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Smooth functional derivatives in Feynman path integrals by time slicing approximation

～小松彦三郎先生の古希を記念して～

工学院大学・工学部・熊ノ郷 直人 (Naoto Kumano-go) [1]
Department of Engineering, Kogakuin University
学習院大学・理学部・藤原 大輔 (Daisuke Fujiwara) [2]
Department of Mathematics, Gakushuin University

1 Introduction

This note is a survey of our recent papers [23] [10] about the theory of Feynman path integrals by time slicing approximation.

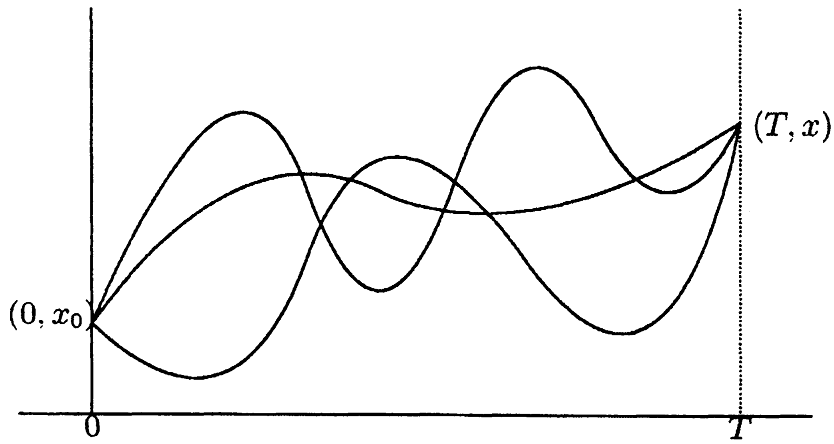
In 1948, R. P. Feynman [3] expressed the integral kernel of the fundamental solution for the Schrödinger equation, using the path integral as follows:

$$\int e^{\frac{i}{\hbar}S[\gamma]}\mathcal{D}[\gamma]\,. \tag{1.1}$$

Here $0<\hbar<1$ is Planck's constant, $\gamma:[0,T]\to\mathbf{R}^d$ is a path with $\gamma(0)=x_0$ and $\gamma(T)=x$, and $S[\gamma]$ is the action along the path γ defined by

$$S[\gamma]=\int_0^T\frac{1}{2}\left|\frac{d\gamma}{dt}\right|^2-V(t,\gamma(t))dt\,. \tag{1.2}$$

The path integral is a new sum of $e^{\frac{i}{\hbar}S[\gamma]}$ over all the paths.



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Feynman explained his new integral (1.1) as a limit of the finite dimensional integral, which is now called the time slicing approximation.

Furthermore, Feynman suggested a new analysis on a path space with the functional integration

$$\int F[\gamma]e^{\frac{i}{\hbar}S[\gamma]}\mathcal{D}[\gamma],$$

and the functional differentiation $(DF)[\gamma][\eta]$. (cf. Feynman-Hibbs[4], L. S. Schulman [27]) However, in 1960, R. H. Cameron [2] proved that the σ-additive measure $e^{\frac{i}{\hbar}S[\gamma]}\mathcal{D}[\gamma]$ does not exist.

Therefore, using the time slicing approximation, we give a fairly general class $\mathcal{F}^\infty$ of functionals $F[\gamma]$ such that the Feynman path integrals

$$\int e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma], \tag{1.3}$$

and the smooth functional derivatives $(DF)[\gamma][\eta]$ exist. More precisely, for any functional $F[\gamma]$ belonging to our class $\mathcal{F}^\infty$, the time slicing approximation of Feynman path integral converges uniformly on compact subsets of the configuration space $\mathbf{R}^{2d}$ of endpoints (x, x_0). Our class of functionals is closed under addition, multiplication, translation, real linear transformation and functional differentiation. The invariance under translation and orthogonal transformation, the integration by parts with respect to functional differentiation, the interchange of the order with Riemann-Stieltjes integrals, the interchange of the order with a limit, the perturbation expansion formula, the semiclassical approximation and the fundamental theorem of calculus are valid in the Feynman path integrals.

There are some mathematical works which prove the time slicing approximation of (1.1) converges uniformly on compact subsets of the configuration space $\mathbf{R}^{2d}$. See D. Fujiwara [5] [7] [8] [9], H. Kitada and H. Kumano-go [18], K. Yajima [30], N. Kumano-go [21], D. Fujiwara and T. Tsuchida [14], and W. Ichinose [15]. However all these works treated (1.1), that is the particular case of (1.3) with $F[\gamma] \equiv 1$.

Many people tried to give a mathematically rigorous meaning to Feynman path integral. E. Nelson [25] succeeded in connecting Feynman path integral to Wiener measure by analytic continuation with respect to a parameter. K. Itô [17] succeeded in defining Feynman path integrals as an improper oscillatory integral over a Hilbert manifold of paths. Albeverio and Høegh Krohn [1], A. Truman [29] and J. Rezende [26] applied Itô's idea and discussed many problems.

2 Main Results

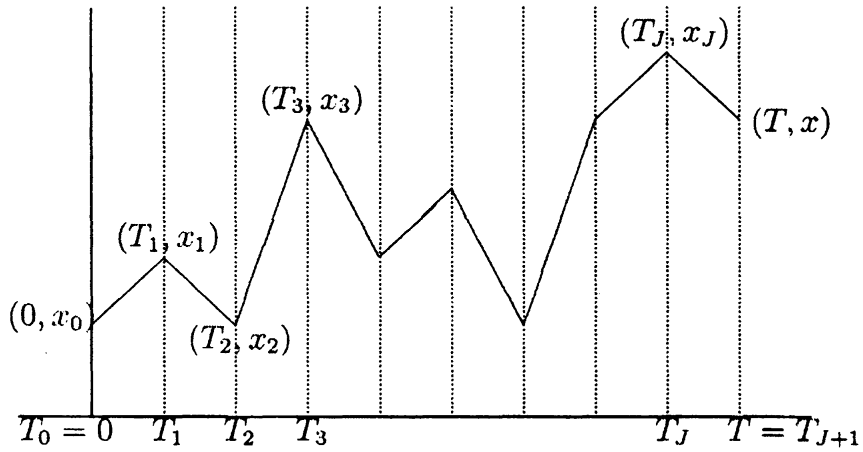
Let $\Delta_{T,0}$ be an arbitrary division of the interval $[0, T]$ into subintervals, i.e.,

$$\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0\,. \tag{2.1}$$

Set $x_{J+1} = x$. Let x_j, $j = 1, 2, \ldots, J$ be arbitrary points of $\mathbf{R}^d$. Let

$$\gamma_{\Delta_{T,0}} = \gamma_{\Delta_{T,0}}(t, x_{J+1}, x_J, \ldots, x_1, x_0)\,, \tag{2.2}$$

be the broken line path which connects (T_j, x_j) and (T_{j-1}, x_{j-1}) by a line segment for any $j = 1, 2, \ldots, J, J+1$. Set $t_j = T_j - T_{j-1}$ and $|\Delta_{T,0}| = \max_{1 \le j \le J+1} t_j$.



As Feynman [3] had first defined by the time slicing approximation, we define the Feynman path integrals (1.3) by

$$\int e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma] = \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]}F[\gamma_{\Delta_{T,0}}] \prod_{j=1}^{J} dx_j\,, \quad (2.3)$$

whenever the limit exists.

Remark 1 *$S[\gamma_{\Delta_{T,0}}]$ and $F[\gamma_{\Delta_{T,0}}]$ are functions of a finite number of variables x_{J+1}, x_J, ..., x_1, x_0, i.e.,*

$$\begin{aligned} S[\gamma_{\Delta_{T,0}}] &= S_{\Delta_{T,0}}(x_{J+1}, x_J, \ldots, x_1, x_0)\,, \\ F[\gamma_{\Delta_{T,0}}] &= F_{\Delta_{T,0}}(x_{J+1}, x_J, \ldots, x_1, x_0)\,. \end{aligned} \quad (2.4)$$

Therefore Feynman omitted the first step $S[\gamma_{\Delta_{T,0}}]$ of the form of functionals and wrote $S_{\Delta_{T,0}}(x_{J+1}, x_J, \ldots, x_1, x_0)$ of the form of functions. Furthermore, many books about Feynman path integrals abandon the first step $S[\gamma_{\Delta_{T,0}}]$ in order to use the Trotter formula, i.e.,

$$\begin{aligned} S[\gamma_{\Delta_{T,0}}] &= \sum_{j=1}^{J+1} \frac{(x_j - x_{j-1})^2}{2t_j} - \sum_{j=1}^{J+1} \int_{T_{j-1}}^{T_j} V(t, \frac{t - T_{j-1}}{T_j - T_{j-1}}x_j + \frac{T_j - t}{T_j - T_{j-1}}x_{j-1})dt \\ &\neq \sum_{j=1}^{J+1} \frac{(x_j - x_{j-1})^2}{2t_j} - \sum_{j=1}^{J+1} V(T_{j-1}, x_{j-1})\,. \end{aligned}$$

However, we keep the first step $S[\gamma_{\Delta_{T,0}}]$, $F[\gamma_{\Delta_{T,0}}]$ in the multi oscillatory integral (2.3).

