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A NEW ASPECT OF THE L^p-EXTENSION THEOREM FOR INHOMOGENEOUS DIFFERENTIAL EQUATIONS

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Weak extension problem

Let

M : real manifold of dimension n,

Z : point in M,

P : differential operator of order m with C^∞ coefficient on M .

For a given distribution $f \in \mathcal{D}'(M)$, assume that $u \in \mathcal{D}'(M \setminus Z)$ is a solution of the equation

$$Pu = f \text{ on } M \setminus Z.$$

Our problem is when $u \in \mathcal{D}'(M \setminus Z)$ can be extended to $\tilde{u} \in \mathcal{D}'(M)$ as a solution of the same equation

$$P\tilde{u} = f \text{ on } M.$$

L^p-category

Let $1 < p < \infty$, and we shall consider this weak extension problem in the L^p-category.

We identify

$$M = \mathbf{R}^n \text{ and } Z = \text{the origin of } \mathbf{R}^n.$$

Without loss of generality, we may assume that all the derivatives of coefficients of P are bounded and the inhomogeneous term f is a tempered distribution, that is, $f \in \mathcal{S}'$.

For $u \in L^p(M \setminus Z)$, $\tilde{u} \in L^p(M)$ always denotes its trivial L^p-extension:

$$\tilde{u}(x) = \begin{cases} u(x) & x \in M \setminus Z \\ 0 & x = Z \end{cases}$$

Homogeneous case

The following answer to this problem with the homogeneous case $f = 0$ is given by Bochner [1] (1956).

Theorem A. If $u \in L^p(M \setminus Z)$, $m \leq n(1 - 1/p)$, and $Pu = 0$ on $M \setminus Z$, then $P\tilde{u} = 0$ on M.

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The inequality $m \leq n(1-1/p)$ in Theorem A cannot be removed if we take account of the fundamental solution of an elliptic differential operator P.

In fact, if P has analytic coefficients, then

$$u(x) = |x|^{m-n}\{A(x) + B(x)\log|x|\}$$

solves $P\tilde{u} = \delta$ on M and $Pu = 0$ on $M \setminus Z$ with some $A(x)$, $B(x)$ bounded in a neighborhood of Z by John [3] (1950). We remark that u belongs to L^p (locally) in the neighborhood for $m > n(1-1/p)$.

Inhomogeneous case

When is the same true for inhomogeneous equations?

We say that the weak extension in the L^p-category holds for a given $f \in \mathcal{S}'$, if

$$(*) \qquad \begin{cases} u \in L^p(M \setminus Z), & m \leq n(1-1/p), \\ Pu = f \quad \text{on} \quad M \setminus Z \end{cases}$$
$$\Longrightarrow \quad P\tilde{u} = f \quad \text{on} \quad M.$$

We have a complete answer:

- $f \notin H_p^{-n(1-1/p)}$ & $(*) \Rightarrow P\tilde{u} \neq f$ on M.
- $f \in H_p^{-n(1-1/p)}$ & $(*) \Rightarrow P\tilde{u} = f$ on M. $\cdots$ S. [6] (2001).

(We do not care about the existence of P and u which satisfy $(*)$.)

But we know more useful criteria which can be easily checked. The weak extension in the L^p-category holds for the following $f \in \mathcal{S}'$:

- $f \in L^1$ $\cdots$ Bochner [1] (1956).
  i.e. $f \in L^1$ & $(*) \Rightarrow P\tilde{u} = f$ on M.
- $f \in L^1$ (microlocally) $\cdots$ S.-Uchida [7] (2000).

We remark

- $f \in L^1 \not\Rightarrow f \in H_p^{-n(1-1/p)}$.
- $f \in L^1$ & $(*) \Rightarrow f \in H_p^{-n(1-1/p)}$.

Example & Question

Let $h(x) \in L^1(\mathbf{R}^n)$ and $g(x') \in \mathcal{S}'(\mathbf{R}^{n-1})$, where $x = (x_1, x')$, $x' = (x_2, \ldots, x_n)$.

