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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2009, 1648: 17-31</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2009-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/140726">http://hdl.handle.net/2433/140726</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
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Analytic singularities of solutions to a radial $p$-Laplacian

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Abstract

Analytic local description of $W^{1,p}(I)$ solutions to a radial $p$-Laplace equation

$$r \left( |U_r|^{p-2} U_r \right)_r + (n-1)|U_r|^{p-2}U_r + r|U|^{q-2}U = 0$$

on $I = [a,b] \subset (0, \infty)$ is given near singular points by a Briot-Bouquet type theorem of two variables, where $1 < p, q < \infty$.

1 Introduction

An $n$-dimensional $p$-elliptic PDE for $u(x)$ is

$$\text{div}(|\nabla u|^{p-2}\nabla u(x)) + |u|^{q-2}u = 0,$$  \hspace{1cm} (1)

where $x \in \mathbb{R}^n$ and $1 < p, q < \infty$.

If $x \in \mathbb{R}^1$, the equation reduces to

$$\left( |u_x|^{p-2} u_x \right)_x + |u|^{q-2}u = 0.$$ \hspace{1cm} (2)

L. Paredes and the present author, making use of a Briot-Bouquet type theorem of one variable, gave analytic expression of solutions to the equation (2) near the singularities ([8]). Our analytic expression readily reproduces differentiability and analyticity obtained by M. Ôtani [6], [7] and by M. Ôtani and T. Idogawa in [4].

If $r = |x|, x \in \mathbb{R}^n$, a radial solution $U(r) = u(x)$ satisfies

$$\left( r^{n-1}|U_r|^{p-2} U_r \right)_r + r^{n-1}|U|^{q-2}U = 0,$$ \hspace{1cm} (3)

or

$$r \left( |U_r|^{p-2} U_r \right)_r + (n-1)|U_r|^{p-2}U_r + r|U|^{q-2}U = 0.$$ \hspace{1cm} (4)

The aim of this report is to extend the results for (2) to the radial $p$ Laplacian (3) by a Briot-Bouquet type theorem of two variables.
Remark 0.1. R. Gérard and H. Tahara studied \( t \frac{\partial u}{\partial t} = \Phi(t, x, u, \frac{\partial u}{\partial x}) \) and generalized it of many variables and of higher order case in [3]. Since our version of a Briot-Bouquet type theorem of two (or several) variables is not covered by theirs, a proof is given, inspired by their work.

2 A Briot-Bouquet type theorem

We recall a classical Briot-Bouquet type theorem of one variable in complex domain. We assume

- \( \Phi(t, h) \) is holomorphic near \((0,0) \in C^2\),
- \( \Phi(0,0) = 0 \),
- \( \frac{\partial \Phi}{\partial h}(0,0) \) is not any positive integers.

Theorem 1 (Briot-Bouquet).

\[
\frac{dh}{dt} = \Phi(t, h)
\]

has a unique holomorphic solution near \( t = 0 \), satisfying \( h(0) = 0 \).

If \( \Phi(t, h) \) is real analytic, so is \( h(t) \), too.

Proof. Set \( a_{\alpha,i} = \frac{1}{\alpha!i!} \frac{\partial^{\alpha+i}\Phi}{\partial t^{\alpha}\partial h^{i}}(0,0) \). Notice we can rewrite the equation by

\[
\frac{dh}{dt} - a_{0,1}h = a_{1,0}t + \sum_{2 \leq \alpha+i} a_{\alpha,i} t^{\alpha} h^{i}.
\]

Moreover, the left hand side satisfies the condition that there exists \( \delta > 0 \) such that

\[
|\alpha - a_{0,1}| \geq \delta
\]

for all \( \alpha \in N^* = \{1, 2, 3, \cdots \} \).

