]} \mathbb{Q}(v)$ is isomorphic to $U_v^{-}(g)$. The involution induced by the Verdier duality functor coincides with the bar involution on $U_v^{-}(g)$.

(ii) The simple perverse sheaves in $\mathcal{P}_{V, \Omega}$ give a lower global basis of $U_v^{-}(g)$.

## 4. Quivers with an Involution $\theta$

### 4.1. Quivers with an involution $\theta$

**Definition 4.1.** A double $\theta$-quiver is a data:

1. a double quiver $(I, H)$,
2. involutions $\theta : I \to I$ and $\theta : H \to H$,

such that

(a) $\text{out}(\theta(h)) = \theta(\text{in}(h))$ and $\text{in}(\theta(h)) = \theta(\text{out}(h))$,

(b) If $\theta(\text{out}(h)) = \text{in}(h)$, then $\theta(h) = h$,

(c) $\theta(h) = \theta(h)$,

(d) There is no $i \in I$ such that $\theta(i) = i$

A $\theta$-orientation is an orientation of $(I, H)$ such that $\Omega$ is stable by $\theta$. For a $\theta$-orientation $\Omega$, we call $(I, \Omega)$ a $\theta$-quiver.

From the assumption (d), any vertex $i$ is a sink with respect to some $\theta$-orientation $\Omega$.

**Example 4.2.** We give two $\theta$-orientations for the case of Example 2.13. The vertex 1 is a sink in the right example.

![Diagram](image)

**Example 4.3.** Our definition of a $\theta$-quiver contains the case of type $A_1^{(1)}$. The following three figures are three $\theta$-orientations in this case.

![Diagram](image)

**Definition 4.4.** A $\theta$-symmetric $I$-graded vector space $V$ is an $I$-graded vector space endowed with a non-degenerate symmetric bilinear form $(\cdot, \cdot) : V \times V \to \mathbb{C}$ such that $V_i$ and $V_j$ are orthogonal if $j \neq \theta(i)$. For an $I$-graded subspace $W$ of $V$, we set $W^\perp := \{v \in V \mid (v, w) = 0 \text{ for any } w \in W\}$.

Hence $(W^\perp)_{\theta(i)} \cong (V_i / W_i)^\ast$. 


Note that if $W \supset W^1$, then $W/W^1$ has a structure of $\theta$-symmetric $I$-graded vector space. Note that two $\theta$-symmetric $I$-graded vector spaces with the same dimension are isomorphic.

**Definition 4.5.** Let $(I, H)$ be a $\theta$-quiver. For a $\theta$-symmetric $I$-graded vector space $V$ and a $\theta$-stable subset $\Omega$ of $H$, we define

$$\theta E_{V, \Omega} := \{ x \in E_{V, \Omega} | x_{\theta(h)} = -^t x_h \in \text{Hom}(V_{\theta(\text{in}(h))}, V_{\theta(\text{out}(h))}) \text{ for any } h \in \Omega \}. $$

The algebraic group $\theta G_{V} := \{ g \in G_{V} | ^tg_{i}^{-1} = g_{\theta(i)} \text{ for any } i \}$ naturally acts on $\theta E_{V, \Omega}$. Set $(\mathbb{C}^\times)^{\Omega, \theta} := \{ (c_h)_{h \in \Omega} | c_h \in \mathbb{C}^\times \text{ and } c_{\theta(h)} = c_h \}$. The group $(\mathbb{C}^\times)^{\Omega, \theta}$ also acts on $\theta E_{V, \Omega}$ by $x_h \mapsto c_h x_h \ (h \in \Omega)$. These two actions commute with each other.

**Definition 4.6.** For a $\theta$-symmetric $I$-graded vector space $V$, a sequence $i = (i_1, \ldots, i_{2m}) \in I^{2m}$ such that $\theta(i_t) = i_{2m-t+1}$ and a sequence $a = (a_1, \ldots, a_{2m}) \in \mathbb{Z}^{2m}_{>0}$ such that $a_{2m-t+1} = a_t$, we say that a flag of $I$-graded subspace of $V$

$$ F = (V = F^0 \supset F^1 \supset \cdots \supset F^m \supset F^{m+1} \supset \cdots \supset F^{2m} = \{0\})$$

is of type $(i, a)$ if

(i) $\dim(F^{\ell-1}/F^{\ell})_{i} = \begin{cases} a_{\ell} & (i = i_{\ell}) \\ 0 & (i \neq i_{\ell}) \end{cases}$,

(ii) $F^{2m-\ell} = (F^{\ell})^\perp$.

Then we have $\text{wt} V = \sum_{1 \leq \ell \leq 2m} a_{\ell} t_{\ell}$. We denote by $\theta F_{i, a}$ the set of flags of type $(i, a)$.

For $x \in \theta E_{V, \Omega}$, a flag $F$ of type $(i, a)$ is $x$-stable if $F^\ell (\ell = 1, \ldots, 2m)$ are $x$-stable. We define

$$\theta \tilde{\mathcal{F}}_{i,a,\Omega} := \{ (x, F) \in \theta E_{V, \Omega} \times \theta F_{i, a} | F \text{ is } x \text{-stable} \}. $$

The group $\theta G_{V}$ naturally acts on $\theta F_{i, a}$ and $\theta \tilde{\mathcal{F}}_{i,a,\Omega}$.

Note that $x : V \rightarrow V \cong V_x$ in $\theta E_{V, \Omega}$ may be regarded as a skew-symmetric form on $V$, and the condition that $F$ is $x$-stable is equivalent to the one $x(F^\ell, F^{2m-\ell}) = 0$ for any $\ell$.

The following lemma is obvious.