- $f = h(x) + (x_1 \pm i0)^{-1} \otimes g(x')$
  $\in L^1$ microlocaly.

Does the weak extension in the L^p-category holds for the following f ($\notin L^1$ microlocally)?

- $f = h(x) + \text{p.v.}\,\frac{1}{x_1} \otimes g(x')$.
- $f = h(x) + \delta(x_1) \otimes g(x')$.

We remark

$$\text{p.v.}\,\frac{1}{x_1} = \frac{1}{2}\left(\frac{1}{x_1 + i0} + \frac{1}{x_1 - i0}\right),$$
$$\delta(x_1) = \frac{1}{2\pi i}\left(\frac{1}{x_1 - i0} - \frac{1}{x_1 + i0}\right).$$

The Class $\mathcal{B}_Z$

We introduce a class of inhomogeneous terms: We use the notation

$$a_\varepsilon(x) = a(x/\varepsilon) \qquad \text{for } \varepsilon > 0.$$

Definition 1. Let $f \in \mathcal{S}'$. We say that

$$f \in \mathcal{B}_Z$$

if there exist a strictly decreasing sequence $\{\varepsilon_\nu\}_{\nu=1}^\infty$ of positive numbers and a cutoff function $a(x)$ of Z such that

$$\lim_{\nu\to\infty} a_{\varepsilon_\nu} f = 0 \text{ in } \mathcal{S}'.$$

Theorem 1. *Let $f \in \mathcal{B}_Z$. If $u \in L^p(M \setminus Z)$, $m \leq n(1-1/p)$, and $Pu = f$ on $M \setminus Z$, then $P\tilde{u} = f$ on M.*

We remark

- $L^1 \subset \mathcal{B}_Z$ $\cdots$ Lebesgue' convergence theorem.

For $h(x) \in L^1(\mathbf{R}^n)$ and $g(x') \in \mathcal{S}'(\mathbf{R}^{n-1})$,

- $f = h(x) + \text{p.v.}\frac{1}{x_1} \otimes g(x') \in \mathcal{B}_Z$.

If fact, the argument can be reduced to show

$$\left\langle a_\varepsilon(x_1)\,\text{p.v.}\frac{1}{x_1},\ \varphi(x_1)\right\rangle \to 0 \quad \text{as } \varepsilon \searrow 0$$

for all test function φ of dimension 1. We take a function $a(x_1)$ such that $a(-x_1) = a(x_1)$. Then we have

$$\begin{aligned}&\left\langle a_\varepsilon(x_1)\,\text{p.v.}\frac{1}{x_1},\ \varphi(x_1)\right\rangle\\ &= \lim_{\delta\searrow 0}\int_{|x_1|\geq\delta} \frac{(a_\varepsilon\varphi)(x_1)}{x_1}\,dx_1\\ &= \lim_{\delta\searrow 0}\int_{|x_1|\geq\delta} \frac{a_\varepsilon(x_1)}{x_1}\,dx_1\cdot\varphi(0) + \int (a_\varepsilon H)(x_1)\,dx_1\end{aligned}$$

with H bounded. The first term vanishes since a_ε is an even function, and the second term tends to 0 as $\varepsilon \searrow 0$.

For $h(x) \in L^1(\mathbf{R}^n)$ and $g(x') \in \mathcal{S}'(\mathbf{R}^{n-1})$ such that $g \notin \mathcal{B}_Z$ (of dimension $n-1$)

- $f = h(x) + \delta(x_1) \otimes g(x') \notin \mathcal{B}_Z$.

The proof of Theorem 1 is based on the argument of Bochner:

For test functions φ, we have

$$\begin{aligned}\langle P\tilde{u} - f, \varphi\rangle &= \langle P\tilde{u} - f, a_\varepsilon\varphi\rangle\\ &= \langle \tilde{a}_\varepsilon\tilde{u}, {}^tP(a_\varepsilon\varphi)\rangle - \langle a_\varepsilon f, \varphi\rangle\\ &\to 0,\end{aligned}$$

where ${}^tP$ is the transpose of P and $\tilde{a}(x)$ is another cutoff function which is equal to 1 on the support of $a(x)$. Here we have used the following facts:

- $\lim_{\varepsilon \searrow 0} \|\tilde{a}_\varepsilon \tilde{u}\|_{L^p} = 0$
- $\sup_{\varepsilon > 0} \|{}^t P(a_\varepsilon \varphi)\|_{L^{p^*}} < \infty$
  if $1/p + 1/p^* = 1$ and $m \leq n(1 - 1/p)$.