Formal solution: Let \( \hat{h}(t) = \sum_{\alpha=1}^{\infty} h_{\alpha} t^{\alpha} \) be a formal solution. Then, we have

\[
(1 - a_{0,1})h_{1} = a_{1,0},
\]

\[
(\alpha - a_{0,1})h_{\alpha} = Q_{\alpha}(a_{\alpha',i}, h_{\alpha''}; \alpha' + i \leq \alpha, \alpha'' \leq \alpha - 1)
\]
for all $\alpha \geq 2$, where $Q_\alpha$ is a polynomial with nonnegative integer coefficients.

**Convergence:** Then, we prove convergence of $\hat{h}(t)$ through the implicit function theorem. An auxiliary equation of $H(t)$ is given by

$$\delta H = |a_{1,0}|t + \sum_{2 \leq \alpha + i} |a_{\alpha,i}| t^\alpha H^i,$$

with $H(0) = 0$. There exists a unique convergent series function $H(t) = \sum_{\alpha=1}^{\infty} H_\alpha t^\alpha$ by the implicit function theorem. Since $\delta H_\alpha = Q_\alpha(|a_{\alpha',i}|, H_{\alpha''}; \alpha' + i \leq \alpha, \alpha'' \leq \alpha - 1)$,

$$|h_1| = |a_{1,0}|/|1 - a_{0,1}|$$
$$\leq |a_{1,0}|/\delta = H_1,$$
$$|h_\alpha| = |Q_\alpha(a_{\alpha'}, h_{\alpha''})/(\alpha - a_{0,1})|$$
$$\leq Q_\alpha(|a_{\alpha'}|, H_{\alpha''})/\delta = H_\alpha$$

for $\alpha \geq 2$ by induction.

\[\square\]

We will make use of a Briot-Bouquet type theorem of two variables for our main results. We state it in a slightly more general form for convenience.

Let $\mathcal{N} = \{0,1,2, \cdots \}$. $B = \{\beta\}$ is a fixed finite subset of $\mathcal{N}^d$, where $\beta = (\beta_1, \cdots, \beta_d)$ is a $d$-dimensional multi-index with $|\beta| \geq 1$. Let $(t, h, \rho_B) = (t_1, \cdots, t_d, h, \{\rho_\beta; \beta \in B\}) \in \mathbb{C}^{d+1+|B|}$ be local variables near the origin, where $|B|$ is the number of the elements in $B$. Let $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{C}^d$ be global variables. $\alpha = (\alpha_1, \cdots, \alpha_d), \beta$ and $\gamma$ denote $d$-dimensional power indices in $\mathcal{N}^d$.

**Theorem 2.** We assume that a holomorphic function $\phi(t, h, \rho_B)$ and a polynomial

$$L(\xi) = \sum_{0 \leq |\gamma| \leq r} l_\gamma \xi^\gamma$$

satisfy

(i) $\phi(t, \rho_B)$ has a power series expansion near $(0, 0)$ without linear
\[ \phi(t, \rho_B) = \sum_{2 \leq |\alpha| + |i_B|} a_{\alpha, i_B} t^\alpha \rho_B^{i_B} = \sum_{2 \leq |\alpha| + |i_B|} a_{\alpha, i_B} t^\alpha \prod_{\beta \in B} \rho_{\beta}^{i_{\beta}}, \]

where \(|i_B| = \sum_{\beta \in B} i_{\beta}\) for a multi-index \(i_B = (i_{\beta})_{\beta \in B}\) and

(ii) there exists a positive constant \(\delta\) such that for all \(d\)-dimensional multi indices \(\alpha\) with \(|\alpha| \geq 1\),

\[
| \sum_{0 \leq |\gamma| \leq r} l_{\gamma} \alpha^\gamma | \geq \delta \max\{1, \alpha^\beta; \beta \in B\},
\]

where \(\alpha^\beta\) denotes the coefficient of \((t \frac{\partial}{\partial t})^{\beta} t^\alpha\).

