<table>
<thead>
<tr>
<th>Title</th>
<th>Analytic singularities of solutions to a radial $p$-Laplacian</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hyperfunctions and linear differential equations 2006. History of Mathematics and Algorithms</td>
<td></td>
</tr>
<tr>
<td>Author(s)</td>
<td>Uchiyama, Koichi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2009), 1648: 17-31</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2009-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/140726">http://hdl.handle.net/2433/140726</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Analytic singularities of solutions to a radial $p$-Laplacian

Koichi Uchiyama
Faculty of Science and Technology,
Sophia University

Abstract
Analytic local description of $W^{1,p}(I)$ solutions to a radial $p$-Laplace equation

$$r \left( |U_r|^{p-2}U_r \right)_r + (n-1)|U_r|^{p-2}U_r + r|U|^{q-2}U = 0$$

on $I = [a, b] \subset (0, \infty)$ is given near singular points by a Briot-Bouquet type theorem of two variables, where $1 < p, q < \infty$.

1 Introduction

An $n$-dimensional $p$-elliptic PDE for $u(x)$ is

$$\text{div}(|\nabla u|^{p-2}\nabla u(x)) + |u|^{q-2}u = 0,$$  \hspace{1cm} (1)

where $x \in \mathbb{R}^n$ and $1 < p, q < \infty$.

If $x \in \mathbb{R}^1$, the equation reduces to

$$\left( |u_x|^{p-2}u_x \right)_x + |u|^{q-2}u = 0.$$ \hspace{1cm} (2)

L. Paredes and the present author, making use of a Briot-Bouquet type theorem of one variable, gave analytic expression of solutions to the equation (2) near the singularities (8). Our analytic expression readily reproduces differentiability and analyticity obtained by M. Ôtani [6], [7] and by M. Ôtani and T. Idogawa in [4].

If $r = |x|, x \in \mathbb{R}^n$,
a radial solution $U(r) = u(x)$ satisfies

$$r^{n-1}|U_r|^{p-2}U_r + r^{n-1}|U|^{q-2}U = 0,$$ \hspace{1cm} (3)
or

$$r \left( |U_r|^{p-2}U_r \right)_r + (n-1)|U_r|^{p-2}U_r + r|U|^{q-2}U = 0.$$ \hspace{1cm} (4)

The aim of this report is to extend the results for (2) to the radial $p$ Laplacian (3) by a Briot-Bouquet type theorem of two variables.
Remark 0.1. R. Gérard and H. Tahara studied $t \frac{\partial u}{\partial t} = \Phi(t, x, u, \frac{\partial u}{\partial x})$ and generalized it of many variables and of higher order case in [3]. Since our version of a Briot-Bouquet type theorem of two (or several) variables is not covered by theirs, a proof is given, inspired by their work.

2 A Briot-Bouquet type theorem

We recall a classical Briot-Bouquet type theorem of one variable in complex domain. We assume

- $\Phi(t, h)$ is holomorphic near $(0,0) \in C^2$,
- $\Phi(0,0) = 0$,
- $\frac{\partial \Phi}{\partial h}(0,0)$ is not any positive integers.

Theorem 1 (Briot-Bouquet).

$t \frac{dh}{dt} = \Phi(t, h)$ has a unique holomorphic solution near $t = 0$, satisfying $h(0) = 0$.

If $\Phi(t, h)$ is real analytic, so is $h(t)$, too.

Proof. Set $a_{\alpha,i} = \frac{1}{\alpha!i!} \frac{\partial^{\alpha+i} \Phi}{\partial t^\alpha \partial h^i}(0,0)$. Notice we can rewrite the equation by

$t \frac{dh}{dt} - a_{0,1}h = a_{1,0}t + \sum_{2 \leq \alpha+i} a_{\alpha,i} t^\alpha h^i$.

Moreover, the left hand side satisfies the condition that there exists $\delta > 0$ such that

$|\alpha - a_{0,1}| \geq \delta$

for all $\alpha \in N^* = \{1, 2, 3, \cdots \}$.

Formal solution: Let $\hat{h}(t) = \sum_{\alpha=1}^{\infty} h_{\alpha} t^\alpha$ be a formal solution. Then, we have

$(1 - a_{0,1})h_1 = a_{1,0},$

$(\alpha - a_{0,1})h_\alpha = Q_\alpha(a_{\alpha',i}, h_{\alpha''}; \alpha' + i \leq \alpha, \alpha'' \leq \alpha - 1)$.
for all $\alpha \geq 2$, where $Q_\alpha$ is a polynomial with nonnegative integer coefficients.