Remark 2 *Even when $F[\gamma] \equiv 1$, the integrals of the right hand side of* (2.3) *does not converge absolutely. We treat multi integrals of this type directly as oscillatory integrals. (cf. H. Kumano-go [19], H. Kumano-go and K. Taniguchi [20], D. Fujiwara, N. Kumano-go and K. Taniguchi [13], N. Kumano-go [22][23])*

Remark 3 *If $|\Delta_{T,0}| \to 0$, the number J of the integrals of the right hand side of* (2.3) *tends to ∞. Therefore, we use the properties of $F[\gamma_{\Delta_{T,0}}]$.*

Remark 4 *If we need the endpoints (x, x_0), we will use the following expression:*

$$\int_{\gamma(0)=x_0,\gamma(T)=x} e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma] = \int e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma] .$$

Our assumption of the potential $V(t, x)$ of (1.2) is the following:

Assumption 1 (Potential) *$V(t, x)$ is a real-valued function of $(t, x) \in \mathbf{R} \times \mathbf{R}^d$, and, for any multi-index α, $\partial_x^\alpha V(t, x)$ is continuous in $\mathbf{R} \times \mathbf{R}^d$. For any integer $k \geq 2$, there exists a positive constant A_k such that for any multi-index α with $|\alpha| = k$,*

$$|\partial_x^\alpha V(t, x)| \leq A_k .$$

In order to state the definition of the class $\mathcal{F}^\infty$ of functionals $F[\gamma]$, we explain the functional derivatives in this paper.

Definition 1 (Functional derivatives) *For any division $\Delta_{T,0}$ of* (2.1)*, assume that*

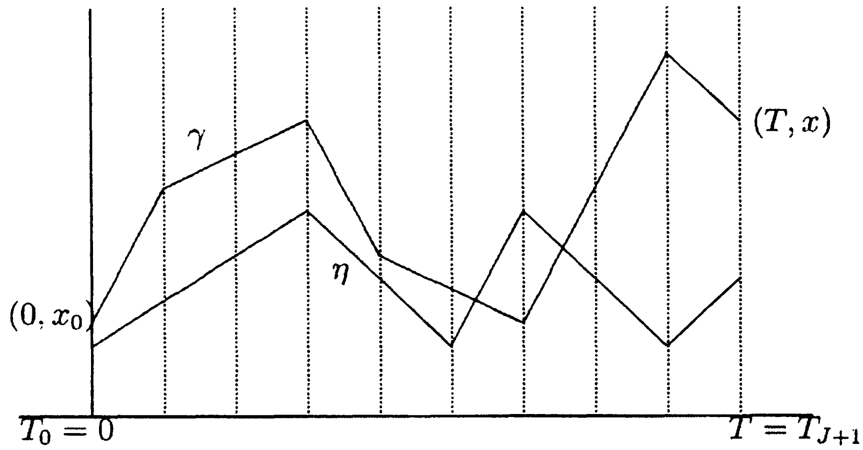
$$F_{\Delta_{T,0}}(x_{J+1}, x_J, \ldots, x_1, x_0) \in C^\infty(\mathbf{R}^{d(J+2)}) .$$

Let $\gamma : [0, T] \to \mathbf{R}^d$ and $\eta_l : [0, T] \to \mathbf{R}^d$, $l = 1, 2, \ldots, L$ be any broken line paths. We define the functional derivative $(D^L F)[\gamma] \prod_{l=1}^L [\eta_l]$ by

$$(D^L F)[\gamma] \prod_{l=1}^{L} [\eta_l] = \left(\prod_{l=1}^{L} \frac{\partial}{\partial \theta_l} \right) F[\gamma + \sum_{l=1}^{L} \theta_l \eta_l] \Big|_{\theta_1=\theta_2=\cdots=\theta_L=0} .$$

When $L = 0$, we also write $(D^L F)[\gamma] \prod_{l=1}^L [\eta_l] = F[\gamma]$.

Remark 5 *Let $\Delta_{T,0}$ of* (2.1) *contain all times when the broken line path γ or the broken line path η breaks. Set $\gamma(T_j) = x_j$ and $\eta(T_j) = y_j$, $j = 0, 1, \ldots, J, J + 1$.*



Then, for any $\theta \in \mathbf{R}$, $\gamma + \theta\eta$ is the broken line path which connects $(T_j, x_j + \theta y_j)$ and $(T_{j-1}, x_{j-1} + \theta y_{j-1})$ by a line segment for $j = 1, 2, \ldots, J, J+1$. Hence we have

$$F[\gamma + \theta\eta] = F_{\Delta_{T,0}}(x_{J+1} + \theta y_{J+1}, x_J + \theta y_J, \ldots, x_1 + \theta y_1, x_0 + \theta y_0)\,. \tag{2.5}$$

Therefore, we can write $(DF)[\gamma][\eta]$ as a finite sum as follows:

$$(DF)[\gamma][\eta] = \frac{d}{d\theta} F[\gamma + \theta\eta]\bigg|_{\theta=0} = \sum_{j=0}^{J+1} (\partial_{x_j} F_{\Delta_{T,0}})(x_{J+1}, x_J, \ldots, x_1, x_0) \cdot y_j\,. \tag{2.6}$$

Note that we 'restrict' the direction of functional derivatives to broken line paths. (cf. Malliavin's derivatives [24].)

Definition 2 (The class $\mathcal{F}^\infty$ of functionals $F[\gamma]$) *Let $F[\gamma]$ be a functional on the path space $C([0,T] \to \mathbf{R}^d)$ such that the domain of $F[\gamma]$ contains all of broken line paths at least. We say that $F[\gamma]$ belongs to the class $\mathcal{F}^\infty$ if $F[\gamma]$ satisfies Assumption 2. For simplicity, we write $F[\gamma] \in \mathcal{F}^\infty$.*

Assumption 2 *Let m be a non-negative integer and $\rho(t)$ be a function of bounded variation on $[0,T]$. For any non-negative integer M, there exists a positive constant C_M such that*

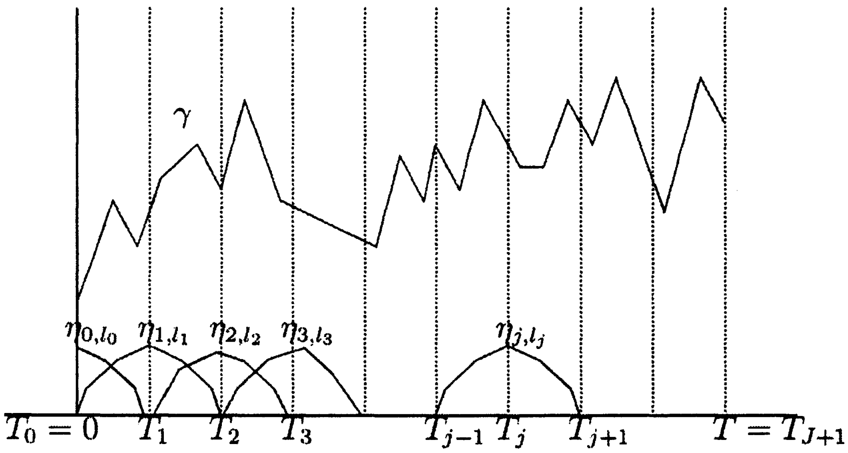
$$\left|\left(D^{\sum_{j=0}^{J+1} L_j} F\right)[\gamma] \prod_{j=0}^{J+1} \prod_{l_j=1}^{L_j} [\eta_{j,l_j}]\right| \le (C_M)^{J+2} (1 + ||\gamma||)^m \prod_{j=0}^{J+1} \prod_{l_j=1}^{L_j} ||\eta_{j,l_j}||\,, \tag{2.7}$$

$$\left|\left(D^{1+\sum_{j=0}^{J+1} L_j} F\right)[\gamma][\eta] \prod_{j=0}^{J+1} \prod_{l_j=1}^{L_j} [\eta_{j,l_j}]\right|$$

$$\le (C_M)^{J+2} (1 + ||\gamma||)^m \int_0^T |\eta(t)| d|\rho|(t) \prod_{j=0}^{J+1} \prod_{l_j=1}^{L_j} ||\eta_{j,l_j}||\,, \tag{2.8}$$

for any division $\Delta_{T,0}$ of (2.1), any $L_j = 0, 1, \ldots, M$, any broken line path $\gamma : [0,T] \to \mathbf{R}^d$, any broken line path $\eta : [0,T] \to \mathbf{R}^d$, and any broken line paths $\eta_{j,l_j} : [0,T] \to \mathbf{R}^d$, $l_j = 1, 2, \ldots, L_j$ whose supports exist in $[T_{j-1}, T_{j+1}]$. Here $0 = T_{-1} = T_0$, $T_{J+1} = T_{J+2} = T$, $\|\gamma\| = \max_{0 \le t \le T} |\gamma(t)|$ and $|\rho|(t)$ is the total variation of $\rho(t)$.