Then, a nonlinear equation

\[
\sum_{0 \leq |\gamma| \leq r} l_{\gamma} \left( t \frac{\partial}{\partial t} \right)^\gamma h(t) = a \cdot t + \sum_{2 \leq |\alpha| + |i_B|} a_{\alpha, i_B} t^\alpha \prod_{\beta \in B} \left( \left( t \frac{\partial}{\partial t} \right)^\beta h(t) \right)^{i_{\beta}}
\]

has a unique holomorphic solution \(h(t)\) near the origin with \(h(0) = 0\).

**Proof.** We will follow the previous proof.

**Construction of \(\hat{h}(t)\):** We set

\[
\hat{h}(t) = \sum_{|\alpha| \geq 1} h_{\alpha} t^\alpha.
\]

Substituting (7) into (6), we have

\[ L(\alpha) h_{\alpha} = a_{\alpha} \quad \text{when} \ |\alpha| = 1, \]
and
\[ L(\alpha)h_\alpha = Q_\alpha(a_{\alpha',i_B}, h_{\alpha''}, (\alpha_\beta)^\beta h_{\alpha_\beta}; \beta \in B) \]
the indices at least satisfy
\[ |\alpha'| + |i_B| \leq |\alpha|, |\alpha''| \leq |\alpha| - 1, \]
\[ |\alpha_\beta| \leq |\alpha| - 1, \]
where \( \alpha', \alpha'', \alpha_\beta \) are copies of \( \alpha \). Thus, \( h_\alpha \) are determined successively.

**Convergence of \( \hat{h} \):** An auxiliary analytic equation (cf. Gérard-Tahara [3])

\[ \delta H = |a_1|t_1 + |a_2|t_2 + \sum_{2\leq|\alpha|+|i_B|} |a_{\alpha,i_B}|t^\alpha(H(t))^{i_B}. \]

Solving this equation of \( H \) by the implicit function theorem, we have a unique holomorphic solution near the origin \( t = 0 \) with \( H(0) = 0 \). We claim

\[ H(t) = \sum_\alpha H_\alpha t^\alpha >> \hat{h}(t). \]

More strongly we claim,

\[ H_\alpha \geq \max\{|h_\alpha|, |\alpha^\beta h_\alpha|; \beta \in B\}. \]

We notice

\[ H_\alpha = \frac{1}{\delta}Q_\alpha(|a_{\alpha',i_B}|, h_{\alpha''}, H_{\alpha_\beta}; \beta \in B), \]

the indices satisfy

\[ |\alpha'| + |i_B| \leq |\alpha|, |\alpha''| \leq |\alpha| - 1, \]
\[ |\alpha_\beta| \leq |\alpha| - 1, \beta \in B. \] (8)

We start with \( |\alpha| = 1 \):

\[ |h_\alpha| = |a_\alpha/L(\alpha)| \leq |a_\alpha|/\delta = H_\alpha. \]

Then, by induction, we have

\[ \max\{1, \alpha^\beta; \beta \in B\} \cdot |h_\alpha| \leq \frac{1}{\delta}Q_\alpha(|a_{\alpha',i_B}|, |h_{\alpha''}|, |(\alpha_\beta)^\beta h_{\alpha_\beta}|; \]

indices satisfy at least
\[ |\alpha'| + |i_B| \leq |\alpha|, |\alpha''| \leq |\alpha| - 1, \]
\[ |\alpha_\beta| \leq |\alpha| - 1 \]
\[ \frac{1}{\delta} Q_{\alpha}(|a_{\alpha',i,i_{B}}|, H_{\alpha'}, H_{\alpha''}); \]
the indices satisfy at least
\[ |\alpha'| + |i_{B}| \leq |\alpha|, |\alpha''| \leq |\alpha| - 1, \]
\[ |\alpha_{\beta}| \leq |\alpha| - 1 \]
Therefore, we have obtained \( \max\{|h_{\alpha}|, |\alpha^{\beta}h_{\alpha}|\} \leq H_{\alpha}. \)

We need this Briot-Bouquet type theorem in case where \( d = 2, r = 2, \max\{|\beta|; \beta \in B\} = 1. \)

\section{Local uniqueness}

To assure that weak solutions in \( W^{1,p}(I) \) are filled locally by our analytical method, we need local uniqueness of solutions to the Cauchy problem to (3). We adapt proofs of uniqueness in J. Benedikt [1] and P. Drábek and M. Ōtani [2] for forth order \( p \)-elliptic equations, completing them with an energy inequality.