**Convergence:** Then, we prove convergence of $\hat{h}(t)$ through the implicit function theorem. An auxiliary equation of $H(t)$ is given by

$$\delta H = |a_{1,0}| t + \sum_{2 \leq \alpha + i} |a_{\alpha,i}| t^\alpha H^i.$$

with $H(0) = 0$.

There exists a unique convergent series function $H(t) = \sum_{\alpha=1}^\infty H_\alpha t^\alpha$ by the implicit function theorem. Since $\delta H_\alpha = Q_\alpha(|a_{\alpha',i}|, H_{\alpha''}; \alpha' + i \leq \alpha, \alpha'' \leq \alpha - 1)$,

$$|h_1| = |a_{1,0}|/|1 - a_{0,1}|$$
$$\leq |a_{1,0}|/\delta = H_1,$$

$$|h_\alpha| = |Q_\alpha(a_{\alpha',i}, H_{\alpha''})/(\alpha - a_{0,1})|$$
$$\leq Q_\alpha(|a_{\alpha',i}|, H_{\alpha''})/\delta = H_\alpha$$

for $\alpha \geq 2$ by induction.

\[\square\]

We will make use of a Briot-Bouquet type theorem of two variables for our main results. We state it in a slightly more general form for convenience.

Let $N = \{0,1,2, \cdots \}$. $B = \{\beta\}$ is a fixed finite subset of $N^d$, where $\beta = (\beta_1, \cdots, \beta_d)$ is a $d$-dimensional multi-index with $|\beta| \geq 1$. Let $(t, h, \rho_B) = (t_1, \cdots, t_d, h, \{\rho_\beta; \beta \in B\}) \in C^{d+1+|B|}$ be local variables near the origin, where $|B|$ is the number of the elements in $B$. Let $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{C}^d$ be global variables. $\alpha = (\alpha_1, \cdots, \alpha_d)$, $\beta$ and $\gamma$ denote $d$-dimensional power indices in $N^d$.

**Theorem 2.** We assume that a holomorphic function $\phi(t, h, \rho_B)$ and a polynomial

$$L(\xi) = \sum_{0 \leq |\gamma| \leq r} l_\gamma \xi^\gamma$$

satisfy

(i) $\phi(t, \rho_B)$ has a power series expansion near $(0,0)$ without linear
parts:

\[ \phi(t, \rho_B) = \sum_{2 \leq |\alpha| + |i_B|} a_{\alpha, i_B} t^\alpha \rho_B^{i_B} \]

\[ = \sum_{2 \leq |\alpha| + |i_B|} a_{\alpha, i_B} t^\alpha \prod_{\beta \in B} \rho_\beta^{i_\beta}, \]

where \(|i_B| = \sum_{\beta \in B} i_\beta\) for a multi-index \(i_B = (i_\beta)_{\beta \in B}\) and

(ii) there exists a positive constant \(\delta\) such that for all d-dimensional multi indices \(\alpha\) with \(|\alpha| \geq 1\),

\[ |\sum_{0 \leq |\gamma| \leq r} l_\gamma \alpha^\gamma| \geq \delta \max\{1, \alpha^\beta; \beta \in B\}, \]

(5)

where \(\alpha^\beta\) denotes the coefficient of \((t \frac{\partial}{\partial t})^\beta t^\alpha\).

Then, a nonlinear equation

\[ \sum_{0 \leq |\gamma| \leq r} l_\gamma \left( t \frac{\partial}{\partial t} \right)^\gamma h(t) = a \cdot t + \]

\[ + \sum_{2 \leq |\alpha| + |i_B|} a_{\alpha, i_B} t^\alpha \]

\[ \cdot \prod_{\beta \in B} \left( t \frac{\partial}{\partial t} h(t) \right)^{i_\beta} \]

has a unique holomorphic solution \(h(t)\) near the origin with \(h(0) = 0\).

Proof. We will follow the previous proof.