**Lemma 4.7.** The variety $\theta \tilde{\mathcal{F}}_{i,a,\Omega}$ is smooth and irreducible. The first projection $\theta \pi_{i, a} : \theta \tilde{\mathcal{F}}_{i,a,\Omega} \rightarrow \theta E_{V, \Omega}$ is $\theta G_{V} \times (\mathbb{C}^\times)^{\Omega, \theta}$-equivariant and projective.

### 4.2. Perverse sheaves on $\theta E_{V, \Omega}$

Let $\Omega$ be a $\theta$-orientation. By Lemma 4.7 and the decomposition theorem [2],

$$\theta L_{i,a,\Omega} := (\theta \pi_{i,a})^{-1}(1_{x \tilde{\mathcal{F}}_{i,a,\Omega}})$$

is a semisimple complex in $\mathcal{D}(\theta E_{V, \Omega})$.

**Definition 4.8.** We define $\mathcal{P}_{V, \Omega}$ as the set of the isomorphism classes of simple perverse sheaves $L$ in $\mathcal{D}(\theta E_{V, \Omega})$ satisfying the property: $L$ appears in $\theta L_{i,a,\Omega}|d$ as a direct summand for some integer $d$ and $(i, a)$. We denote by $\mathcal{P}_{V, \Omega}$ the full subcategory of $\mathcal{D}(\theta E_{V, \Omega})$ consisting of objects which are isomorphic to finite direct sums of $L[d]$ with $L \in \mathcal{P}_{V, \Omega}$ and $d \in \mathbb{Z}$.

Note that any object in $\mathcal{P}_{V, \Omega}$ is $\theta G_{V} \times (\mathbb{C}^\times)^{\Omega, \theta}$-equivariant.

### 4.3. Multiplications and Restrictions

Fix $\theta$-symmetric and $I$-graded vector spaces $V$ and $W$, and an $I$-graded vector space $T$ such that $\text{wt}(V) = \text{wt}(W) + \text{wt}(T) + \theta(\text{wt}(T))$.

We consider the following diagram

$$E_{T, \Omega} \times \theta E_{W, \Omega} \xrightarrow{p_1} \theta E'_{\Omega} \xrightarrow{p_2} \theta E''_{\Omega} \xrightarrow{p_3} \theta E_{V, \Omega}.$$
Here $\mathcal{E}_{V,W}$ is the variety of $(x, V)$ where $x \in \mathcal{E}_{V,W}$ and $V$ is an $x$-stable $I$-graded subspace of $V$ such that $V \supset V_{\perp}$ and $\text{wt}(V/V) = \text{wt}(T)$, and we denote by $\mathcal{E}_{V,W}$ the variety of $(x, V, \varphi^{W}, \varphi^{T})$ where $(x, V) \in \mathcal{E}_{V,W}$, $\varphi^{W} : W \to V/V_{\perp}$ is an isomorphism of $\theta$-symmetric $I$-graded vector spaces and $\varphi^{T} : T \to V/V_{\perp}$ is an isomorphism of $I$-graded vector spaces. We define $p_{1}, p_{2}$ and $p_{3}$ by $p_{1}(x, V, \varphi^{W}, \varphi^{T}) = (x^{T}, x^{W})$, $p_{2}(x, V, \varphi^{W}, \varphi^{T}) = (x, V)$ and $p_{3}(x, V) = x$. Here the morphism $x^{W}, x^{T}$ are defined by

$$x^{W}_{h} = \varphi^{-1}_{\text{in}(h)} \circ (x|_{V/V_{\perp}})_{h} \circ \varphi_{\text{out}(h)}, \quad x^{T}_{h} = \varphi_{\text{in}(h)}^{-1} \circ (x|_{V/V_{\perp}})_{h} \circ \varphi_{\text{out}(h)}.$$ 

Then $p_{1}$ is smooth with connected fibers, $p_{2}$ is a principal $G_{T} \times G_{W}$-bundle and $p_{3}$ is projective.

For a $G_{T}$-equivariant semisimple object $K_{T} \in \mathcal{D}_{T,W}(\Omega)$ and a $G_{W}$-equivariant semisimple object $K_{W} \in \mathcal{D}_{W}(\Omega)$, there exists a unique semisimple object $K'' \in \mathcal{D}(\mathcal{E}_{V,W})$ satisfying $p_{1}^{*}(K_{T} \otimes K_{W}) = p_{2}^{*}K''$.

**Definition 4.9.** We define $K_{T} \ast K_{W} = p_{3}^{*}(K'') \in \mathcal{D}(\mathcal{E}_{V,W})$.

Next, we fix an $I$-graded vector space $U$ such that $V \supset U \supset V_{\perp} \supset \{0\}$.

We also fix an isomorphism $W \cong U/U_{\perp}$ as $\theta$-symmetric $I$-graded vector spaces and an isomorphism $T \cong V/U$ as $I$-graded vector spaces. We consider the following diagram

$$\mathcal{E}_{T,W} = \mathcal{E}_{V,W} \leftarrow \mathcal{E}_{W,V} \rightarrow \mathcal{E}_{V,W},$$

where

$$\mathcal{E}_{W,V} = \{x \in \mathcal{E}_{V,W} \mid U \text{ is } x\text{-stable}\}.$$

We have $p(x) = (x^{T}, x^{W}), \iota(x) = x$.

**Definition 4.10.** For $K \in \mathcal{D}(\mathcal{E}_{V,W})$, we define $\text{Res}_{T,W}(K) := p_{3}^{*}(K)$.

**Proposition 4.11.** Let $V$ and $W$ be $\theta$-symmetric $I$-graded vector spaces such that $\text{wt} V = \text{wt} W + \alpha_{i} + \alpha_{\theta(i)}$. For $a \in \mathbb{Z}_{\geq 0}$, let $S_{i}^{a}$ be an $I$-graded vector space such that $\text{wt}(S_{i}^{a}) = a\alpha_{i}$.