Remark 6 *Note that the support of the broken line path η_{j,l_j} exists in $[T_{j-1}, T_{j+1}]$ for any $j = 0, 1, \ldots, J, J+1$. Roughly speaking, the broken line paths η_{j,l_j}, $j = 0, 1, \ldots, J, J+1$ slice the time interval $[0,T]$.*



Remark 7 *About the process how we were making up Assumption 2, see D. Fujiwara [6], N. Kumano-go [23] and D. Fujiwara-N. Kumano-go [10] in this order.*

Theorem 1 (Existence of Feynman path integral) *Let T be sufficiently small. Then, for any $F[\gamma] \in \mathcal{F}^\infty$, the right hand side of (2.3) converges uniformly on any compact set of the configuration space $(x, x_0) \in \mathbf{R}^{2d}$, together with all its derivatives in x and x_0.*

Remark 8 *Through this note, the size of sufficiently small T depends only on d and A_k of Assumption 1.*

Theorem 2 (Smooth algebra) *For any $F[\gamma]$, $G[\gamma] \in \mathcal{F}^\infty$, any broken line path $\zeta : [0,T] \to \mathbf{R}^d$ and any real $d \times d$ matrix P, we have the following.*

(1) $F[\gamma] + G[\gamma] \in \mathcal{F}^\infty$, $\quad F[\gamma]G[\gamma] \in \mathcal{F}^\infty$.

(2) $F[\gamma + \zeta] \in \mathcal{F}^\infty$, $\quad F[P\gamma] \in \mathcal{F}^\infty$.

(3) $(DF)[\gamma][\zeta] \in \mathcal{F}^\infty$.

Remark 9 *In other words, $\mathcal{F}^\infty$ is closed under addition, multiplication, translation, real linear transformation and functional differentiation. Applying Theorem 2 to the examples of Theorem 3 (1)(2), Theorem 4 (1), Theorem 6 and Theorem 8, the reader can produce many functionals $F[\gamma] \in \mathcal{F}^\infty$.*

Assumption 3 *Let m be a non-negative integer. $B(t,x)$ is a function of $(t,x) \in \mathbf{R} \times \mathbf{R}^d$. For any multi-index α, $\partial_x^\alpha B(t,x)$ is continuous on $\mathbf{R} \times \mathbf{R}^d$, and there exists a positive constant C_α such that*

$$|\partial_x^\alpha B(t,x)| \le C_\alpha (1+|x|)^m .$$

Theorem 3 (Interchange of the order with Riemann-Stieltjes integrals) *Let $0 \le T' \le T'' \le T$ and $0 \le t \le T$. Let $\rho(t)$ be a function of bounded variation on $[T', T'']$. Suppose $B(t,x)$ satisfy Assumption 3. Then we have the following.*

(1) *The value at a fixed time t*

$$F[\gamma] = B(t, \gamma(t)) \in \mathcal{F}^\infty .$$

(2) *The Riemann-Stieltjes integral*

$$F[\gamma] = \int_{T'}^{T''} B(t,\gamma(t)) d\rho(t) \in \mathcal{F}^\infty .$$

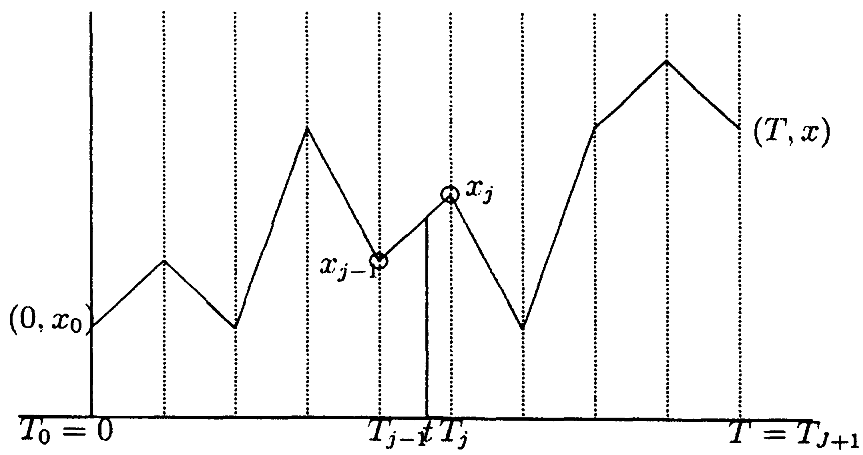
(3) *Let T be sufficiently small. Then we have*

$$\int_{T'}^{T''} \left(\int e^{\frac{i}{\hbar}S[\gamma]} B(t,\gamma(t)) \mathcal{D}[\gamma] \right) d\rho(t) = \int e^{\frac{i}{\hbar}S[\gamma]} \left(\int_{T'}^{T''} B(t,\gamma(t)) d\rho(t) \right) \mathcal{D}[\gamma] .$$

Furthermore, for any $F[\gamma] \in \mathcal{F}^\infty$, we have

$$\int_{T'}^{T''} \left(\int e^{\frac{i}{\hbar}S[\gamma]} B(t,\gamma(t)) F[\gamma] \mathcal{D}[\gamma] \right) d\rho(t) = \int e^{\frac{i}{\hbar}S[\gamma]} \left(\int_{T'}^{T''} B(t,\gamma(t)) d\rho(t) \right) F[\gamma] \mathcal{D}[\gamma] .$$

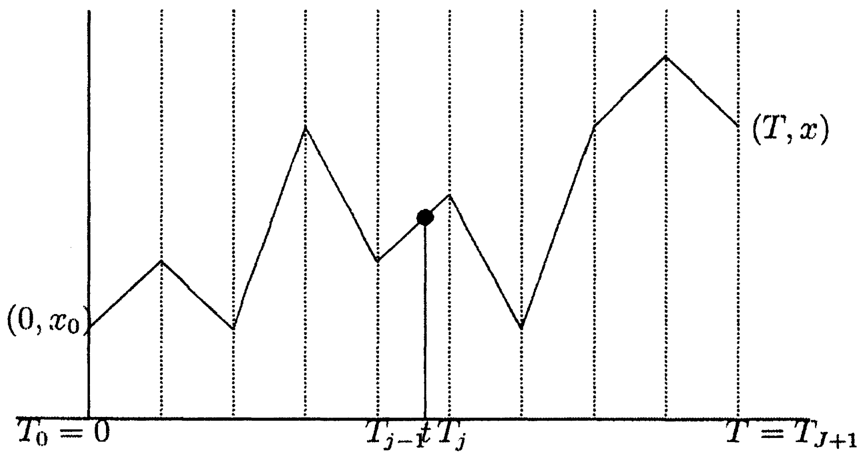
Remark 10 *We explain the key of the proof of Theorem 3 (3) roughly. In order to use the Trotter formula, many books about Feynman path integrals approximate the position of the particle at time t by the endpoint x_j or x_{j-1}.*



On the other hand, using the number j so that $T_{j-1} < t \le T_j$, we keep the position of the particle at time t, i.e.,

$$\gamma_{\Delta_{T,0}}(t) = \frac{t - T_{j-1}}{T_j - T_{j-1}} x_j + \frac{T_j - t}{T_j - T_{j-1}} x_{j-1} ,$$

inside the finite dimensional oscillatory integral of (2.3)*. Therefore, we can use the continuity of the broken line path $\gamma_{\Delta_{T,0}}(t)$ with respect to t.*



Proof of Theorem 3 (3).
(1) Note that $B(t,\gamma_{\Delta_{T,0}}(t))$ is a continuous function of t on $[T', T'']$, together with all its derivatives in x_j, $j = 0, 1, \dots, J, J+1$.
(2) By Lebesgue's dominated convergence theorem after integrating by parts by x_j, $j = 1, 2, \dots, J$ (Oscillatory integrals), for any division $\Delta_{T,0}$,

$$\prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} B(t,\gamma_{\Delta_{T,0}}(t)) \prod_{j=1}^{J} dx_j$$

is also a continuous function of t on $[T', T'']$.
(3) By Theorem 1, the convergence of

$$\int e^{\frac{i}{\hbar}S[\gamma]} B(t,\gamma(t))\mathcal{D}[\gamma] = \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} B(t,\gamma_{\Delta_{T,0}}(t)) \prod_{j=1}^{J} dx_j ,$$

is uniform with respect to t on $[T', T'']$.
(4) Therefore,

$$\int e^{\frac{i}{\hbar}S[\gamma]} B(t,\gamma(t))\mathcal{D}[\gamma]$$

is also a continuous function of t on $[T', T'']$ and Riemann-Stieltjes integrable.