Let \( I = [a, b] \) be a compact interval in \((0, \infty)\).

\textbf{Proposition 1 (Local uniqueness).} Let \( r_0 \) be an arbitrary positive constant in \( I \). Local solutions on \( I \) are uniquely determined near \( r_0 \) by initial data \( U(r_0) \) and \( U_{r}(r_0) \).

\textbf{Proof.} Set \( V(r) = r^{n-1}|U_{r}(r)|^{p-2}U_{r}(r), \) \( p' = p/(p-1) \) and \( f_{p}(X) = |X|^{p-2}X. \) Notice \( f_{p,X}(X) = (p-1)|X|^{p-2}. \) Then, we have from the equation,

\begin{align*}
V_{r}(r) &= -r^{n-1}U(r)q^{-2}U(r) = -r^{n-1}f_{q}(U(r)), \\
U_{r}(r) &= r^{\frac{1-n}{p-1}}|V(r)|^{p'-2}V(r) = f_{p'}(r^{1-n}V(r)).
\end{align*}

(9)

Suppose
\[ V_{1}(r_0) = V_{2}(r_0), \quad U_{1}(r_0) = U_{2}(r_0) \]
\[ |U_{0}| + |V_{0}| > 0. \]

We will show that there exists a positive constant \( \epsilon \) such that \( U_{1}(r) = U_{2}(r) \) on \( J(\epsilon) = [r_{0} - \epsilon, r_{0} + \epsilon] \), as proved in Benedikt [1] in 4th order \( p \) elliptic ordinary differential equation.
Case (i): $1 < p \leq 2$ and $2 \leq q$.

\[ V_1(r) - V_2(r) = r^{n-1}\{f_p(U_{1,r}(r)) - f_p(U_{2,r}(r))\} \]
\[ = r^{n-1}\int_{U_{1,r}(r)}^{U_{2,r}(r)} f_p(x) \, dx \]

We set $K_1 = \max\{|U_{i,r}(r)|; r \in I, i = 1,2\}$. Noticing $f_p(x)$ is positive decreasing on $(0, \infty)$, when $1 < p < 2$, we have

\[ |r^{n-1}\{f_p(U_{1,r}(r)) - f_p(U_{2,r}(r))\}| \geq (r_0 - \epsilon)^{n-1}(p-1)K_1^{p-2}|U_{1,r}(r) - U_{2,r}(r)|. \]

We set $K_0 = \max\{|U_{i}(r)|; r \in I, i = 1,2\}$.

\[ V_1(r) - V_2(r) = -\int_{r_0}^{r} \tau^{n-1}\{f_q(U_{1}(\tau)) - f_q(U_{2}(\tau))\} \, d\tau. \]

On the other hand, we have

\[ |f_q(U_{1}(\tau)) - f_q(U_{2}(\tau))| = \left| \int_{U_{2}(\tau)}^{U_{1}(\tau)} f_q(x) \, dx \right| \]
\[ \leq (q-1)K_2^{q-2}|U_{1}(\tau) - U_{2}(\tau)| \]
\[ \leq (q-1)K_2^{q-2}\int_{r_0}^{\tau} |U_{1,r}(\sigma) - U_{2,r}(\sigma)| \, d\sigma \]
\[ \leq (q-1)K_2^{q-2}\epsilon \| U_{1,r} - U_{2,r} \|_{J(\epsilon)}, \]

where $\| U \|_{J(\epsilon)} = \max_{|r-r_0| \leq \epsilon} |U(r)|$.