Construction of \(\hat{h}(t)\): We set

\[ \hat{h}(t) = \sum_{|\alpha| \geq 1} h_\alpha t^\alpha. \]

(7)

Substituting (7) into (6), we have

\[ L(\alpha)h_\alpha = a_\alpha \text{ when } |\alpha| = 1, \]
and
\[ L(\alpha)h_\alpha = Q_\alpha(a_{\alpha',i_B}, h_{\alpha''}, (\alpha_{\beta})^\beta h_{\alpha_{\beta}}; \beta \in B \]

the indices at least satisfy
\[ |\alpha'| + |i_B| \leq |\alpha|, |\alpha''| \leq |\alpha| - 1, \]
\[ |\alpha_{\beta}| \leq |\alpha| - 1, \]

where \( \alpha', \alpha'', \alpha_{\beta} \) are copies of \( \alpha \). Thus, \( h_\alpha \) are determined successively.

**Convergence of \( \hat{h} \):** An auxiliary analytic equation (cf. Gérard-Tahara [3])
\[
\delta H = |a_1|t_1 + |a_2|t_2 + \sum_{2 \leq |\alpha| + |i_B|} |a_{\alpha,i_B}|t^\alpha (H(t))^{i_B}.
\]

Solving this equation of \( H \) by the implicit function theorem, we have a unique holomorphic solution near the origin \( t = 0 \) with \( H(0) = 0 \). We claim
\[ H(t) = \sum_\alpha H_\alpha t^\alpha >> \hat{h}(t). \]

More strongly we claim,
\[ H_\alpha \geq \max\{|h_\alpha|, |\alpha^\beta h_\alpha|; \beta \in B\}. \]

We notice
\[ H_\alpha = \frac{1}{\delta}Q_\alpha(|a_{\alpha',i_B}|, H_{\alpha''}, H_{\alpha_{\beta}}; \text{the indices satisfy} \]
\[ \text{at least } |\alpha'| + |i_B| \leq |\alpha|, |\alpha''| \leq |\alpha| - 1, \]
\[ |\alpha_{\beta}| \leq |\alpha| - 1, \beta \in B). \]

We start with \( |\alpha| = 1 \):
\[ |h_\alpha| = |a_\alpha/L(\alpha)| \leq |a_\alpha|/\delta = H_\alpha. \]

Then, by induction, we have
\[ \max\{1, \alpha^\beta; \beta \in B\} \cdot |h_\alpha| \leq \frac{1}{\delta}Q_\alpha(|a_{\alpha',i_B}|, |h_{\alpha''}|, |(\alpha_{\beta})^\beta h_{\alpha_{\beta}}|; \text{indices satisfy at least} \]
\[ |\alpha'| + |i_B| \leq |\alpha|, |\alpha''| \leq |\alpha| - 1, \]
\[ |\alpha_{\beta}| \leq |\alpha| - 1 \]
\[
\leq \frac{1}{\delta} Q_{\alpha'}(|a_{\alpha',i,i_B}|, H_{\alpha'}, H_{\alpha''};
\]
the indices satisfy at least
\[
|\alpha'| + |i_B| \leq |\alpha|, |\alpha''| \leq |\alpha| - 1,
|\alpha_{\beta}| \leq |\alpha| - 1
\]

Therefore, we have obtained \( \max\{|h_{\alpha}|, |\alpha^\beta h_{\alpha}|\} \leq H_{\alpha}. \)

We need this Briot-Bouquet type theorem in case where \( d = 2, \)
\( r = 2, \max\{|\beta|; \beta \in B\} = 1. \)

3 Local uniqueness

To assure that weak solutions in \( W^{1,p}(I) \) are filled locally by our analytical method, we need local uniqueness of solutions to the Cauchy problem to (3). We adapt proofs of uniqueness in J. Benedikt [1]

Let \( I = [a, b] \) be a compact interval in \((0, \infty)\).

**Proposition 1** (Local uniqueness). Let \( r_0 \) be an arbitrary positive constant in \( I \). Local solutions on \( I \) are uniquely determined near \( r_0 \) by initial data \( U(r_0) \) and \( U_r(r_0) \).