(5) Furthermore, by the uniform convergence, we can interchange the order of $\int_{T'}^{T''} \cdots dt$ and $\lim_{|\Delta_{T,0}|\to 0}$.

$$\int_{T'}^{T''} \left(\int e^{\frac{i}{\hbar}S[\gamma]} B(t,\gamma(t)) \mathcal{D}[\gamma] \right) dt$$

$$\equiv \int_{T'}^{T''} \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} B(t,\gamma_{\Delta_{T,0}}(t)) \prod_{j=1}^{J} dx_j dt$$

$$= \lim_{|\Delta_{T,0}|\to 0} \int_{T'}^{T''} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} B(t,\gamma_{\Delta_{T,0}}(t)) \prod_{j=1}^{J} dx_j dt\,.$$

By Fubini's theorem after integrating by parts by x_j, $j = 1, 2, \ldots, J$ (Oscillatory integrals), we have

$$= \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i \hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} \int_{T'}^{T''} B(t,\gamma_{\Delta_{T,0}}(t)) dt \prod_{j=1}^{J} dx_j$$

$$= \int e^{\frac{i}{\hbar}S[\gamma]} \left(\int_{T'}^{T''} B(t,\gamma(t)) dt \right) \mathcal{D}[\gamma]\,.\Box$$

Assumption 4 *$f(b)$ is an analytic function of $b \in \mathbf{C}$ on a neighborhood of zero, i.e., there exist positive constants $\mu > 0$, $A > 0$ such that*

$$||f||_{\mu,A} = \sup_{n,\,|b|\le\mu} \frac{|\partial_b^n f(b)|}{A^n n!} < \infty\,.$$

Theorem 4 (Interchange of the order with a limit) *Let $0 \le T' \le T'' \le T$. Let $\rho(t)$ be a function of bounded variation on $[T', T'']$. Suppose $B(t,x)$ satisfy Assumption 3 with $m = 0$. Let $f(b)$ and $f_k(b)$, $k = 1, 2, 3, \ldots$ be analytic functions such that $\lim_{k\to\infty} ||f_k - f||_{\mu,A} = 0$. Then we have the following.*

(1)

$$F[\gamma] = f\left(\int_{T'}^{T''} B(t,\gamma(t)) d\rho(t) \right) \in \mathcal{F}^\infty\,.$$

(2) *Let T be sufficiently small. Then we have*

$$\lim_{k\to\infty} \int e^{\frac{i}{\hbar}S[\gamma]} f_k \left(\int_{T'}^{T''} B(t,\gamma(t)) d\rho(t) \right) \mathcal{D}[\gamma] = \int e^{\frac{i}{\hbar}S[\gamma]} f \left(\int_{T'}^{T''} B(t,\gamma(t)) d\rho(t) \right) \mathcal{D}[\gamma]\,.$$

Furthermore, for any $F[\gamma] \in \mathcal{F}^\infty$, we have

$$\lim_{k\to\infty} \int e^{\frac{i}{\hbar}S[\gamma]} f_k \left(\int_{T'}^{T''} B(t,\gamma(t)) d\rho(t) \right) F[\gamma]\mathcal{D}[\gamma]$$

$$= \int e^{\frac{i}{\hbar}S[\gamma]} f \left(\int_{T'}^{T''} B(t,\gamma(t)) d\rho(t) \right) F[\gamma]\mathcal{D}[\gamma]\,.$$

Corollary 1 (Perturbation expansion formula) *Let T be sufficiently small. Let $\rho(t)$ and $B(t,x)$ be the same as in Theorem 4. Then we have*

$$\int e^{\frac{i}{\hbar}S[\gamma]+\frac{i}{\hbar}\int_{T'}^{T''}B(\tau,\gamma(\tau))d\rho(\tau)}\mathcal{D}[\gamma]=\sum_{n=1}^{\infty}\left(\frac{i}{\hbar}\right)^n\int_{T'}^{T''}d\rho(\tau_n)\int_{T'}^{\tau_n}d\rho(\tau_{n-1})\cdots\int_{T'}^{\tau_2}d\rho(\tau_1)$$
$$\times\int e^{\frac{i}{\hbar}S[\gamma]}B(\tau_n,\gamma(\tau_n))B(\tau_{n-1},\gamma(\tau_{n-1}))\cdots B(\tau_1,\gamma(\tau_1))\mathcal{D}[\gamma]\,.$$

Theorem 5 (Semiclassical approximation) *Let T be sufficiently small. Let $F[\gamma]\in\mathcal{F}^\infty$ and the domain of $F[\gamma]$ be continuously extended to $C([0,T]\to\mathbf{R}^d)$ with respect to the norm $\|\gamma\|=\max_{0\le t\le T}|\gamma(t)|$. Let γ^{cl} be the classical path with $\gamma^{cl}(0)=x_0$ and $\gamma^{cl}(T)=x$, and $D(T,x,x_0)$ be the Morette-Van Vleck determinant. Define $\Upsilon(\hbar,T,x,x_0)$ by*

$$\int e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma]=\left(\frac{1}{2\pi i\hbar T}\right)^{d/2}e^{\frac{i}{\hbar}S[\gamma^{cl}]}\left(D(T,x,x_0)^{-1/2}F[\gamma^{cl}]+\hbar\Upsilon(\hbar,T,x,x_0)\right).$$

Then, for any multi-indices α, β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha\partial_{x_0}^\beta\Upsilon(\hbar,T,x,x_0)|\le C_{\alpha,\beta}(1+|x|+|x_0|)^m\,.$$

Remark 11 *If $\hbar\to 0$, the remainder term $\hbar\Upsilon(\hbar,T,x,x_0)\to 0$.*

Theorem 6 (New curvilinear integrals along path on path space) *Let $0\le T'\le T''\le T$. Let m be a non-negative integer. Let $Z(t,x)$ be a vector-valued function of $(t,x)\in\mathbf{R}\times\mathbf{R}^d$ into $\mathbf{R}^d$ such that, for any multi-index α, $\partial_x^\alpha Z(t,x)$ and $\partial_x^\alpha\partial_t Z(t,x)$ are continuous on $[0,T]\times\mathbf{R}^d$, and there exists a positive constant C_α such that*

$$|\partial_x^\alpha Z(t,x)|+|\partial_x^\alpha\partial_t Z(t,x)|\le C_\alpha(1+|x|)^m\,,$$

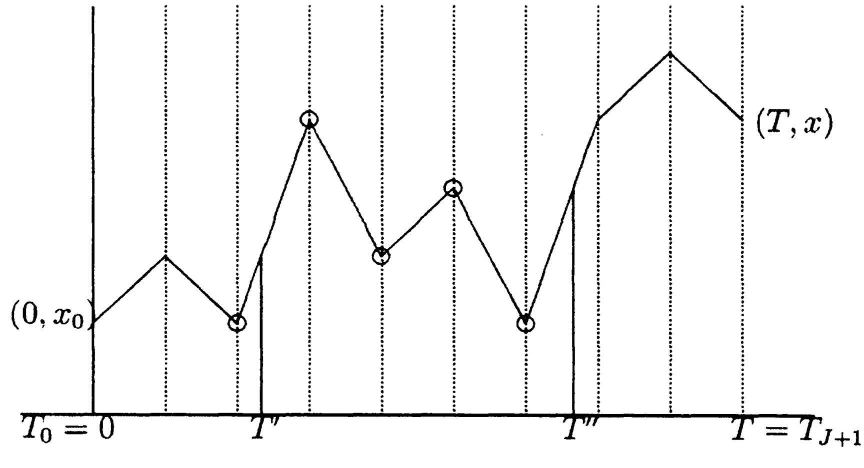
and $\partial_x Z(t,x)$ is a symmetric matrix, i.e., ${}^t(\partial_x Z)=\partial_x Z$. Then the curvilinear integrals along paths of Feynman path integral

$$F[\gamma]=\int_{T'}^{T''}Z(t,\gamma(t))\cdot d\gamma(t)\in\mathcal{F}^\infty\,.$$

Here $Z\cdot d\gamma$ is the inner product of Z and $d\gamma$ in $\mathbf{R}^d$.