Choosing $\epsilon$ sufficiently small, we conclude

\[ \| U_{1,r} - U_{2,r} \|_{J(\epsilon)} = 0. \]

Hence, $V_1 = V_2$ on $J(\epsilon)$, therefore, $V_{1,r} = V_{2,r}$, which gives $U_{1}(r) = U_{2}(r)$ on $J(\epsilon)$.

Since we can proceed the rest as in [1], we show only classification of cases.

Case (ii): $1 < p \leq 2$ and $1 < q < 2$

Subcase (ii-1): We assume also $U_0 \neq 0$.

Subcase (ii-2): $U_0 = 0, V_0 \neq 0$. 
Case (iii): $2 < p$, and $2 \leq q$.

Subcase (iii-1): $V_1(r_0) = V_2(r_0) \neq 0$.

Subcase (iii-2): $V_1(r_0) = V_2(r_0) = 0$ and $U_1(r_0) = U_2(r_0) \neq 0$.

Case (iv): $2 < p$ and $1 < q < 2$.

Subcase (iv-1): $U_0 = 0$, and $V_0 \neq 0$.

Subcase (iv-2): $U_0 \neq 0$, and $V_0 = 0$.

We need different arguments, when $U_0 = V_0 = 0$. We assume $U_0 = V_0 = 0$.

To complete the proof, we use an energy inequality.

**Proposition 2.** (i) Every nonzero $W^{1,p}(I)$ solution $U$ on $I$ has $C^1(\overline{I})$ regularity.

(ii) Then, it satisfies the energy equality

$$\frac{p-1}{p}|U_r(r)|^p + \frac{1}{q}|U(r)|^q = \frac{p-1}{p}|U_r(c)|^p + \frac{1}{q}|U(c)|^q - (n-1)\int_c^r \frac{1}{\sigma}|U_r(\sigma)|^p d\sigma$$

for all $r, c \in I$.

(iii) If $U(r_0) = U_r(r_0) = 0$ for some $r_0 \in I$, then, $U(r) = 0$ on $I$.

**Remark 2.1.** (i) is due to M. Ôtani [6].

When $n = 1$, (ii) is the energy equality.

When $n = 1$, (iii) is trivial (for any $p, q > 1$) in virtue of the energy equality. When $n > 1$, this completes the proof of local uniqueness.

**Proof of (iii).** $U(r) = 0$ for all $r \in [r_0, b]$ in virtue of the energy inequality.

For all $r \in [a, r_0]$ we have

$$\frac{p-1}{p}|U_r(r)|^p + \frac{1}{q}|U(r)|^q = (n-1)\int_r^{r_0} \frac{1}{\sigma}|U_r(\sigma)|^p d\sigma,$$

therefore,

$$\frac{p-1}{p}|U_r(r)|^p \leq (n-1)\int_r^{r_0} \frac{1}{\sigma}|U_r(\sigma)|^p d\sigma.$$
By Gronwall’s lemma, we have $U_r(r) = 0$ on $I$. Since $U(r_0) = 0$, it implies $U(r) = 0$ on $I$.

Remark 2.2. We note a different proof, when $q \geq p$ as in [2].

We note
\[ r^{n-1} f_p(U_r(r)) = V(r) = \int_{r_0}^{r} V_r(\tau) d\tau = - \int_{r_0}^{r} \tau^{n-1} f_q(U(\tau)) d\tau. \]

We have $r_0^{n-1} \| U_r \|_{J(\epsilon)}^{p-1} \leq \epsilon \| V_r \|_{J(\epsilon)} \leq \epsilon (r_0 + \epsilon)^{n-1} \| U \|_{J(\epsilon)}^{q-1} \leq \epsilon^q (r_0 + \epsilon)^{n-1} \| U_r \|_{J(\epsilon)}^{q-1}$. If we assume $\| U \|_{J(\epsilon)} > 0$, we have $r_0^{n-1} \leq \epsilon^q \| U \|_{J(\epsilon)}^{q-p}$ for any small positive $\epsilon$. This gives contradiction, hence $\| U \|_{J(\epsilon)} = 0$.