**Proof.** Set \( V(r) = r^{n-1}|U(r)|^{q-2}U(r), p' = p/(p-1) \) and \( f_p(X) = |X|^{p-2}X. \) Notice \( f_{p,X}(X) = (p-1)|X|^{p-2} \). Then, we have from the equation,

\[
\begin{cases}
V_r(r) = -r^{n-1}|U(r)|^{q-2}U(r) = -r^{n-1}f_q(U(r)), \\
U_r(r) = r^{1-n/p}|V(r)|^{p'-2}V(r) = f_{p'}(r^{1-n}V(r)).
\end{cases}
\]

(9)

Suppose

\[
V_1(r_0) = V_2(r_0), \quad U_1(r_0) = U_2(r_0)
\]

\(|U_0| + |V_0| > 0.\)

We will show that there exists a positive constant \( \epsilon \) such that \( U_1(r) = U_2(r) \) on \( J(\epsilon) = [r_0 - \epsilon, r_0 + \epsilon], \) as proved in Benedikt [1] in 4th order \( p \) elliptic ordinary differential equation.
Case (i): $1 < p \leq 2$ and $2 \leq q$.

$$V_1(r) - V_2(r) = r^{n-1}\{f_p(U_{1,r}(r)) - f_p(U_{2,r}(r))\}$$

$$= r^{n-1} \int_{U_{2,r}(r)}^{U_{1,r}(r)} f_{p,X}(\tau) d\tau$$

We set $K_1 = \max\{|U_{i,r}(r)|; r \in I, i = 1, 2\}$. Noticing $f_{p,X}(X)$ is positive decreasing on $(0, \infty)$, when $1 < p < 2$, we have

$$|r^{n-1}\{f_p(U_{1,r}(r)) - f_p(U_{2,r}(r))\}| \geq (r_0 - \epsilon)^{n-1}(p - 1)K_1^{p-2}|U_{1,r}(r) - U_{2,r}(r)|.$$

We set $K_0 = \max\{|U_i(r)|; r \in I, i = 1, 2\}$.

$$V_1(r) - V_2(r) = -\int_{r_0}^{r} \tau^{n-1}\{f_q(U_1(\tau)) - f_q(U_2(\tau))\}d\tau.$$

On the other hand, we have

$$|f_q(U_1(\tau)) - f_q(U_2(\tau))| = \left| \int_{U_2(\tau)}^{U_1(\tau)} f_{q,X}(\sigma) d\sigma \right|$$

$$\leq (q - 1)K_2^{q-2}|U_1(\tau) - U_2(\tau)|$$

$$\leq (q - 1)K_2^{q-2} \left| \int_{r_0}^{\tau} \{U_{1,r}(\sigma) - U_{2,r}(\sigma)\} d\sigma \right|$$

$$\leq (q - 1)K_2^{q-2}\epsilon\|U_{1,r} - U_{2,r}\|_{J(\epsilon)},$$

where $\|U\|_{J(\epsilon)} = \max_{|r - r_0| \leq \epsilon} |U(r)|$.

Choosing $\epsilon$ sufficiently small, we conclude

$$\|U_{1,r} - U_{2,r}\|_{J(\epsilon)} = 0.$$

Hence, $V_1 = V_2$ on $J(\epsilon)$, therefore, $V_{1,r} = V_{2,r}$, which gives $U_1(r) = U_2(r)$ on $J(\epsilon)$.

Since we can proceed the rest as in [1], we show only classification of cases.

Case (ii): $1 < p \leq 2$ and $1 < q < 2$

Subcase (ii-1): We assume also $U_0 \neq 0$.

Subcase (ii-2): $U_0 = 0$, $V_0 \neq 0$. 
Case (iii): $2 < p$, and $2 \leq q$.
Subcase (iii-1): $V_1(r_0) = V_2(r_0) \neq 0$.
Subcase (iii-2): $V_1(r_0) = V_2(r_0) = 0$ and $U_1(r_0) = U_2(r_0) \neq 0$.

Case (iv): $2 < p$ and $1 < q < 2$.
Subcase (iv-1): $U_0 = 0$, and $V_0 \neq 0$.
Subcase (iv-2): $U_0 \neq 0$, and $V_0 = 0$.

We need different arguments, when $U_0 = V_0 = 0$. We assume $U_0 = V_0 = 0$.
To complete the proof, we use an energy inequality.

**Proposition 2.** (i) Every nonzero $W^{1,p}(I)$ solution $U$ on $I$ has $C^1(\overline{I})$ regularity.
(ii) Then, it satisfies the energy equality
\[
\frac{p-1}{p} |U_r(r)|^p + \frac{1}{q} |U(r)|^q = \frac{p-1}{p} |U_r(c)|^p + \frac{1}{q} |U(c)|^q
- (n-1) \int_c^r \frac{1}{\sigma} |U_r(\sigma)|^p d\sigma
\]  

for all $r, c \in I$.
(iii) If $U(r_0) = U_r(r_0) = 0$ for some $r_0 \in I$, then, $U(r) = 0$ on $I$.