(i) Suppose $\mathcal{L} \in \mathcal{D}(\mathcal{E}_{V,W})$. We have

$$1_{S_{i}^{a}}(\mathcal{L}) = \mathcal{L}(i,i,\theta(i)),(a,a,a).$$

for $a \in \mathbb{Z}_{\geq 0}$.

(ii) Suppose $\mathcal{L} \in \mathcal{D}(\mathcal{E}_{V,W})$ and $a_{i} > 0$ for all $i$ such that $i_{\ell} = i$. For $1 \leq k \leq 2m$ such that $i_{k} = i$, we define $a^{(k)} = (a_{1}^{(k)}, \cdots, a_{2m}^{(k)})$ by $a_{l}^{(k)} = a_{l} - \delta_{l,k} - \delta_{l,2m-k+1}$ and we set

$$M_{i,k}(i,a^{(k)}) = \sum_{l \leq i, k < l < \ell \in \text{out}(h) = i_{\text{in}(h)} = i_{k}} a_{l}^{(k)}.$$

Then we have

$$\text{Res}_{T,W}(\mathcal{L}) = \bigoplus_{i_{\ell} \in \mathbb{Z}_{\geq 0}} \mathcal{L}(i,a^{(k)})(-2M_{i,k}(i,a^{(k)}))$$

**Lemma 4.12.** Let $T^{1}$ and $T^{2}$ be $I$-graded vector spaces. Let $W$ and $V$ be $\theta$-symmetric $I$-graded vector spaces such that $\text{wt} V = \text{wt} T^{1} + \theta(\text{wt} T^{1}) + \text{wt} T^{2} + \theta(\text{wt} T^{2}) + \text{wt} W$.

For $G_{T}$-equivariant semisimple objects $L_{j} \in \mathcal{D}(\mathcal{E}_{T,W}(\Omega)) (j = 1, 2)$ and a $G_{W}$-equivariant semisimple object $L \in \mathcal{D}(\mathcal{E}_{W,V}(\Omega))$, we have $(L_{1} \ast L_{2}) \ast L \cong L_{1} \ast (L_{2} \ast L)$. Here, $L_{1} \ast L_{2}$ is the Lusztig's multiplication defined in Section 3.
4.4. Restriction functor $E_i$, Induction functors $F_i$ and $F_i^{(a)}$

We consider the following diagram

$$E_{T,\Omega} \times \theta E_{W,\Omega} \xleftarrow{p_1} \theta E_{\Omega} \xrightarrow{p_2} \theta E_{\Omega} \xrightarrow{p_3} \theta E_{V,\Omega}.$$ 

Lemma 4.13. Suppose $T = S_i$. Let $d_{p_1}$ and $d_{p_2}$ be the dimension of the fibers of $p_1$ and $p_2$, respectively. The we have

$$d_{p_1} - d_{p_2} = \dim \theta E_{\Omega} - \dim \theta E_{W,\Omega} = \dim W_i + \sum_{h \in \Omega : \text{out}(h) = i} \dim W_{\text{in}(h)}.$$ 

Definition 4.14.

(i) For $T = S_i$ and a $G_W$-equivariant semisimple object $K$ in $\mathcal{D}_{W,\Omega}$, we define the operator $F_i$ by

$$F_i(K) = (1_{S_i} \ast K)[d_{F_i}].$$

where

$$d_{F_i} = d_{p_1} - d_{p_2} = \dim W_i + \sum_{h \in \Omega : \text{out}(h) = i} \dim W_{\text{in}(h)}.$$ 

(ii) For $T = S_i$, we define the functor $E_i : \mathcal{D}(\theta E_{V,\Omega}) \to \mathcal{D}(\theta E_{W,\Omega})$ by

$$E_i(K) = \text{Res}_{S_i, W}(K)[d_{E_i}].$$

where

$$d_{E_i} = d_{F_i} - 2\dim W_i = -\dim W_i + \sum_{h \in \Omega : \text{out}(h) = i} \dim W_{\text{in}(h)}.$$ 

By Proposition 4.11, $E_i$ and $F_i$ induce the restriction functor $\mathcal{D}_{V,\Omega} \to \mathcal{D}_{W,\Omega}$, induction functor $\mathcal{D}_{W,\Omega} \to \mathcal{D}_{V,\Omega}$, respectively.

Definition 4.15. For $a \in \mathbb{Z}_{>0}$, let $W$ and $V$ be $\theta$-symmetric $I$-graded vector spaces such that $\text{wt}(V) = \text{wt}(W) + a(\alpha_i + \alpha_{\theta(i)})$. For a $G_W$-equivariant semisimple object $L \in \mathcal{D}_{W,\Omega}$, we define $F_i^{(a)}(L) = 1_{S_i} \ast L[d_a]$ where

$$d_a = a \left( \dim W_i + \sum_{h \in \Omega : \text{out}(h) = i} \dim W_{\text{in}(h)} \right) + \frac{a(a-1)}{2} \# \{ h \in \Omega : \text{out}(h) = i, \text{in}(h) = \theta(i) \}.$$ 

We call $F_i^{(a)}$ the $a$-th divided power of $F_i$.

By Proposition 4.11(1), we have the following lemma.

Lemma 4.16. The object $L_{1,a,\Omega}$ is isomorphic to $F_i^{(a_1)} F_i^{(a_2)} \cdots F_i^{(a_m)} 1_{pt}$ up to shift.

Lemma 4.17. The operator $F_i^{(a)}$ gives a functor $\mathcal{D}_{W,\Omega} \to \mathcal{D}_{V,\Omega}$ and satisfy $F_i F_i^{(a)} = F_i^{(a)} F_i = [a + 1]_i F_i^{(a+1)}$.