4 Analytic singularities

We shall now describe local analytic singularities of the solution $U(r)$ to (4).

When $n = 1$, a classical Briot-Bouquet type theorem of one variable was sufficient to obtain the unique existence of analytic solution to the nonlinear ordinary differential equation ([8]).

When $U(r_0) \neq 0$ and $U_r(r_0) \neq 0$, $r_0$ is an analytic point of solution $U(r)$. Hence, we consider two types of singularities:
• $r_0 = \sigma$ where $U(\sigma) = 0$ and $U_r(\sigma) = A \neq 0$,
• $r_0 = \tau$ where $U(\tau) = A \neq 0$ and $U_r(\tau) = 0$.

CASE 1. $\sigma$ where $U(\sigma) = 0$ and $U_r(\sigma) = A \neq 0$. We assume $A > 0$ without loss of generality. We treat the case where $\sigma > 0$.

Theorem 3. For $1 < p, q < \infty$, there exists a unique analytic function $F(t, s)$ in a neighborhood of the origin such that we have near $r = \sigma$
\[ U(r) = (r - \sigma)F(r - \sigma, |r - \sigma|^q). \quad (11) \]

$F(t, s)$ is a holomorphic solution to
\[ (p - 1)(\sigma + t)\{F(t, s) + tF_t(r, s) + q s F_s(t, s)\}^{p-2} \]
\[ \{tF_t(t, s) + q s F_s(t, s) \]
\[ + t(tF_t(t, s))_t + 2 q t s F_{t,s}(t, s) + q^2 s(sF_s(t, s))_s \} \]
\[ + (\sigma + t)s(F(t, s))^{q-1} \quad (12) \]
\[ + (n - 1)t \{ F(t, s) + tF_{t}(t, s) + qsF_{s}(t, s) \}^{p-1} \]
\[ = 0 \]  
(13)

with

\[ F(0, 0) = A. \]  
(14)

Consequently, we can compute the expansion of \( U(r) \) at \( r = \sigma \):

\[ U(r) = (r - \sigma) \left\{ A - \frac{n - 1}{2\sigma(p - 1)}A(r - \sigma) \right. \]
\[ - \frac{A^{q-p+1}}{(p - 1)q(q + 1)}|r - \sigma|^q + \cdots \left\} \]  
(15)

Proof. We reduce equation (13) by change of the unknown function

\[ F(t, s) = A + h(t, s) \]

into

\[ L \left( t \frac{\partial}{\partial t}, s \frac{\partial}{\partial s} \right) h(t, s) = -\frac{1}{(p - 1)(\sigma + t)} \]
\[ \times \{ A + h(t, s) + th_{t}(t, s) + qsh_{s}(t, s) \}^{2-p} \]
\[ \times [(\sigma + t)s(A + h(t, s))^{q-1} + (n - 1)t \]
\[ \{ A + h(t, s) + th_{t}(t, s) + qsh_{s}(t, s) \}^{p-1} \}
\[ = a_{1,0}t + a_{0,1}s \]
\[ + \sum_{2 \leq p + q + i + j + k} a_{p,q,i,j,k}t^{p}s^{q} \]
\[ \times (h(t, s))^{i} \left( t \frac{\partial h}{\partial t}(t, s) \right)^{j} \left( s \frac{\partial h}{\partial s}(t, s) \right)^{k}, \]  
(16)

where

\[ a_{1,0} = -\frac{(n - 1)A}{(p - 1)\sigma} \]
\[ a_{0,1} = -\frac{1}{p - 1}A^{q-p+1}. \]  
(17)

We have \( L(\alpha, \beta) = \alpha + q\beta + \alpha^2 + 2q\alpha\beta + q^2\beta^2 \).
Since $L(1, 0) = 2$ and $L(0, 1) = q(q + 1)$, $F_t(0, 0)$ and $F_s(0, 0)$ are determined and so on. Thus, the unique existence of the solution is obtained by the B-B type theorem of two variables.