**Remark 2.1.** (i) is due to M. Ôtani [6].
When $n = 1$, (ii) is the energy equality.
When $n = 1$, (iii) is trivial (for any $p, q > 1$) in virtue of the energy equality. When $n > 1$, this completes the proof of local uniqueness.

**Proof of (iii).** $U(r) = 0$ for all $r \in [r_0, b]$ in virtue of the energy inequality.

For all $r \in [a, r_0]$ we have
\[
\frac{p-1}{p} |U_r(r)|^p + \frac{1}{q} |U(r)|^q = (n-1) \int_r^{r_0} \frac{1}{\sigma} |U_r(\sigma)|^p d\sigma,
\]
therefore,
\[
\frac{p-1}{p} |U_r(r)|^p \leq (n-1) \int_r^{r_0} \frac{1}{\sigma} |U_r(\sigma)|^p d\sigma.
\]
By Gronwall's lemma, we have $U_r(r) = 0$ on $I$. Since $U(r_0) = 0$, it implies $U(r) = 0$ on $I$.

Remark 2.2. We note a different proof, when $q \geq p$ as in [2].

We note

$$r^{n-1} f_p(U_r(r)) = V(r) = \int_{r_0}^{r} V_r(\tau) d\tau = -\int_{r_0}^{r} \tau^{n-1} f_q(U(\tau)) d\tau.$$ 

We have $r_0^{n-1} || U_r ||_{J(\epsilon)}^{p-1} \leq \epsilon || V_r ||_{J(\epsilon)} \leq \epsilon (r_0 + \epsilon)^{n-1} || U ||_{J(\epsilon)}^{q-1}$. If we assume $|| U ||_{J(\epsilon)} > 0$, we have $r_0^{n-1} \leq \epsilon^q || U_r ||_{J(\epsilon)}^{q-p}$ for any small positive $\epsilon$. This gives contradiction, hence $|| U ||_{J(\epsilon)} = 0$.

4 Analytic singularities

We shall now describe local analytic singularities of the solution $U(r)$ to (4).

When $n = 1$, a classical Briot-Bouquet type theorem of one variable was sufficient to obtain the unique existence of analytic solution to the nonlinear ordinary differential equation ([8]).

When $U(r_0) \neq 0$ and $U_r(r_0) \neq 0$, $r_0$ is an analytic point of solution $U(r)$. Hence, we consider two types of singularities:

• $r_0 = \sigma$ where $U(\sigma) = 0$ and $U_r(\sigma) = A \neq 0$,

• $r_0 = \tau$ where $U(\tau) = A \neq 0$ and $U_r(\tau) = 0$.

CASE 1. $\sigma$ where $U(\sigma) = 0$ and $U_r(\sigma) = A \neq 0$. We assume $A > 0$ without loss of generality. We treat the case where $\sigma > 0$.

Theorem 3. For $1 < p, q < \infty$, there exists a unique analytic function $F(t, s)$ in a neighborhood of the origin such that we have near $r = \sigma$

$$U(r) = (r - \sigma)F(r - \sigma, |r - \sigma|^q). \quad (11)$$

$F(t, s)$ is a holomorphic solution to

$$(p - 1)(\sigma + t)\{F(t, s) + t F_t(r, s) + q s F_s(t, s)\}^{p-2}$$

$$\{t F_t(t, s) + q s F_s(t, s)$$

$$+ t(t F_t(t, s)) + 2 q t s F_{t,s}(t, s) + q^2 s(s F_s(t, s))\}$$

$$+ (\sigma + t)s(F(t, s))^{q-1} \quad (12)$$

(continued)
\[ + (n - 1)t \{ F(t, s) + tF_t(t, s) + qsF_s(t, s) \}^{p-1} = 0 \]  

with

\[ F(0, 0) = A. \]  

Consequently, we can compute the expansion of \( U(r) \) at \( r = \sigma \):

\[ U(r) = (r - \sigma) \left\{ A - \frac{n - 1}{2\sigma(p - 1)} A(r - \sigma) \right\} \left\{ \frac{A^{q-p+1}}{(p - 1)q(q + 1)} |r - \sigma|^q + \cdots \right\}. \]