Remark 12 *In order to explain the difference with known curvilinear integrals on a path space, please forgive very rough sketch. As examples of curvilinear integrals for paths on a path space, Itô integral [16] and Stratonovich integral [28] for the Brownian motion $\mathbf{B}(t)$ are successful in stochastic analysis. (cf. P. Malliavin [24]) If we can set $\mathbf{B}(T_j)=x_j$, Itô integral is approximated by initial points, i.e.,*

$$\int_{T'}^{T''}Z(t,\mathbf{B}(t))\cdot d\mathbf{B}(t)\approx\sum_j Z(T_{j-1},x_{j-1})\cdot(x_j-x_{j-1})\,.$$



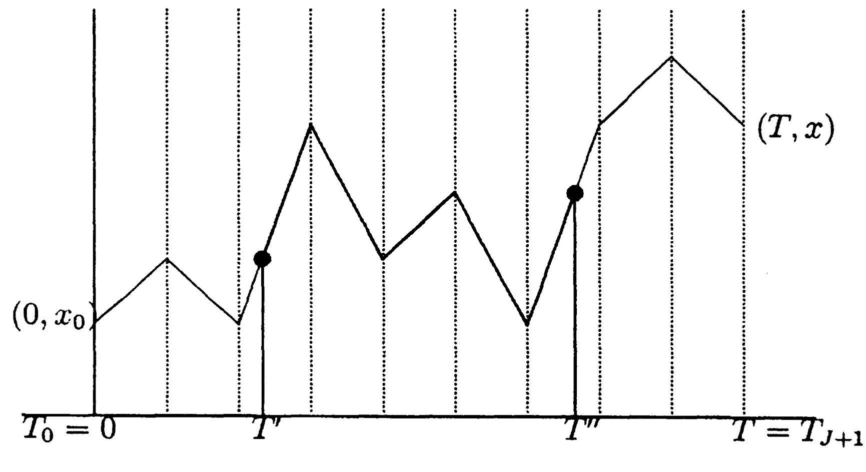
and Stratonovich integral is approximated by middle points, i.e.,

$$\int_{T'}^{T''} Z(t,\mathbf{B}(t))\circ d\mathbf{B}(t) \approx \sum_j Z\left(\frac{T_j+T_{j-1}}{2},\frac{x_j+x_{j-1}}{2}\right)\cdot(x_j-x_{j-1})\,.$$

And many books about Feynman path integrals use endpoints or middle points.

On the other hand, if $\gamma=\gamma_{\Delta_{T,0}}$, *our new curvilinear integrals is the classical curvilinear integrals itself along the broken line path* $\gamma_{\Delta_{T,0}}$, *i.e.,*

$$\int_{T'}^{T''} Z(t,\gamma_{\Delta_{T,0}}(t))\cdot d\gamma_{\Delta_{T,0}}(t)\,. \tag{2.9}$$



In other words, Itô integral and Stratonovich integral are some limits of the Riemann sums. On the other hand, our new integral is a limit of curvilinear integrals.

Theorem 7 (Fundamental theorem of calculus) *Let m be a non-negative integer and $0 \le T' \le T'' \le T$. $g(t,x)$ is a function of $(t,x) \in \mathbf{R} \times \mathbf{R}^d$ such that $g(t,x)$ and $\partial_t g(t,x)$ satisfy Assumption 3. Let T be sufficiently small. Then we have*

$$\int e^{\frac{i}{\hbar}S[\gamma]}\Big(g\left(T'',\gamma(T'')\right) - g\left(T',\gamma(T')\right)\Big)\mathcal{D}[\gamma] \tag{2.10}$$
$$= \int e^{\frac{i}{\hbar}S[\gamma]}\Big(\int_{T'}^{T''}(\partial_x g)(t,\gamma(t))\cdot d\gamma(t) + \int_{T'}^{T''}(\partial_t g)(t,\gamma(t))dt\Big)\mathcal{D}[\gamma]\,.$$

Furthermore, for any $F[\gamma] \in \mathcal{F}^\infty$, we have

$$\int e^{\frac{i}{\hbar}S[\gamma]}\Big(g\left(T'',\gamma(T'')\right) - g\left(T',\gamma(T')\right)\Big)F[\gamma]\mathcal{D}[\gamma]$$
$$= \int e^{\frac{i}{\hbar}S[\gamma]}\Big(\int_{T'}^{T''}(\partial_x g)(t,\gamma(t))\cdot d\gamma(t) + \int_{T'}^{T''}(\partial_t g)(t,\gamma(t))dt\Big)F[\gamma]\mathcal{D}[\gamma]\,.$$

Remark 13 (2.9) *is the key of the proof of Theorem 7.*

Proof of Theorem 7. By Theorem 3(1) and Theorem 2(1), we have

$$G_1[\gamma] = g\left(T'',\gamma(T'')\right) - g\left(T',\gamma(T')\right) \in \mathcal{F}^\infty\,.$$

We note that ${}^t(\partial_x^2 g) = (\partial_x^2 g)$. By Theorem 6, Theorem 3(2) and Theorem 2(1), we have

$$G_2[\gamma] = \int_{T'}^{T''}(\partial_x g)(t,\gamma(t))\cdot d\gamma(t) + \int_{T'}^{T''}(\partial_t g)(t,\gamma(t))dt \in \mathcal{F}^\infty\,.$$

By the fundamental theorem of calculus, we have $G_1[\gamma_{\Delta_{T,0}}] = G_2[\gamma_{\Delta_{T,0}}]$ for any broken line path $\gamma_{\Delta_{T,0}}$. By Theorem 1, we get

$$\int e^{\frac{i}{\hbar}S[\gamma]}G_1[\gamma]\mathcal{D}[\gamma] = \lim_{|\Delta_{T,0}|\to 0}\prod_{j=1}^{J+1}\left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2}\int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]}G_1[\gamma_{\Delta_{T,0}}]\prod_{j=1}^{J}dx_j$$
$$= \lim_{|\Delta_{T,0}|\to 0}\prod_{j=1}^{J+1}\left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2}\int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]}G_2[\gamma_{\Delta_{T,0}}]\prod_{j=1}^{J}dx_j = \int e^{\frac{i}{\hbar}S[\gamma]}G_2[\gamma]\mathcal{D}[\gamma]\,.\square$$

Theorem 8 *For any broken line path $\zeta : [0,T] \to \mathbf{R}^d$, we have the following:*

(1) $(DS)[\gamma][\zeta] \in \mathcal{F}^\infty$.

(2) $e^{\frac{i}{\hbar}(S[\gamma+\zeta]-S[\gamma])} \in \mathcal{F}^\infty$.
(*The constants $C_{\alpha,\beta}$, $C'_{\alpha,\beta}$ of Theorems 5, 16, 17 depend on $\hbar$.*)

Theorem 9 (Translation) *Let T be sufficiently small. For any $F[\gamma] \in \mathcal{F}^\infty$ and any broken line path $\eta : [0,T] \to \mathbf{R}^d$,*

$$\int_{\gamma(0)=x_0,\gamma(T)=x} e^{\frac{i}{\hbar}S[\gamma+\eta]}F[\gamma+\eta]\mathcal{D}[\gamma] = \int_{\gamma(0)=x_0+\eta(0),\gamma(T)=x+\eta(T)} e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma]\,.$$

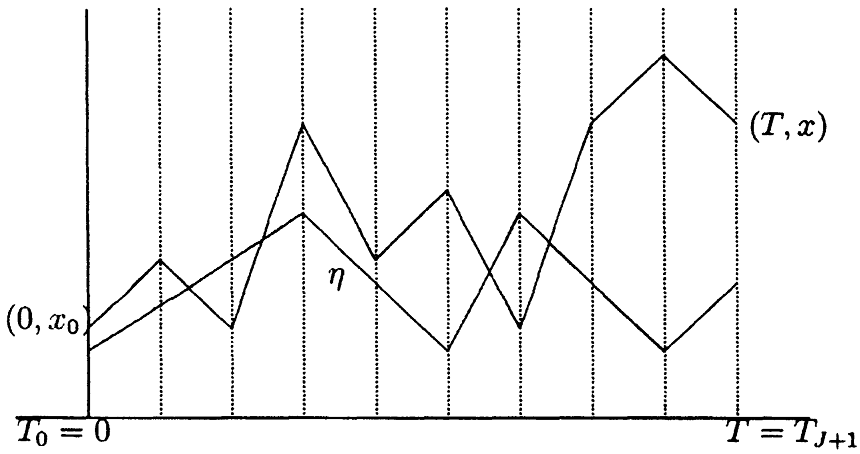
Corollary 2 (Invariance under translation) *Let T be sufficiently small. For any $F[\gamma] \in \mathcal{F}^\infty$ and any broken line path $\eta : [0,T] \to \mathbf{R}^d$ with $\eta(0) = \eta(T) = 0$,*

$$\int_{\gamma(0)=x_0,\gamma(T)=x} e^{\frac{i}{\hbar}S[\gamma+\eta]} F[\gamma+\eta]\mathcal{D}[\gamma] = \int_{\gamma(0)=x_0,\gamma(T)=x} e^{\frac{i}{\hbar}S[\gamma]} F[\gamma]\mathcal{D}[\gamma] .$$