4.5. Commutativity with Fourier transforms

For two $\theta$-orientations $\Omega$ and $\Omega'$, we have $\Omega \setminus \Omega' = \Omega' \setminus \Omega$. Then we can regard $\theta E_{V,\Omega} \to \theta E_{V,\Omega' \cap \Omega}$ and $\theta E_{V,\Omega'} \to \theta E_{V,\Omega \cap \Omega'}$ as vector bundles and they are the dual vector bundle to each other by the form $\sum_{h \in \Omega \cap \Omega'} \text{tr}(h \otimes h')$ on $\theta E_{V,\Omega} \times \theta E_{V,\Omega'}$. We say that $L \in \mathcal{D}(\theta E_{V,\Omega})$ is $(C^\times)^{\Omega,\delta}$-monodromic if $H^j(L)$ is locally constant on every $(C^\times)^{\Omega,\delta}$-orbit on $\theta E_{V,\Omega}$. Let $\mathcal{D}(C^\times)^{\Omega,\delta}$-mono($\theta E_{V,\Omega})$ be the full subcategory of $\mathcal{D}(\theta E_{V,\Omega})$ consisting of $(C^\times)^{\Omega,\delta}$-monodromic objects. Hence we have the Fourier transform

$$\Phi^\Omega_{\Omega'} : \mathcal{D}(C^\times)^{\Omega,\delta}$-mono($\theta E_{V,\Omega}) \to \mathcal{D}(C^\times)^{\Omega,\delta}$-mono($\theta E_{V,\Omega'}).$$

The following lemma is obvious.
Lemma 4.18. For three $\theta$-orientations $\Omega, \Omega'$ and $\Omega''$, we have
\[ \Phi_{V}^{\Omega''} \circ \Phi_{V}^{\Omega} \cong a^{*} \circ \Phi_{V}^{\Omega''} : D_{(C^{x})^{n,s}_{-\text{mon}}}(E_{V}, \Omega) \to D_{(C^{x})^{n,s}_{-\text{mon}}}(E_{V}, \Omega'') \]
where $a : E_{V, \Omega''} \to E_{V, \Omega'}$ is defined by $x_{h} \mapsto -x_{h}$ or $x_{h}$ according that $h \in \Omega' \cap \Omega'' \cap \Omega$ or not. In particular, $D_{(C^{x})^{n,s}_{-\text{mon}}}(E_{V}, \Omega)$ does not depend on $\Omega$.

Since any object in $\mathcal{D}_{(C^{x})^{n,s}_{-\text{mon}}}(E_{V}, \Omega)$ is $\theta$-$G_{V} \times (C^{x})^{\Omega, \theta}$-equivariant, it is a monodromic object. By the commutativity between $E_{i}$, $F_{i}$ and $(C^{x})^{\Omega, \theta}$-action, the functors $E_{i}$ and $F_{i}$ preserve the category $(C^{x})^{\Omega, \theta}$-monodromic objects.

**Theorem 4.19.** Let $V$ and $W$ be $\theta$-symmetric 1-graded vector spaces such that $\text{wt} V = \text{wt} W + \alpha_{i} + \alpha_{\theta(i)}$, and $\Omega$ and $\Omega'$ be two $\theta$-symmetric orientations.

1. Let $F_{i}^{\Omega}$ and $F_{i}^{\Omega'}$ be the induction functors with respect to $\Omega$ and $\Omega'$, respectively. For a $\theta$-$G_{W}$-equivariant semisimple object $L \in \mathcal{D}_{(C^{x})^{n,s}_{-\text{mon}}}(E_{V}, \Omega)$ we have $\Phi_{V}^{\Omega'} \circ F_{i}^{\Omega}(L) \cong F_{i}^{\Omega'} \circ \Phi_{V}^{\Omega}(L)$.

2. Let $E_{i}^{\Omega}$ and $E_{i}^{\Omega'}$ be the restriction functors with respect to $\Omega$ and $\Omega'$, respectively. For a $\theta$-$G_{V}$-equivariant semisimple object $K \in \mathcal{D}_{(C^{x})^{n,s}_{-\text{mon}}}(E_{V}, \Omega)$ we have $\Phi_{W}^{\Omega'} \circ E_{i}^{\Omega}(K) \cong E_{i}^{\Omega'} \circ \Phi_{V}^{\Omega'}(K)$.

3. The Fourier transform $\Phi_{V}^{\Omega'}$ gives an isomorphism between $\mathcal{D}_{(C^{x})^{n,s}_{-\text{mon}}}(E_{V}, \Omega)$ and $\mathcal{D}_{(C^{x})^{n,s}_{-\text{mon}}}(E_{V}, \Omega')$.

Similarly, we can prove the commutativity of $F_{i}^{(a)}$'s and the Fourier transforms.

**Proposition 4.20.** Let $W$ and $V$ be $\theta$-symmetric 1-graded vector spaces such that $\text{wt}(V) = \text{wt}(W) + a(\alpha_{i} + \alpha_{\theta(i)})$. Let $F_{i}^{(a)\Omega}$ and $F_{i}^{(a)\Omega'}$ be the $a$-th divided powers with respect to $\theta$-orientations $\Omega$ and $\Omega'$, respectively. For a $\theta$-$G_{W}$-equivariant semisimple object $L \in \mathcal{D}_{(C^{x})^{n,s}_{-\text{mon}}}(E_{V}, \Omega)$ we have $\Phi_{V}^{\Omega'} \circ F_{i}^{(a)\Omega}(L) \cong F_{i}^{(a)\Omega'} \circ \Phi_{V}^{\Omega'}(L)$.