**Proof.** We reduce equation (13) by change of the unknown function

\[ F(t, s) = A + h(t, s) \]

into

\[
L \left( t \frac{\partial}{\partial t}, s \frac{\partial}{\partial s} \right) h(t, s) = \frac{1}{(p - 1)(\sigma + t)} \times \{ A + h(t, s) + th_t(t, s) + qsh_s(t, s) \}^{2-p} \times [(\sigma + t)s(A + h(t, s))^{q-1} + (n - 1)t \{ A + h(t, s) + th_t(t, s) + qsh_s(t, s) \}^{p-1}] \\
= a_{1,0}t + a_{0,1}s \\
+ \sum_{2 \leq p+q+i+j+k} a_{p,q,i,j,k} t^p s^q \\
\times (h(t, s))^i \left( t \frac{\partial h}{\partial t}(t, s) \right)^j \left( s \frac{\partial h}{\partial s}(t, s) \right)^k,
\]

where

\[ a_{1,0} = - \frac{(n - 1)A}{(p - 1)\sigma} \]

\[ a_{0,1} = - \frac{1}{p - 1} A^{q-p+1}. \]

We have \( L(\alpha, \beta) = \alpha + q\beta + \alpha^2 + 2q\alpha\beta + q^2\beta^2 \).
Since $L(1,0) = 2$ and $L(0,1) = q(q+1)$, $F_t(0,0)$ and $F_s(0,0)$ are determined and so on. Thus, the unique existence of the solution is obtained by the B-B type theorem of two variables.

Next, $(r - \sigma) F(r - \sigma, |r - \sigma|^q)$ is a $C^2$ function near $\sigma$. It satisfies (1) with the prescribed Cauchy data. By Proposition 1, it is equal to the unique solution $U(r)$ with the same Cauchy data. 

**Corollary 1** ([4], [7],[8]). (i) When $q$ is an even integer more than 1, the solution $U(r)$ is real analytic near $\sigma$.

(ii) When $q$ is not an even integer, the solution $U(r)$ is of class $C^{<q}$ at $\sigma$, where $<x>$ is the least integer greater than or equal to $x$.

**CASE 2.** $\tau$ where $U(\tau) = A$ and $U_\tau(\tau) = 0$.

As in the case 1, we can assume without loss of generality that $A > 0$.

**Theorem 4.** For any $p$ and $q$ satisfying $1 < p, q < \infty$, there exists a unique analytic function $G(t,s)$ in a neighborhood of the origin such that we have near $r = \tau$

$$U(r) = A + |r - \tau|^\frac{p}{p-1}G \left(r - \tau, |r - \tau|^\frac{p}{p-1}\right), \quad (19)$$

where $G(t,s)$ is a holomorphic solution to the nonlinear equation:

$$(p-1)(t+\tau)\left\{-\left(\frac{p}{p-1}\right)G(t,s) + tG_t(t,s) + \frac{p}{p-1}sG_s(t,s)\right\}^{p-2} \times \left\{ \frac{p}{(p-1)^2}G(t,s) + \frac{p+1}{p-1}tG_t(t,s) + \frac{p(p+1)}{(p-1)^2}sG_s(t,s) + t(tG_t)_t(t,s) + \frac{2p}{p-1}tsG_{t,s}(t,s) + \frac{p^2}{(p-1)^2}s(sG_s)_s(t,s) \right\} + (\tau+t)(A+sG(t,s))^{q-1} - (n-1)t \left\{-\left(\frac{p}{p-1}\right)G(t,s) + tG_t(t,s) + \frac{p}{p-1}sG_s(t,s)\right\}^{p-1} = 0 \quad (21)$$
with

\[ G(0, 0) = -\frac{p-1}{p} A^{\frac{2}{p-1}}. \]

Consequently, we have a convergent expansion near \( r = \tau \):

\[
U(r) = A + B |r - \tau|^\frac{p}{p-1} + C(r - \tau)|r - \tau|^\frac{p}{p-1} \\
+ D|r - \tau|^\frac{2p}{p-1} + \cdots ,
\]

(22)

where

\[
B = -\frac{p-1}{p} A^{\frac{q-1}{p-1}} \text{ and}
\]

(23)

\[
C = \frac{(n-1)}{2(2p-1)} A^{\frac{q-1}{p-1}}
\]

(24)

\[
D = \frac{q-1}{2(2p-1)} \left( \frac{p-1}{p} \right)^2 A^{1+\frac{2(q-1)}{p-1}}.
\]

(25)

Proof. We show, at first, unique existence of the solution \( G(t, s) \).