Hence we get $P\tilde{u} = f$.

The Class $\mathcal{M}$

Here is another class of inhomogeneous terms:

Definition 2. Let $f \in \mathcal{S}'$. We say that

$$f \in \mathcal{M}$$

if $\hat{f}$ is a function which satisfies

$$|\hat{f}(\xi)| \to 0 \qquad (|\xi| \to \infty)$$

uniformly in a direction, that is, for a point on the sphere S^{n-1}, there exists a conic neighborhood Γ such that

$$\sup_{|\xi| > R, \xi \in \Gamma} |\hat{f}(\xi)| \to 0 \quad (R \to \infty).$$

Theorem 2. *Let $f \in \mathcal{M}$. If $u \in L^p(M \setminus Z)$, $m \leq n(1 - 1/p)$, and $Pu = f$ on $M \setminus Z$, then $P\tilde{u} = f$ on M.*

We remark

- $L^1 \subset \mathcal{M} \cdots$ Riemann-Lebesgue's theorem.

For $h(x) \in L^1(\mathbf{R}^n)$, $g(x') \in \mathcal{M}$ (of dimension $n - 1$), and $c^\pm \in \mathbf{C}$,

- $f = h(x) + \text{p. v.} \frac{1}{x_1} \otimes g(x') \in \mathcal{M}$.
- $f = h(x) + \delta(x_1) \otimes g(x') \in \mathcal{M}$.
- $f = \left(\frac{c^+}{x_1 + i0} + \frac{c^-}{x_1 - i0}\right) \otimes g(x') \in \mathcal{M}$.

More generally, for $x = (x_1, x_2, \ldots, x_n)$,

- $f = g_1(x_1) \otimes g_2(x_2) \cdots \otimes g_n(x_n) \in \mathcal{M}$

if at least one of g_j $(j = 1, 2, \ldots, n)$ belongs to $\mathcal{M}$ of dimension 1 and all other g_l is a linear combination of $(t \pm i0)^{-1}$.

Furthermore, if such $g_j(t)$ admits a nice regularity and up to $(k-1)$-th derivatives of it are integrable again, then we have a stronger decaying property

$$\sup_{|t| > R} \left|t^{k-1} \widehat{g_j}(t)\right| \to 0 \qquad (R \to \infty).$$

In this case, linear combinations of more general homogeneous distributions

$$(t \pm i0)^{-1}, (t \pm i0)^{-2}, \ldots, (t \pm i0)^{-k}$$

are allowed for all other $g_l(t)$ since their Fourier transforms are polynomial of order up to $k - 1$ in each direction.

For smooth function $\psi(t)$ of $t \geq 0$, which is equal to 0 for $0 \leq t \leq 1$ and 1 for $t \geq 2$,

• $f = F^{-1}\big[e^{i|\xi|^\gamma}|\xi|^{-n\gamma/2}\psi(|\xi|)\big] \in \mathcal{M}$
$\in H_p^{-n(1-1/p)} \cdots$? (*"Yes"* by S.Sjöstrand [5] 1970)

We can prove $f \notin L^1$ for $0 < \gamma < 1$. In fact, by Ishii [2] (1974), f is of the form

$$f(x) = K(|x|)|x|^{-n} + O(|x|^\omega)$$

as $|x| \to 0$, where $|K(|x|)|$ is a non-zero constant and $\omega > -n$.

The proof of Theorem 2 is based on the argument by S.-Uchida [7]:

If $Pu = f$ on $M \setminus Z$, then we have

$$P\tilde{u} = f + Q(D)\delta \in H_p^{-m}$$

with a polynomial Q by the structure theorem and the mapping property of P.