Next, $(r - \sigma)F(r - \sigma, |r - \sigma|^q)$ is a $C^2$ function near $\sigma$. It satisfies (1) with the prescribed Cauchy data. By Proposition 1, it is equal to the unique solution $U(r)$ with the same Cauchy data. □

**Corollary 1** ([4], [7],[8]). (i) When $q$ is an even integer more than 1, the solution $U(r)$ is real analytic near $\sigma$.

(ii) When $q$ is not an even integer, the solution $U(r)$ is of class $C^{<q}$ at $\sigma$, where $< x >$ is the least integer greater than or equal to $x$.

**CASE 2.** $\tau$ where $U(\tau) = A$ and $U_\tau(\tau) = 0$.

As in the case 1, we can assume without loss of generality that $A > 0$.

**Theorem 4.** For any $p$ and $q$ satisfying $1 < p, q < \infty$, there exists a unique analytic function $G(t, s)$ in a neighborhood of the origin such that we have near $r = \tau$

$$U(r) = A + |r - \tau|^\underline{p}G(r - \tau, |r - \tau|^\underline{p-1}),$$

where $G(t, s)$ is a holomorphic solution to the nonlinear equation:

$$
\begin{align*}
(p - 1)(t + \tau)\{&-\left(\frac{p}{p - 1}G(t, s) \\
+ tG_t(t, s) + \frac{p}{p - 1}sG_s(t, s)\right)\}\}^{p-2} \\
\times \left\{&\frac{p}{(p - 1)^2}G(t, s) + \frac{p + 1}{p - 1}tG_t(t, s) \\
+ \frac{p(p + 1)}{(p - 1)^2}sG_s(t, s) + t(tG_t)_t(t, s) \\
+ \frac{2p}{p - 1}tsG_{t,s}(t, s) + \frac{p^2}{(p - 1)^2}s(sG_s)_s(t, s)\right\} \\
+ (\tau + t)(A + sG(t, s))^{q-1} \\
- (n - 1)t \left\{-\left(\frac{p}{p - 1}G(t, s) \\
+ tG_t(t, s) + \frac{p}{p - 1}sG_s(t, s)\right)\}\}^{p-1} = 0
\end{align*}
$$
with
\[ G(0, 0) = -\frac{p-1}{p} A^{\frac{q-1}{p-1}}. \]

Consequently, we have a convergent expansion near \( r = \tau \):
\[ U(r) = A + B |r - \tau|^\frac{p}{p-1} + C(r - \tau)|r - \tau|^\frac{p}{p-1} \]
\[ + D |r - \tau|^\frac{2p}{p-1} + \cdots, \]
where
\[ B = -\frac{p-1}{p} A^{\frac{q-1}{p-1}} \quad \text{and} \]
\[ C = \frac{(n-1)}{2(2p-1)} A^{\frac{q-1}{p-1}} \]
\[ D = \frac{q-1}{2(2p-1)} \left( \frac{p-1}{p} \right)^2 A^{1+\frac{2(q-p)}{p-1}}. \]

Proof. We show, at first, unique existence of the solution \( G(t, s) \).
We reduce the equation by change of unknown function by
\[ G(t, s) = B + h(t, s), \]
into
\[ \frac{p}{(p-1)^2} h(t, s) + \frac{p+1}{p-1} th_t(t, s) + \frac{p(p+1)}{(p-1)^2} sh_s(t, s) \]
\[ + t(th_t)_t(t, s) + \frac{2p}{p-1} tsh_{t,s}(t, s) \]
\[ + \frac{p^2}{(p-1)^2} s(sh_s)_s(t, s) \]
\[ = -\frac{p}{(p-1)^2} B - \frac{1}{(p-1)(t+\tau)} \]
\[ \times \left\{ \left( \frac{pB}{p-1} - \frac{p}{p-1} h(t, s) \right) \right. \]
\[ \left. - th_t(t, s) - \frac{p}{p-1} sh_s(t, s) \right\}^{2-p} \]
\[ \times \left\{ (\tau + t)(A + sB + sh(t, s))^{q-1} \right. \]
\[-(n-1)t \left( -\frac{p}{p-1} B - \frac{p}{p-1} h(t, s) \right) - th_t(t, s) - \frac{p}{p-1} sh_s(t, s) \right)^{p-1}\]  