5. A quiver construction of symmetric crystals

5.1. Grothendieck group

For a $\theta$-orientation $\Omega$ and a $\theta$-symmetric and 1-graded vector space $V$, we define $\theta K_{V, \Omega}$ as the Grothendieck group of $\mathcal{D}_{V, \Omega}$. Namely $\theta K_{V, \Omega}$ is generated by $(L)$ for $L \in \mathcal{D}_{V, \Omega}$ with the relation $(L) = (L') + (L'')$ when $L \cong L' \oplus L''$. This is a $Z[v, v^{-1}]$-module by $v(L) = (L[1])$ and $v^{-1}(L) = (L[-1])$ for $L \in \mathcal{D}_{V, \Omega}$. Hence, $\theta K_{V, \Omega}$ is a free $Z[v, v^{-1}]$-module with a basis $\{(L) \mid L \in \mathcal{D}_{V, \Omega}\}$. For another $\theta$-symmetric and 1-graded vector space $V'$ such that $\text{wt} V = \text{wt} V'$, we have $\theta K_{V, \Omega} \cong \theta K_{V', \Omega'}$. We define
\[ \theta K_{\Omega} := \bigoplus_{V} \theta K_{V, \Omega} \]
where $V$ runs over the isomorphism classes of $\theta$-symmetric 1-graded vector spaces. For two $\theta$-orientations $\Omega$ and $\Omega'$, the Fourier transform induces an equivalence $\mathcal{D}_{V, \Omega} \to \mathcal{D}_{V, \Omega'}$ and the isomorphism $\theta K_{V, \Omega} \cong \theta K_{V, \Omega'}$. Therefore $\theta K_{\Omega} \cong \theta K_{\Omega'}$.

We set $\theta K = \theta K_{\Omega}$, $\mathcal{D}_{\theta} = \mathcal{D}_{V, \Omega}$. By Lemma 4.18, they are well-defined.

5.2. Actions of $E_{i}$ and $F_{i}$

The functors $E_{i}$ and $F_{i}^{(a)}$ induce the action on $\theta K$. Since $E_{i}$ and $F_{i}$ commute with the Fourier transforms, they also act on $\theta K$. The submodule $\theta K' := \sum_{i,s} Z[v, v^{-1}](\theta L_{i,s, \Omega}) \subset \theta K$ is stable by $E_{i}$ and $F_{i}$ by Proposition 4.11. We define
\[ T_{i}|_{\theta K_{V, \Omega}} = v^{-(\alpha_{i}, \text{wt} V)}|_{\theta K_{V, \Omega}}. \]

**Proposition 5.1.** The operators $E_{i}$, $F_{i}$ and $T_{i}$ (i $\in I$) regarded as operators on $\theta K'$ satisfy
\[ E_{i}F_{j} - v^{-(\alpha_{i}, \alpha_{j})}F_{j}E_{i} = \delta_{ij} + \delta_{\theta(i), j}T_{i}, \]
and
\[ T_{i}E_{j}T_{i}^{-1} = v^{(\alpha_{i}, \alpha_{j} + \delta_{\theta(i), j} \alpha_{j})}E_{j}, \quad T_{i}F_{j}T_{i}^{-1} = v^{(\alpha_{i}, \alpha_{j} - \delta_{\theta(i), j} \alpha_{j})}F_{j}. \]
5.3. Key estimates of coefficients

Let $\Omega$ be a $\theta$-orientation and suppose that a vertex $i$ is a sink. For a $\theta$-symmetric $I$-graded vector space $V$ and $r \in \mathbb{Z}_{\geq 0}$, we define

$$\theta^{i}E_{V,\Omega,r} := \left\{ x \in \theta^{i}E_{V,\Omega} \mid \dim \ker \left( \bigoplus_{h \in \Omega, \text{in}(h) = i} V_{\text{out}(h)} \to V_{i} \right) = r \right\}.$$ 

Then we have $\theta^{i}E_{V,\Omega} = \bigcup_{r \geq 0} \theta^{i}E_{V,\Omega,r}$, and $\theta^{i}E_{V,\Omega,r} := \bigcup_{r \geq r'} \theta^{i}E_{V,\Omega,r'}$ is a closed subset of $\theta^{i}E_{V,\Omega}$.

**Definition 5.2.** For $L \in \theta^{i}P$ and $i \in I$, choose a $\theta$-orientation $\Omega$ such that $i$ is a sink with respect to $\Omega$, and regard $L$ as an element of $\theta^{i}P$. We define $\epsilon_{i}(L)$ as the largest integer $r$ satisfying $\text{Supp}(L) \subset \theta^{i}E_{V,\Omega,r}$. This does not depend on the choice of $\Omega$.

Note that $0 \leq \epsilon_{i}(L) \leq \dim V_{i}$.

We shall prove the following key estimates with respect to $F_{i}(L)$ and $E_{i}(L)$.

**Theorem 5.3.** Assume that $\theta$-symmetric and $I$-graded vector spaces $V$ and $W$ satisfy $\text{wt} V = \text{wt} W + \alpha_{i} + \alpha_{\theta(i)}$. Fix a $\theta$-orientation $\Omega$ such that the vertex $i$ is a sink.

1. For $L \in \theta^{i}P$, there exists a unique simple perverse sheaf $L_{0} \in \theta^{i}P$ such that $\epsilon_{i}(L_{0}) = \epsilon_{i}(L) + 1$ and

$$F_{i}(L) = \left[ \epsilon_{i}(L) + 1 \right]_{\text{wt}}(L_{0}) + \sum_{L' \in \theta^{i}P : \epsilon_{i}(L') > \epsilon_{i}(L) + 1} a_{L'}(L')$$

for $a_{L'} \in \mathbb{Z}[V_{i}]$. We define the map $\tilde{F}_{i} : \theta^{i}P \cong \theta^{i}P \to \theta^{i}P$ by $\tilde{F}_{i}(L) = L_{0}$. It does not depend on the choice of $\Omega$.