Furthermore, since $f = \hat{f}(D)\delta$, we have

$$\widetilde{Q}(D)\delta \in H_p^{-m}, \qquad \widetilde{Q}(\xi) = Q(\xi) + \hat{f}(\xi).$$

If the polynomial $Q \neq 0$, then $Q(D)$ is microlocally elliptic in a direction. The same is true for $\widetilde{Q}(D)$ since $\hat{f}(\xi)$ is just a perturbation. Then we have $\delta \in H_p^{-m}$ (microlocally), which implies $m > n(1-1/p)$. Hence $m \leq n(1-1/p)$ yields $Q = 0$ and we can conclude $P\tilde{u} = f$.

The classes $\mathcal{U}_Z^p$ and $\mathcal{V}_Z^p$

We introduce some other classes of inhomogeneous terms:

For $f \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, we set

$$T_f\varphi = (F^{-1}f) * \varphi.$$

We symbolically write $T_f = f(D)$, which can be regarded as the operator from $\mathcal{S}$ to $\mathcal{S}'$. We set

$$M_p = \{f \in \mathcal{S}'; f(D) \in \mathcal{L}(L^p)\},$$
$$\hat{M}_p = \{f \in \mathcal{S}'; \hat{f}(D) \in \mathcal{L}(L^p)\}$$

which are Banach spaces with the norms

$$\|f\|_{M_p} = \|f(D)\|_{\mathcal{L}(L^p)},$$
$$\|f\|_{\hat{M}_p} = \|\hat{f}(D)\|_{\mathcal{L}(L^p)}$$

respectively.

We use the notation

$$a_\varepsilon(x) = a(x/\varepsilon) \qquad \text{for } \varepsilon > 0.$$

Definition 3. Let $f \in \hat{M}_p$. We say that

$$f \in \mathcal{U}_Z^p$$

if there exist a strictly decreasing sequence $\{\varepsilon_\nu\}_{\nu=1}^\infty$ of positive numbers and a cutoff function $a(x)$ of Z such that $a_{\varepsilon_\nu} f \in \hat{M}_p$ $(\nu = 1, 2, \ldots)$ and

$$\lim_{\nu\to\infty} \widehat{a_{\varepsilon_\nu} f}(D) = 0 \text{ in } \mathcal{L}(L^p).$$

Definition 4. Let $f \in \hat{M}_p$. We say that

$$f \in \mathcal{V}_Z^p$$

if there exist a strictly decreasing sequence $\{\varepsilon_\nu\}_{\nu=1}^{\infty}$ of positive numbers and a cutoff function $a(x)$ of Z such that

$$\lim_{\nu \to \infty} a_{\varepsilon_\nu}(X)\hat{f}(D) = 0 \text{ in } \mathcal{L}(L^p).$$

Theorem 3. *Let $f \in \mathcal{U}_Z \cup \mathcal{V}_Z$. If $u \in L^p(M \setminus Z)$, $m \leq n(1-1/p)$, and $Pu = f$ on $M \setminus Z$, then $P\tilde{u} = f$ on M.*

We remark

- $L^1 \subset \mathcal{U}_Z^p$.

In fact, for $f \in L^1$, $\varphi \in \mathcal{S}$, and a cutoff function a of Z, we have

$$\widehat{a_\varepsilon f}(D)\varphi = (a_\varepsilon f) * \varphi.$$

Hence

$$\begin{aligned}\left\|\widehat{a_\varepsilon f}(D)\varphi\right\|_{L^p} &\leq \|(a_\varepsilon f) * \varphi\|_{L^p} \\ &\leq \|a_\varepsilon f\|_{L^1}\|\varphi\|_{L^p}.\end{aligned}$$

Hence we have

$$\left\|\widehat{a_\varepsilon f}(D)\right\|_{\mathcal{L}(L^p)} \leq \|a_\varepsilon f\|_{L^1} \to 0$$

as $\varepsilon \searrow 0$.