We reduce the equation by change of unknown function by

\[ G(t, s) = B + h(t, s), \]

into

\[
\frac{p}{(p-1)^2} h(t, s) + \frac{p+1}{p-1} th_t(t, s) + \frac{p(p+1)}{(p-1)^2} sh_s(t, s) \\
+ t(th_t)_t(t, s) + \frac{2p}{p-1} tsh_{t,s}(t, s)
\]

(26)

\[ + \frac{p^2}{(p-1)^2} s(sh_s)_s(t, s) \]

\[ = -\frac{p}{(p-1)^2} B - \frac{1}{(p-1)(t+\tau)} \]

\[ \times \left\{ -\frac{pB}{p-1} - \frac{p}{p-1} h(t, s) \\
- th_t(t, s) - \frac{p}{p-1} sh_s(t, s) \right\}^{2-p} \]

\[ \times \left\{ (\tau+t)(A+sB+sh(t, s))^{q-1} \right\}
\]
Developing the right hand side with respect ot $(t, s, h, \rho, \theta)$, where \( \rho = sh_s \) and \( \theta = sh_s \), we, at first, obtain

\[
B = -\frac{p - 1}{p} A^{\frac{s-1}{p-1}}
\]

by the condition that the constant term vanishes:

\[
-\frac{p}{(p-1)^2} B - \frac{1}{p-1} \left\{ -\frac{p}{p-1} B \right\}^{2-p} A^{q-1} = 0.
\]

Then, we have the development of the R.H.S. is

\[
R.H.S. = a_{1,0,0,0,0} t + a_{0,1,0,0,0} s + \frac{2p - p^2}{(p-1)^2} h(t, s) + \frac{2 - p}{p-1} t h_t(t, s)
\]

\[
+ \frac{2p - p^2}{(p-1)^2} s h_s(t, s)
\]

\[
+ \sum_{2 \leq \alpha + \beta + i + j + k} a_{\alpha,\beta,i,j,k} t^\alpha s^\beta
\]

\[
\times (h(t, s))^i \left( t \frac{\partial h}{\partial t} (t, s) \right)^j \left( s \frac{\partial h}{\partial s} (t, s) \right)^k.
\]

Set

\[
l_{0,0} = \frac{p}{(p-1)^2} - \frac{2p - p^2}{(p-1)^2} = \frac{p}{p - 1},
\]

\[
l_{1,0} = \frac{p + 1}{p - 1} - \frac{2 - p}{p - 1} = \frac{2p - 1}{p - 1},
\]

\[
l_{0,1} = \frac{p^2 + p}{(p-1)^2} - \frac{2p - p^2}{(p-1)^2} = \frac{p(2p - 1)}{(p - 1)^2},
\]

\[
l_{2,0} = 1, l_{1,1} = \frac{2p}{p - 1}, l_{0,2} = \frac{p^2}{(p - 1)^2}.
\]
Thus, we have the reduced equation

\[
L\left(t \frac{\partial}{\partial t}, s \frac{\partial}{\partial s}\right) h(t, s) = a_{1,0,0,0}t + a_{0,1,0,0}s + \sum_{2 \leq \alpha + \beta + i + j + k} a_{\alpha, \beta, i, j, k} t^\alpha s^\beta \\
\times (h(t, s))^i \left(t \frac{\partial h}{\partial t}(t, s)\right)^j \left(s \frac{\partial h}{\partial s}(t, s)\right)^k.
\]

Since \(L\) satisfies the condition (5), the unique existence of the solution to (21) is obtained by our Briot-Bouquet type theorem.

\[\square\]

**Corollary 2** ([4], [7],[8]). (i) If \(p/(p-1)\) is an even integer, i.e. \(p = (2m + 2)/(2m + 1)\) (\(m = 0, 1, 2, \cdots\)), \(u(x)\) is real analytic at \(\tau\).

(ii) If \(p/(p-1)\) is not an even integer, the solution \(U(r)\) is of class \(C_{<x>}^{2}+1\) at \(\tau\), where \(<x>\) is the least integer greater than or equal to \(x\). Especially, when \(1 < p \leq 2\), \(U(r)\) is of class \(C^2\) at \(\tau\). When \(2 < p\), \(U(r)\) is not of class \(C^2\) at \(\tau\).

**References**