2. Let $K \in \theta^{i}P$. If $\epsilon_{i}(K) > 0$, there exists a unique simple perverse sheaf $K_{0} \in \theta^{i}P$ such that $\epsilon_{i}(K_{0}) = \epsilon_{i}(K) - 1$ and

$$E_{i}(K) = v^{1-\epsilon_{i}(K)(K_{0})} + \sum_{K' \in \theta^{i}P : \epsilon_{i}(K') > \epsilon_{i}(K) - 1} b_{K'}(K')$$

for $b_{K'} \in \mathbb{Z}[V_{i}]$. Here we regard $K_{0} = 0$ if $\epsilon_{i}(K) = 0$.

We define the map $\tilde{E}_{i} : \theta^{i}P \cong \theta^{i}P \to \theta^{i}P \cup \{0\} \cong \theta^{i}P \cup \{0\}$ by $\tilde{E}_{i}(K) = K_{0}$ if $\epsilon_{i}(K) > 0$ and $\tilde{E}_{i}(K) = 0$ if $\epsilon_{i}(K) = 0$. It does not depend on the choice of $\Omega$.

**Lemma 5.4.** Suppose $\text{wt} V \neq 0$. For any $L \in \theta^{i}P$, there exists $i \in I$ such that $\epsilon_{i}(L) > 0$.

**Proof.** If $V \neq \{0\}$, there exists an integer $d$, $i = (i_{1}, \ldots, i_{2m})$ and a such that $L[d]$ appears in a direct summand of $\theta^{i}L_{1,i}.\Omega$. We may assume $a_{1} > 0$. Then, taking $\Omega$ such that $i_{1}$ is a sink, we have $\text{Supp}(L) \subset \text{Supp}(\theta^{i}L_{1,i}.\Omega) \subset \theta^{i}E_{V,\Omega,r}$. By the definition of $\epsilon_{i}$, we have $\epsilon_{i_{1}}(L) \neq 0$. \hfill $\square$

**Lemma 5.5.** For $L \in \theta^{i}P$, we have $\tilde{E}_{i} \tilde{F}_{i}(L) = (L)$, and if $\tilde{E}_{i}(L) \neq 0$, we have $\tilde{F}_{i} \tilde{E}_{i}(L) = L$.

5.4. Verdier duality functor

The Verdier duality functor $D : \mathcal{D}(\theta^{i}E_{V,\Omega}) \to \mathcal{D}(\theta^{i}E_{V,\Omega})$ satisfies $D(L[d]) = D(L)[-d]$ for $L \in \mathcal{D}(\theta^{i}E_{V,\Omega})$, $d \in \mathbb{Z}$. Then $D$ induces the involution $v^{\pm 1} \mapsto v^{\mp 1}$.

**Proposition 5.6.**

(i) $D(\theta^{i}L_{1,i}.\Omega) = \theta^{i}L_{1,i}.\Omega[2 \dim \theta^{i}F_{i}L_{1,i.\Omega}]$.

(ii) For any $L \in \theta^{i}P$, we have $D(F_{i}L) = F_{i}D(L)$.

(iii) For any $L \in \theta^{i}P$, we have $D(L) \cong L$. 

Proof. (i) and (ii) follow from the general property of the Verdier duality functor. To prove (iii), we use the induction on \( wt V \).

When \( wt V = 0 \), the claim is clear by \( \mathcal{P}_{1,V,0} = \{1_{pt}\} \) and \( D(1_{pt}) = 1_{pt} \).

Suppose \( wt V \neq 0 \). By Lemma 5.4, there exists \( i \) such that \( \varepsilon_i(L) > 0 \). We shall prove \( D(L) = L \) by the descending induction on \( \varepsilon_i(L) \). By Theorem 5.3 and Lemma 5.5, we have

\[
F_i(\tilde{E}_i L) = [\varepsilon_i(L)]_v(L) + \sum_{L' \in \mathcal{P}_{V,0} : \varepsilon_i(L') > \varepsilon_i(L)} a_{L'}(L').
\]

By the induction hypothesis on \( wt V \), \( D(\tilde{E}_i L) = \tilde{E}_i L \). Hence the lefthand side is \( D \)-invariant by (ii). We restrict \( F_i(\tilde{E}_i L) \) on the open subset \( \theta \mathcal{P}_{V,0,\leq \varepsilon_i(L)} \). When \( \varepsilon_i(L) = 0 \), the claim is clear by \( \theta \mathcal{P}_{V,0} = \{1_{pt}\} \) and \( D(1_{pt}) = 1_{pt} \).

Suppose \( \varepsilon_i(L) \neq 0 \). By Lemma 5.4, there exists \( i \) such that \( \epsilon_i(L) > 0 \).

We shall prove \( D(L) = L \) by the descending induction on \( \varepsilon_i(L) \).

By Theorem 5.3 and Lemma 5.5, we have

\[
F_i(\tilde{E}_i L) = \left[ \varepsilon_i(L) + a \right]_v(\tilde{F}_i^a L) + \sum_{L' : \varepsilon_i(L') > \varepsilon_i(L) + a} c_{L'}(L').
\]

Thus \( L \in \mathcal{P}_{V,0,\leq \varepsilon_i(L)} \) is \( D \)-invariant.

\( \square \)

Remark 5.7. By the result of (iii), we have \( a_{L'}(v) = a_{L'}(v^{-1}) \) in Theorem 5.3 (1).

Lemma 5.8. For \( L \in \mathcal{P}_{V,\Omega} \), we have

\[
F_i^{(a)}(L) = \left[ \varepsilon_i(L) + a \right]_v(\tilde{F}_i^a L) + \sum_{L' : \varepsilon_i(L') > \varepsilon_i(L) + a} c_{L'}(L')
\]

with \( c_{L'} \in \mathbb{Z}[v,v^{-1}] \).