Proof of Theorem 9. By Theorem 8(1) and Theorem 2(1)(2), we have $e^{\frac{i}{\hbar}(S[\gamma+\eta]-S[\gamma])}F[\gamma+\eta] \in \mathcal{F}^\infty$. By Theorem 1,

$$\int_{\gamma(0)=x_0,\gamma(T)=x} e^{\frac{i}{\hbar}S[\gamma+\eta]} F[\gamma+\eta]\mathcal{D}[\gamma] = \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2}$$
$$\times \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} \cdot e^{\frac{i}{\hbar}(S[\gamma_{\Delta_{T,0}}+\eta]-S[\gamma_{\Delta_{T,0}}])} F[\gamma_{\Delta_{T,0}}+\eta] \prod_{j=1}^{J} dx_j$$

exists. Choose $\Delta_{T,0}$ which contains all times when the broken line path η breaks. Set $\eta(T_j) = y_j$, $j = 0, 1, \ldots, J, J+1$.



Since $\gamma_{\Delta_{T,0}} + \eta$ is the broken line path which connects $(T_j, x_j + y_j)$ and $(T_{j-1}, x_{j-1} + y_{j-1})$ by a line segment for $j = 1, 2, \ldots, J, J+1$, we can write

$$= \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S_{\Delta_{T,0}}(x_{J+1}+y_{J+1},x_J+y_J,\ldots,x_1+y_1,x_0+y_0)}$$
$$\times F_{\Delta_{T,0}}(x_{J+1}+y_{J+1}, x_J+y_J, \ldots, x_1+y_1, x_0+y_0) \prod_{j=1}^{J} dx_j .$$

By the change of variables: $x_j + y_j \to x_j$, $j = 1, 2, \ldots, J$, we have

$$= \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S_{\Delta_{T,0}}(x_{J+1}+y_{J+1},x_J,\ldots,x_1,x_0+y_0)}$$

$$\times F_{\Delta_{T,0}}(x_{J+1}+y_{J+1},x_J,\ldots,x_1,x_0+y_0)\prod_{j=1}^{J}dx_j$$
$$=\int_{\gamma(0)=x_0+\eta(0),\gamma(T)=x+\eta(T)} e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma]\,.\square$$

Theorem 10 (Taylor's expansion formula) *Let T be sufficiently small. For any $F[\gamma]\in\mathcal{F}^\infty$ and any broken line path $\eta:[0,T]\to\mathbf{R}^d$,*

$$\int e^{\frac{i}{\hbar}S[\gamma]}F[\gamma+\eta]\mathcal{D}[\gamma]=\sum_{l=0}^{L}\frac{1}{l!}\int e^{\frac{i}{\hbar}S[\gamma]}(D^lF)[\gamma][\eta]\cdots[\eta]\mathcal{D}[\gamma]$$
$$+\int_0^1\frac{(1-\theta)^L}{L!}\int e^{\frac{i}{\hbar}S[\gamma]}(D^{L+1}F)[\gamma+\theta\eta][\eta]\cdots[\eta]\mathcal{D}[\gamma]d\theta\,. \tag{2.11}$$

Proof of Theorem 10. Using Taylor's expansion formula of (2.5) with respect to $0\le\theta\le1$, we have

$$F[\gamma+\eta]=\sum_{l=0}^{L}\frac{1}{l!}(D^lF)[\gamma][\eta]\cdots[\eta]+\int_0^1\frac{(1-\theta)^L}{L!}(D^{L+1}F)[\gamma+\theta\eta][\eta]\cdots[\eta]d\theta\,,$$

for any broken line path γ. By (2.3), we get (2.11). $\square$

Theorem 11 (Integration by parts) *Let T be sufficiently small. For any $F[\gamma]\in\mathcal{F}^\infty$ and any broken line path $\eta:[0,T]\to\mathbf{R}^d$ with $\eta(0)=\eta(T)=0$,*

$$\int e^{\frac{i}{\hbar}S[\gamma]}(DF)[\gamma][\eta]\mathcal{D}[\gamma]=-\frac{i}{\hbar}\int e^{\frac{i}{\hbar}S[\gamma]}(DS)[\gamma][\eta]F[\gamma]\mathcal{D}[\gamma]\,. \tag{2.12}$$

Proof of Theorem 11. Choose $\Delta_{T,0}$ which contains all times when the broken line path η breaks. Set $\gamma_{\Delta_{T,0}}(T_j)=x_j$ and $\eta(T_j)=y_j$, $j=0,1,\ldots,J,J+1$. By Theorem 9 with $\eta(0)=\eta(T)=0$, we have

$$\begin{aligned}0&=\lim_{|\Delta_{T,0}|\to0}\prod_{j=1}^{J+1}\left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2}\int_{\mathbf{R}^{dJ}}\left(e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}+\eta]}F[\gamma_{\Delta_{T,0}}+\eta]-e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]}F[\gamma_{\Delta_{T,0}}]\right)\prod_{j=1}^{J}dx_j\\&=\lim_{|\Delta_{T,0}|\to0}\int_0^1\prod_{j=1}^{J+1}\left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2}\int_{\mathbf{R}^{dJ}}e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}+\theta\eta]}\\&\quad\times\left(\frac{i}{\hbar}(DS)[\gamma_{\Delta_{T,0}}+\theta\eta][\eta]F[\gamma_{\Delta_{T,0}}+\theta\eta]+(DF)[\gamma_{\Delta_{T,0}}+\theta\eta][\eta]\right)\prod_{j=1}^{J}dx_jd\theta\,.\end{aligned}$$

Note (2.5) and $y_0=y_{J+1}=0$. By the change of variables: $x_j+\theta y_j\to x_j$, $j=1,2,\ldots,J$, we have

$$=\lim_{|\Delta_{T,0}|\to0}\int_0^1\prod_{j=1}^{J+1}\left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2}\int_{\mathbf{R}^{dJ}}e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]}$$

$$\times\left(\frac{i}{\hbar}(DS)[\gamma_{\Delta_{T,0}}][\eta]F[\gamma_{\Delta_{T,0}}]+(DF)[\gamma_{\Delta_{T,0}}][\eta]\right)\prod_{j=1}^{J}dx_jd\theta$$
$$=\frac{i}{\hbar}\int e^{\frac{i}{\hbar}S[\gamma]}(DS)[\gamma][\eta]F[\gamma]\mathcal{D}[\gamma]+\int e^{\frac{i}{\hbar}S[\gamma]}(DF)[\gamma][\eta]\mathcal{D}[\gamma]\,.\square$$

Theorem 12 (Orthogonal transformation) *Let T be sufficiently small. For any $F[\gamma]\in\mathcal{F}^\infty$ and any $d\times d$ orthogonal matrix Q,*

$$\int_{\gamma(0)=x_0,\gamma(T)=x}e^{\frac{i}{\hbar}S[Q\gamma]}F[Q\gamma]\mathcal{D}[\gamma]=\int_{\gamma(0)=Qx_0,\gamma(T)=Qx}e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma]\,. \tag{2.13}$$

Corollary 3 (Invariance under orthogonal transformation) *Let T be sufficiently small. For any $F[\gamma]\in\mathcal{F}^\infty$, any $d\times d$ orthogonal matrix Q and any broken line path $\eta:[0,T]\to\mathbf{R}^d$,*

$$\int_{\gamma(0)=0,\gamma(T)=0}e^{\frac{i}{\hbar}S[Q\gamma+\eta]}F[Q\gamma+\eta]\mathcal{D}[\gamma]=\int_{\gamma(0)=\eta(0),\gamma(T)=\eta(T)}e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma]\,.$$