(28)

Developing the right hand side with respect ot \((t, s, h, \rho, \theta)\), where \(\rho = sh_s\) and \(\theta = sh_s\), we, at first, obtain

\[B = -\frac{p-1}{p} A^{\frac{s-1}{p-1}}\]

by the condition that the constant term vanishes:

\[-\frac{p}{(p-1)^2} B - \frac{1}{p-1} \left\{ -\frac{p}{p-1} B \right\}^{2-p} A^{q-1} = 0.\]

Then, we have the development of the R.H.S. is

\[R.H.S. = a_{1,0,0,0,0} t + a_{0,1,0,0,0} s + \frac{2p - p^2}{(p-1)^2} h(t, s) + \frac{2-p}{p-1} th_t(t, s) + \frac{2p - p^2}{(p-1)^2} sh_s(t, s) + \sum_{2\leq\alpha+\beta+i+j+k} a_{\alpha,\beta,i,j,k} t^\alpha s^\beta \times (h(t, s))^{i} \left( \frac{\partial h}{\partial t}(t, s) \right)^{j} \left( \frac{\partial h}{\partial s}(t, s) \right)^{k}\]

(29)

(30)

Set

\[l_{0,0} = \frac{p}{(p-1)^2} - \frac{2p - p^2}{(p-1)^2} = \frac{p}{p-1},\]

\[l_{1,0} = \frac{p+1}{p-1} - \frac{2-p}{p-1} = \frac{2p-1}{p-1},\]

\[l_{0,1} = \frac{p^2 + p}{(p-1)^2} - \frac{2p - p^2}{(p-1)^2} = \frac{p(2p-1)}{(p-1)^2},\]

\[l_{2,0} = 1, l_{1,1} = \frac{2p}{p-1}, l_{0,2} = \frac{p^2}{(p-1)^2}.\]
Thus, we have the reduced equation

\[
L \left( t \frac{\partial}{\partial t}, s \frac{\partial}{\partial s} \right) h(t, s) = a_{1,0,0,0} t + a_{0,1,0,0} s + \sum_{2 \leq \alpha + \beta + i + j + k} a_{\alpha, \beta, i, j, k} t^{\alpha} s^{\beta}
\]

\[\times (h(t, s))^{i} \left( t \frac{\partial h}{\partial t}(t, s) \right)^{j} \left( s \frac{\partial h}{\partial s}(t, s) \right)^{k}. \tag{32}\]

Since \( L \) satisfies the condition (5), the unique existence of the solution to (21) is obtained by our Briot-Bouquet type theorem.

\[\square\]

**Corollary 2** ([4], [7],[8]). (i) If \( p/(p - 1) \) is an even integer, i.e. \( p = (2m + 2)/(2m + 1) \) \((m = 0, 1, 2, \ldots)\), \( u(x) \) is real analytic at \( \tau \).

(ii) If \( p/(p - 1) \) is not an even integer, the solution \( U(r) \) is of class \( C^{<\frac{2-p}{p-1}>+1} \) at \( \tau \), where \( <x> \) is the least integer greater than or equal to \( x \). Especially, when \( 1 < p \leq 2 \), \( U(r) \) is of class \( C^{2} \) at \( \tau \). When \( 2 < p \), \( U(r) \) is not of class \( C^{2} \) at \( \tau \).

**References**