Proof. We shall prove the claim by the induction on \( a \). If \( a = 1 \), the claim follows from Theorem 5.3. If \( a > 1 \), by the induction hypothesis and Theorem 5.3, we have

\[
F_i F_i^{(a)}(L) = \left[ \varepsilon_i(L) + a + 1 \right]_v(\tilde{F}_i^{a+1} L) + \sum_{L'' : \varepsilon_i(L'') > \varepsilon_i(L) + a + 1} d_{L''}(L'').
\]

On the other hand, since \( F_i^{(a+1)} L = 1_{S_{l_i}^{a+1}} \ast L \in \mathbb{Z}[v,v^{-1}] \), we conclude \( d_{L''} \in \mathbb{Z}[v,v^{-1}] \).

\( \square \)

Proposition 5.9. We have \( ^gK = \sum \mathbb{Z}[v,v^{-1}] F_{i_1}^{(a_1)} \cdots F_{i_k}^{(a_k)} 1_{\{pt\}} \).

Proof. For \( L \in \mathcal{P}_{V,\Omega} \) such that \( wt V \neq 0 \), there exists \( i \) such that \( \varepsilon_i(L) > 0 \). We shall prove that \( L \) is contained in \( \sum \mathbb{Z}[v,v^{-1}] F_{i_1}^{(a_1)} \cdots F_{i_k}^{(a_k)} 1_{\{pt\}} \) by the induction on \( wt V \) and the descending induction on \( \varepsilon_i(L) \). We have

\[
F_i^{(\varepsilon_i(L))} (\tilde{E}_i^{\varepsilon_i(L)} L) = (L) + \sum_{L' \in \mathcal{P}_{V,0} : \varepsilon_i(L') \geq \varepsilon_i(L)} c_{L'}(L').
\]

by Lemma 5.8 and Lemma 5.5. By the induction hypothesis, we have \( c_{L'}(L') \) and \( \tilde{E}_i^{\varepsilon_i(L)} L \) are contained in \( \sum \mathbb{Z}[v,v^{-1}] F_{i_1}^{(a_1)} \cdots F_{i_k}^{(a_k)} 1_{\{pt\}} \).

Thus \( L \in \sum \mathbb{Z}[v,v^{-1}] F_{i_1}^{(a_1)} \cdots F_{i_k}^{(a_k)} 1_{\{pt\}} \).

\( \square \)

5.5. Main Theorem

Let us recall

\[
^gK' = \sum_{(i,a)} \mathbb{Z}[v,v^{-1}] F_{i_1}^{(a_1)} \cdots F_{i_k}^{(a_k)} 1_{\{pt\}} \subset ^gK.
\]

Theorem 5.10.
(i) \( K = K' \).

(ii) For \( L \in \mathfrak{P}_V \), we define \( \text{wt}(L) = - \text{wt} V \). Then \((\omega, \tilde{E}_i, \tilde{F}_i, \epsilon_i)\) gives a crystal structure on \( \mathfrak{P}_V \) in the sense of section 2.3. Here \( V \) runs over all isomorphism classes of \( \theta \)-symmetric \( I \)-graded vector spaces.

(iii) Let \( \mathcal{L} \) be the \( A_0 \)-submodule \( \sum_{(L) \in \mathfrak{P}_V} A_0 \mathfrak{P}(L) / \theta K \). Then \( (\{L\} \mod v L) / \theta \mathfrak{P}_V \) gives a crystal basis of \( \theta K \). Especially, the actions of modified root operators \( \tilde{E}_i \) and \( \tilde{F}_i \) on \( \mathcal{L} / v \mathcal{L} \) are compatible with the actions of \( \tilde{E}_i \) and \( \tilde{F}_i \) on \( \mathfrak{P}_V \) introduced in Theorem 5.3.

Proof. (i) is nothing but Proposition 5.9.

(ii) By the definition of \( \epsilon_i \), \( \tilde{E}_i \) and \( \tilde{F}_i \), and Lemma 5.5, we conclude that \((\omega, \tilde{E}_i, \tilde{F}_i, \epsilon_i)\) gives a crystal structure on \( \mathfrak{P}_V \) in the sense of section 2.3(i)-(iv). By the estimates in Theorem 5.3, the actions of \( \tilde{E}_i \) and \( \tilde{F}_i \) on \( \{L\} \mod v L \) satisfy the conditions (2)-(7) in section 2.3. Thus we obtain the claim.

(iii) follows from Theorem 2.14.

Lemma 5.11. We have \( \{v \in \theta K \mid \text{wt} V = 0 \} \) for any \( i \in I \) is \( Z[v, v^{-1}]1_{\{pt\}} \).

Proof. Suppose that \( E_i(\sum c_i(L)) = 0 \) for any \( L \). Then \( \text{wt} V = c_i v^{-c_i} \in Z[v] \) for some \( c_i \). Put \( \tilde{a}_c = v^{-c} c_i \in Z[v] \). Specializing \( v \) to 0, we have \( \tilde{a}_c(0) = 0 \) if \( E_i \neq 0 \). But for any \( L \) such that \( \text{wt} V \neq 0 \), there exists \( i \in I \) such that \( \epsilon_i(L) > 0 \). Hence we obtain \( \tilde{a}_c(0) = 0 \) and hence \( a_i 
eq 0 \in Z[v] \). By the induction on \( c \), we have \( a_i \neq 0 \) for any \( c \). Thus we conclude \( \text{wt} V = 0 \neq 0 \).

Theorem 5.12.

(i) \( \theta K \otimes_{Z[v, v^{-1}]} \mathfrak{Q}(v) \cong V_\theta(0) \) as a \( B_\theta(g) \)-module. The involution induced by the Verdier duality functor coincides with the bar involution on \( V_\theta(0) \).

(ii) \( \{L \mid L \in \theta \mathfrak{P}_V \} \) gives the lower global basis on \( V_\theta(0) \).