Proof of Theorem 12. By Theorem 1,

$$\int_{\gamma(0)=x_0,\gamma(T)=x}e^{\frac{i}{\hbar}S[Q\gamma]}F[Q\gamma]\mathcal{D}[\gamma]$$
$$=\lim_{|\Delta_{T,0}|\to 0}\prod_{j=1}^{J+1}\left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2}\int_{\mathbf{R}^{dJ}}e^{\frac{i}{\hbar}S[Q\gamma_{\Delta_{T,0}}]}F[Q\gamma_{\Delta_{T,0}}]\prod_{j=1}^{J}dx_j\,,$$

exists. $Q\gamma_{\Delta_{T,0}}$ is the broken line path which connects (T_j,Qx_j) and (T_{j-1},Qx_{j-1}) by a line segment for $j=1,2,\ldots,J,J+1$. By the change of variables: $Qx_j\to x_j$, $j=1,2,\ldots,J$ and $|\det Q|=1$, we have

$$=\lim_{|\Delta_{T,0}|\to 0}\prod_{j=1}^{J+1}\left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2}\int_{\mathbf{R}^{dJ}}e^{\frac{i}{\hbar}S_{\Delta_{T,0}}(Qx_{J+1},x_J,\ldots,x_1,Qx_0)}$$
$$\times F_{\Delta_{T,0}}(Qx_{J+1},x_J,\ldots,x_1,Qx_0)\prod_{j=1}^{J}dx_j$$
$$=\int_{\gamma(0)=Qx_0,\gamma(T)=Qx}e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma]\,.\square$$

3 The details about the convergence of Theorem 1

For simplicity, for any $0\le l\le L\le J+1$, we set

$$x_{L,l}=(x_L,x_{L-1},\ldots,x_l)\,.$$

Let T satisfy $4A_2dT^2<1$ where A_2 is a constant in Assumption 1. Then we can define $x^\dagger_{J,1}=x^\dagger_{J,1}(x_{J+1},x_0)$ by

$$(\partial_{x_{J,1}}S_{\Delta_{T,0}})(x_{J+1},x^\dagger_{J,1},x_0)=0\,. \tag{3.1}$$

For any given function $f = f(x_{J+1}, x_{J,1}, x_0)$, let $f^\dagger$ be the function obtained by pushing $x_{J,1} = x_{J,1}^\dagger$ into f, i.e.,

$$f^\dagger = f^\dagger(x_{J+1}, x_0) = f(x_{J+1}, x_{J,1}^\dagger, x_0) .$$

Then we have

$$\begin{aligned} \gamma_{\Delta_{T,0}}^\dagger &= \gamma_{\Delta_{T,0}}(t, x_{J+1}, x_{J,1}^\dagger, x_0) , \\ S[\gamma_{\Delta_{T,0}}^\dagger] &= S_{\Delta_{T,0}}^\dagger(x_{J+1}, x_0) = S_{\Delta_{T,0}}(x_{J+1}, x_{J,1}^\dagger, x_0) , \\ F[\gamma_{\Delta_{T,0}}^\dagger] &= F_{\Delta_{T,0}}^\dagger(x_{J+1}, x_0) = F_{\Delta_{T,0}}(x_{J+1}, x_{J,1}^\dagger, x_0) . \end{aligned}$$

We define $D_{\Delta_{T,0}}(x_{J+1}, x_0)$ by

$$D_{\Delta_{T,0}}(x_{J+1}, x_0) = \det(\partial_{x_{J,1}}^2 S_{\Delta_{T,0}})^\dagger \times \left(\frac{\prod_{j=1}^{J+1} t_j}{T_{J+1}} \right)^d .$$

Furthermore we define the remainder term $\Upsilon_{\Delta_{T,0}}(\hbar, x_{J+1}, x_0)$ by

$$\begin{aligned} &\prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar} S[\gamma_{\Delta_{T,0}}]} F[\gamma_{\Delta_{T,0}}] \prod_{j=1}^{J} dx_j \\ &= \left(\frac{1}{2\pi i\hbar T} \right)^{d/2} e^{\frac{i}{\hbar} S[\gamma_{\Delta_{T,0}}^\dagger]} \left(D_{\Delta_{T,0}}(x, x_0)^{-1/2} F[\gamma_{\Delta_{T,0}}^\dagger] + \hbar \Upsilon_{\Delta_{T,0}}(\hbar, x, x_0) \right) . \end{aligned}$$

We can prove the convergence of (2.3) in the following order.

Theorem 13 (Convergence of Path) *Let $4A_2 dT^2 < 1/2$. Then, for any multi-indices α, β, there exist positive constants $C_{\alpha,\beta}$, $C'_{\alpha,\beta}$ independent of $\Delta_{T,0}$ such that*

$$\begin{aligned} \left\| \partial_x^\alpha \partial_{x_0}^\beta \gamma_{\Delta_{T,0}}^\dagger \right\| &\le C_{\alpha,\beta} (1 + |x| + |x_0|)^{\max(1-|\alpha+\beta|,0)} , \\ \left\| \partial_x^\alpha \partial_{x_0}^\beta \left(\gamma_{\Delta_{T,0}}^\dagger - \gamma^{cl} \right) \right\| &\le C'_{\alpha,\beta} |\Delta_{T,0}| T (1 + |x| + |x_0|) , \end{aligned}$$

where $\gamma^{cl} = \gamma^{cl}(t, x, x_0)$ is the classical path with $\gamma^{cl}(0) = x_0$ and $\gamma^{cl}(T) = x$, and $\|\gamma\| = \max_{0 \le t \le T} |\gamma(t)|$.

Theorem 14 (Convergence of Phase) *Let $4A_2 dT^2 < 1/2$. Then, for any multi-indices α, β, there exist positive constants $C_{\alpha,\beta}$, $C'_{\alpha,\beta}$ independent of $\Delta_{T,0}$ such that*

$$\begin{aligned} \left| \partial_x^\alpha \partial_{x_0}^\beta \left(S_{\Delta_{T,0}}^\dagger(x, x_0) - \frac{(x - x_0)^2}{2T} \right) \right| &\le C_{\alpha,\beta} T (1 + |x| + |x_0|)^{\max(2-|\alpha+\beta|,0)} , \\ \left| \partial_x^\alpha \partial_{x_0}^\beta \left(S_{\Delta_{T,0}}^\dagger(x, x_0) - S(T, x, x_0) \right) \right| &\le C'_{\alpha,\beta} |\Delta_{T,0}|^2 T (1 + |x| + |x_0|)^{1+\max(1-|\alpha+\beta|,0)} , \end{aligned}$$

where $S(T, x, x_0) = S[\gamma^{cl}]$ is the action along the classical path γ^{cl} with $\gamma^{cl}(0) = x_0$ and $\gamma^{cl}(T) = x$.

Theorem 15 (Convergence of Main term 1) *Let T be sufficiently small. Then, for any multi-indices α, β, there exist positive constants $C_{\alpha,\beta}$, $C'_{\alpha,\beta}$ independent of $\Delta_{T,0}$ such that*

$$\left|\partial_x^\alpha \partial_{x_0}^\beta \left(D_{\Delta_{T,0}}(x, x_0) - 1\right)\right| \le C_{\alpha,\beta} T^2 ,$$
$$\left|\partial_x^\alpha \partial_{x_0}^\beta \left(D_{\Delta_{T,0}}(x, x_0) - D(T, x, x_0)\right)\right| \le C'_{\alpha,\beta} |\Delta_{T,0}| T(1 + |x| + |x_0|) ,$$

where $D(T, x, x_0)$ is the Morette-Van Vleck determinant.

Theorem 16 (Convergence of Main term 2) *Let T be sufficiently small. Let $F[\gamma] \in \mathcal{F}^\infty$. Then, for any multi-indices α, β, there exist positive constants $C_{\alpha,\beta}$, $C'_{\alpha,\beta}$ independent of $\Delta_{T,0}$ such that*

$$\left|\partial_x^\alpha \partial_{x_0}^\beta F^\dagger_{\Delta_{T,0}}(x, x_0)\right| \le C_{\alpha,\beta}(1 + |x| + |x_0|)^m ,$$
$$\left|\partial_x^\alpha \partial_{x_0}^\beta \left(F^\dagger_{\Delta_{T,0}}(x, x_0) - F(T, x, x_0)\right)\right| \le C'_{\alpha,\beta} |\Delta_{T,0}| T(1 + |x| + |x_0|)^{m+1} .$$

with a function $F(T, x, x_0)$. Furthermore, if the domain of $F[\gamma]$ is continuously extended to $C([0, T] \to \mathbf{R}^d)$ with respect to the norm $||\gamma|| = \max_{0\le t\le T} |\gamma(t)|$, then $F(T, x, x_0) = F[\gamma^{cl}]$.

Theorem 17 (Convergence of Remainder term) *Let T be sufficiently small. Let $F[\gamma] \in \mathcal{F}^\infty$. Then, for any multi-indices α, β, there exist positive constants $C_{\alpha,\beta}$, $C'_{\alpha,\beta}$ independent of $\Delta_{T,0}$ and $\hbar$ such that*

$$\left|\partial_x^\alpha \partial_{x_0}^\beta \Upsilon_{\Delta_{T,0}}(\hbar, x, x_0)\right| \le C_{\alpha,\beta} T(U + T^2)(1 + |x| + |x_0|)^m ,$$
$$\left|\partial_x^\alpha \partial_{x_0}^\beta \left(\Upsilon_{\Delta_{T,0}}(\hbar, x, x_0) - \Upsilon(\hbar, T, x, x_0)\right)\right| \le C'_{\alpha,\beta} |\Delta_{T,0}| (U + T^2)(1 + |x| + |x_0|)^{m+1} ,$$

with a function $\Upsilon(\hbar, T, x, x_0)$.