For a smooth function $\psi(t)$ of $t \geq 0$, which is equal to 0 for $0 \leq t \leq 1$ and 1 for $t \geq 2$, and for $0 < \gamma < 1$, we have

- $f = F^{-1}\left[e^{i|\xi|^\gamma}|\xi|^{-n\gamma/2}\psi(|\xi|)\right] \in \mathcal{V}_Z^p$

In fact, for $\varphi \in \mathcal{S}$ and a cutoff function a of Z, we have

$$\begin{aligned}\|a_{\varepsilon_\nu}(X)\hat{f}(D)\varphi\|_{L^p} &\leq \|a_{\varepsilon_\nu}\|_{L^r}\|\hat{f}(D)\varphi\|_{L^q} \\ &\leq C\|a_{\varepsilon_\nu}\|_{L^r}\||D|^\alpha\hat{f}(D)\varphi\|_{L^p} \\ &\leq C\|a_{\varepsilon_\nu}\|_{L^r}\|\varphi\|_{L^p},\end{aligned}$$

hence

$$\|a_{\varepsilon_\nu}(X)\hat{f}(D)\|_{\mathcal{L}(L^p)} \leq C\|a_{\varepsilon_\nu}\|_{L^r} \to 0$$

as $\varepsilon \searrow 0$. Here we have used:

* Hölder's inequality with

$$1/p = 1/r + 1/q, \quad 1 < q < \infty,$$

* Hardy-Littlewood-Sobolev inequality with

$$1/q = 1/p - \alpha/n, \quad 0 < \alpha < n,$$

* The L^p-boundedness of $|D|^\alpha\hat{f}(D)$ with

$$n\gamma/2 - \alpha = n\gamma|1/p - 1/2|$$

by Miyachi [4] (1980).

The proof of Theorem 3 is just a repetition of the argument in that of Theorem 2:

If $Pu = f$ on $M \setminus Z$, then we have

$$P\tilde{u} = f + Q(D)\delta \in H_p^{-m}$$

with a polynomial Q. Multiplying $a_\varepsilon(x)$, we have

$$Q(D)\delta + a_\varepsilon f \in H_p^{-m}.$$

Noticing

$$a_\varepsilon f = \widehat{a_\varepsilon f}(D)\delta = a_\varepsilon(X)\hat{f}(D)\delta,$$

we can rewrite it as

$$(Q(D) + R_\varepsilon)\delta \in H_p^{-m},$$

where $R_\epsilon = \widehat{a_\varepsilon f}(D)$ or $R_\epsilon = a_\varepsilon(X)\hat{f}(D)$.

If $Q \neq 0$, we can construct a (microlocal) inverses $(Q(D) + R_\varepsilon)^{-1}$ in the space $\mathcal{L}(H_p^{-m})$ since R_ϵ is small as $\mathcal{L}(H_p^{-m})$. Then we have

$$\delta \in H_p^{-m}$$

(microlocally) again which contradicts to $m \leq n(1-1/p)$, and we can conclude $Q = 0$, hence $P\tilde{u} = f$.

References

[1] Bochner, S., *Weak solutions of linear partial differential equations*, J. Math. Pures Appl. **35** (1956), 193–202.

[2] Ishii, H., *On some Fourier multipliers and partial differential equations*, Math. Japon. **19** (1974), 139–163.

[3] John, F., *The fundamental solution of linear elliptic differential equations with analytic coefficients*, Comm. Pure. Appl. Math. **3** (1950), 273–304.

[4] Miyachi, A., *On some Fourier multipliers for $H^p(\mathbf{R}^n)$*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **27** (1980), 157–179.

[5] Sjöstrand, S., *On the Riesz means of the solutions of the Schrödinger equation*, Ann. Scuola Norm. Sup. Pisa **24** (1970), 331–348.

[6] Sugimoto, M, *A weak extension theorem for inhomogeneous differential equations*, Forum Math. **13** (2001), 323–334.

[7] Sugimoto, M. and Uchida, M., *A generalization of Bochner's extension theorem and its application*, Ark. Mat. **38** (2000), 399–409.