Proof. (i) By Proposition 5.1, to check the defining relations of \( B_\theta(g) \), we only need to prove the \( v \)-Serre relations. Put \( S_e = \sum_{k=0}^{b} (-1)^{k} E_i^{(k)} E_j E_i^{(b-k)} \), \( S_f = \sum_{k=0}^{b} (-1)^{k} F_i^{(k)} F_j F_i^{(b-k)} \)

and note that \( F_k S_e = S_e F_k \) and \( E_k S_f = S_f E_k \) for any \( k \in I \).

Since \( \theta K_0 \) is generated by \( F_k^{(n)} \) from \( \phi = 1_{\{pt\}} \) and \( S_e \phi = 0 \), we have \( S_e v = 0 \) for any \( v = \theta K_0 \). We show \( S_e(L) = 0 \) for any \( L \in \theta \mathfrak{P}_V \) by the induction on \( \text{wt} V \). If \( \text{wt}(S_i(L)) \neq 0 \), we have \( E_k S_f(L) = S_f E_k(L) \) for any \( k \in I \) by applying the induction hypothesis to \( E_k(L) \). Since \( \text{wt}(S_i(L)) \neq 0 \), we have \( S_f(L) = 0 \) by Lemma 5.11. Hence \( \theta K \) is a \( B_\theta(g) \)-module. Note that \( T_1 = 1_{\{pt\}} \) for any \( i \in I \). We conclude \( \theta K \cong V_\theta(0) \) by Lemma 5.11 and the characterization of \( V_\theta(0) \) in Proposition 2.10.

(ii) We already know that \( \mathcal{L} \cap \mathfrak{P}_V \) is a crystal lattice and \( \{L\} \mod v L \) is the basis of \( \mathcal{L} / v L \). Note that \( \sum_{L \in \mathfrak{P}_V} Z[v, v^{-1}]_{\mathcal{L}}(L) \) is stable under the actions of \( E_k^{(n)} \) and \( F_k^{(n)} \) by Lemma 5.8 and \( L \) is \( D \)-invariant, namely bar-invariant. Moreover \( \{(L) \mid L \in \theta \mathfrak{P}_V \} \) is a basis of the \( A_0 \)-module \( \mathcal{L} \) and also a basis of the \( Z[v, v^{-1}] \)-module \( \theta K \). Hence we conclude \( \{L\} \mod v L \) gives the lower global basis on \( V_\theta(0) \).

Corollary 5.13. For any Kac-Moody algebra \( g \) with a symmetric Cartan matrix, the \( B_\theta(g) \)-module \( V_\theta(0) \) has a crystal basis and a lower global basis, namely Conjecture 2.11 and Conjecture 2.12 is true if \( \lambda = 0 \).

Example 5.14. Let us consider the case \( g = sl_2, I = \{ \pm 1 \} \) and \( \theta(\pm 1) = -1 \). Fix a \( \theta \)-symmetric orientation \( -1 \rightarrow 1 \). For a \( \theta \)-symmetric \( I \)-graded vector space \( V \) such that \( \text{wt}(V) = n(\alpha_{-1} + \alpha_1) \), \( \theta V_{\alpha} \) is the set of skew symmetric matrix \( x \) of size \( n \). Its \( \theta V_{\alpha} \)-orbits are parametrized by the rank \( 2r \) \( (0 \leq r \leq \lfloor \frac{n}{2} \rfloor ) \) of \( x \). We denote \( O^\theta_{2r} \) by the orbit consisting of \( n \times n \) skew symmetric matrices \( x \) of rank \( 2r \). Note that any \( \theta V_{\alpha} \)-equivariant simple local system on each \( \theta V_{\alpha} \)-orbit is trivial. Let us
denote $IC^n_r$ by the simple perverse sheaves corresponding to the orbit $O^n_r$. Note that $\varepsilon_1(IC^n_r) = n - 2r$. Let $W$ be a $\theta$-symmetric $I$-graded vector space such that $\text{wt}(W) = (n-1)(\alpha_{-1} + \alpha_1)$. We consider the diagram:

$$\theta E_{W, \Omega} \xrightarrow{p_1} \theta E'_{\Omega} \xrightarrow{p_2} \theta E''_{\Omega} \xrightarrow{p_3} \theta E_{V, \Omega}.$$ 

Note that the fibers of $p_3$ on $O^n_r$ is isomorphic to $\mathbb{P}^{n-1-2r}$. Then

$$F_1(IC^n_{r-1}) = [n-2r]_v(IC^n_r) + \sum_{k=0}^{r-1} a_k(IC^n_k)$$

where $a_k \in v^{2-n+2k}\mathbb{Z}[v]$. We obtain the crystal graph:

$$\begin{array}{ccccccc}
IC_0 & -1 & IC_0 & -1 & IC_3 & 1 & IC_6 & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & \\
IC_2 & -1 & IC_1 & -1 & IC_4 & -1 & IC_7 & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & \\
IC_1 & -1 & IC_2 & -1 & IC_5 & -1 & IC_8 & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & \\
IC_0 & -1 & IC_0 & -1 & IC_3 & -1 & IC_6 & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & \\
\end{array}$$

In this case, all indecomposable representations are described by

$$C \xrightarrow{0} C \quad \text{and} \quad C^2 \xrightarrow{J} C^2$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We denote $(1)$ and $(-1,1)$ by above indecomposable representations, respectively. Thus we can parametrized $\theta G_V$-orbit in $\theta E_{V, \Omega}$ and associated simple perverse sheaves by $a(1) + b(-1,1)$ ($a, b \in \mathbb{Z}_{\geq 0}$), especially $O^n_r$ (and $IC^n_r$) corresponds to $(n-2r)(1) + r(-1,1)$. Therefore we recover the crystal graph parametrized by "$\theta$-restricted multi-segments" in [6, Example 4.7 (1)].

References