We explain the key of these proofs on the convergence:

Let $(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})$ be the coarser division defined by

$$T = T_{J+1} > T_J > \cdots > T_{N+1} > T_{n-1} > \cdots > T_1 > T_0 = 0 . \tag{3.2}$$

In order to prove that the sequences of the functions of Theorems 13, 14, 15, 16, 17 are the Cauchy sequences with respect to the division $\Delta_{T,0}$, we need to compare the function for the division $\Delta_{T,0}$ and the function for the division $(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})$. The two functions are different in the number of variables. However we can connect the two functions with a broken line path as follows.

Lemma 3.1 *For any $1 \le n \le N \le J$, define $x^\triangleleft_{N,n} = x^\triangleleft_{N,n}(x_{N+1}, x_{n-1})$ by*

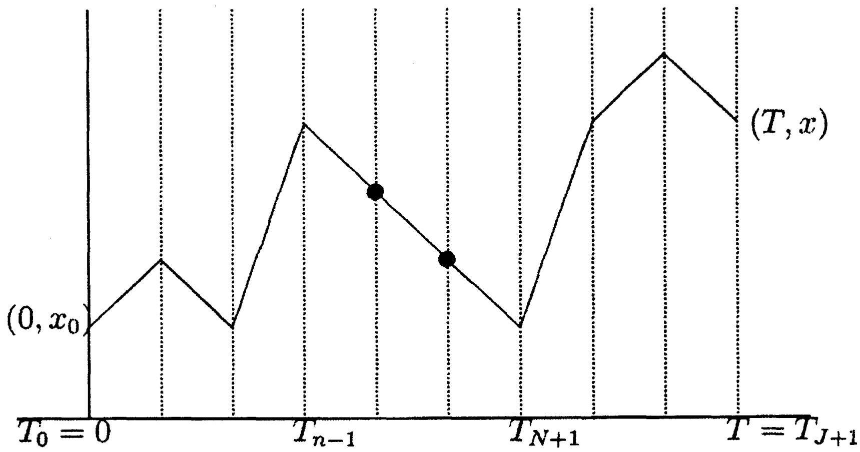
$$x^\triangleleft_j = \frac{T_j - T_{n-1}}{T_{N+1} - T_{n-1}} x_{N+1} + \frac{T_{N+1} - T_j}{T_{N+1} - T_{n-1}} x_{n-1} , \quad j = n, n+1, \ldots, N .$$

Then, for any functional $F[\gamma]$ whose domain contains all of broken line paths, we have

$$F_{\Delta_{T,0}}(x_{J+1,N+1}, x^\triangleleft_{N,n}, x_{n-1,0}) = F_{(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})}(x_{J+1,N+1}, x_{n-1,0}) .$$

Proof of Lemma 3.1. Set $x_{N,n} = x^{\triangleleft}_{N,n}$. Then the broken line path $\gamma_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}$ becomes the broken line path $\gamma_{\Delta_{T,0}}$. Therefore we have

$$\begin{aligned} &F_{\Delta_{T,0}}(x_{J+1,N+1}, x^{\triangleleft}_{N,n}, x_{n-1,0}) = F[\gamma_{\Delta_{T,0}}] \\ &= F[\gamma_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}] = F_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}(x_{J+1,N+1}, x_{n-1,0})\,.\square \end{aligned} \tag{3.3}$$



4 Assumption 2' by 'piecewise classical paths'

As a remark on Assumption 2, we state Assumption 2' under the time slicing approximation by 'piecewise classical paths'.

Let γ^{cl} be the classical path with $\gamma^{cl}(0) = x_0$ and $\gamma^{cl}(T) = x$, i.e., γ^{cl} satisfies the Euler equation

$$\frac{d^2}{dt^2}\gamma^{cl}(t) - (\partial_x V)(t, \gamma^{cl}(t)) = 0\,.$$

For any division $\Delta_{T,0}$ of (2.1), let

$$\gamma_{\Delta_{T,0}} = \gamma_{\Delta_{T,0}}(t, x_{J+1}, x_J, \dots, x_1, x_0)\,,$$

be the 'piecewise classical path' which connects (T_j, x_j) and (T_{j-1}, x_{j-1}) by a classical path for any $j = 1, 2, \dots, J+1$. Set

$$\begin{aligned} S[\gamma_{\Delta_{T,0}}] &= S_{\Delta_{T,0}}(x_{J+1}, x_J, \dots, x_1, x_0)\,, \\ F[\gamma_{\Delta_{T,0}}] &= F_{\Delta_{T,0}}(x_{J+1}, x_J, \dots, x_1, x_0)\,. \end{aligned}$$

Assumption 2' Let m and U be non-negative integers and u_j, $j = 1, 2, \dots, J, J+1$ be non-negative parameters depending on $\Delta_{T,0}$ such that $\sum_{j=1}^{J+1} u_j \le U < \infty$. For any non-negative integer M, there exists a positive constant C_M such that for any division

$\Delta_{T,0}$, any $|\alpha_j| \le M$, $j = 0, 1, \ldots, J, J+1$, and any $1 \le k \le J$,

$$\left|\left(\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j}\right) F_{\Delta_{T,0}}(x_{J+1}, x_J, \ldots, x_1, x_0)\right| \le (C_M)^{J+1}(1 + \sum_{j=0}^{J+1} |x_j|)^m ,$$

$$\left|\left(\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j}\right) \partial_{x_k} F_{\Delta_{T,0}}(x_{J+1}, x_J, \ldots, x_1, x_0)\right| \le (C_M)^{J+1}(u_{k+1} + u_k)(1 + \sum_{j=0}^{J+1} |x_j|)^m .$$

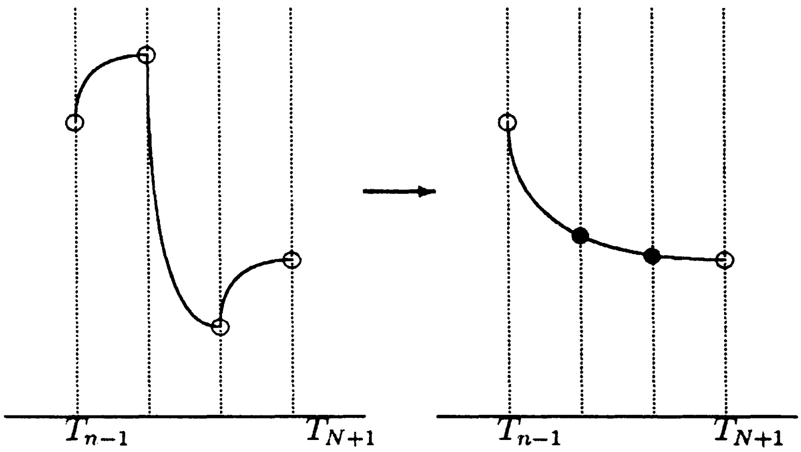
The class $\mathcal{F}'$ of functionals $F[\gamma]$ defined by Assumption 2' also satisfies Theorem 1, Theorem 2(1), Theorem 3, Theorem 4, Theorem 5, Theorem 6 and Theorem 7. Furthermore, the convergence of the time slicing approximation by 'piecewise classical paths' is much sharper than the convergence of the time slicing approximation by broken line paths.

Sketch of Proof. Let $x_{N,n}^{\dagger} = x_{N,n}^{\dagger}(x_{N+1}, x_{n-1})$ be the critical point defined by

$$(\partial_{x_{J,1}} S_{\Delta_{T,0}})(x_{J+1,N+1}, x_{N,n}^{\dagger}, x_{n-1,0}) = 0 .$$

Note that if we push the critical points into a piecewise classical path, the piecewise classical path changes to a single classical path. Then we can hide all critical points inside single classical paths, i.e.,

$$F_{\Delta_{T,0}}(x_{J+1,N+1}, x_{N,n}^{\dagger}, x_{n-1,0}) = F_{(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})}(x_{J+1,N+1}, x_{n-1,0}) . \square$$



For the details of the proofs, see D. Fujiwara and N. Kumano-go [11] [12]. Furthermore, in [12], Fujiwara wrote down the second term of the semi-classical asymptotic expansion of Feynman path integrals (1.3) with the integrand $F[\gamma]$. If $F[\gamma] \equiv 1$, the second terms coincides with the one given by G. D. Birkhoff.

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