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Analytic singularities of solutions to a radial p-Laplacian

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Abstract

Analytic local description of $W^{1,p}(I)$ solutions to a radial p-Laplace equation

$$r\left(|U_r|^{p-2}U_r\right)_r+(n-1)|U_r|^{p-2}U_r+r|U|^{q-2}U=0$$

on $I=[a,b]\subset(0,\infty)$ is given near singular points by a Briot-Bouquet type theorem of two variables, where $1<p,q<\infty$.

1 Introduction

An n-dimensional p-elliptic PDE for $u(x)$ is

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u(x))+|u|^{q-2}u=0, \tag{1}$$

where $x\in\boldsymbol{R}^n$ and $1<p,q<\infty$.

If $x\in\boldsymbol{R}^1$, the equation reduces to

$$(|u_x|^{p-2}u_x)_x+|u|^{q-2}u=0. \tag{2}$$

L. Paredes and the present author, making use of a Briot-Bouquet type theorem of one variable, gave analytic expression of solutions to the equation (2) near the singularities ([8]). Our analytic expression readily reproduces differentiability and analyticity obtained by M. Ôtani [6], [7] and by M. Ôtani and T. Idogawa in [4].

If $r=|x|, x\in\boldsymbol{R}^n$,
a radial solution $U(r)=u(x)$ satisfies

$$\left(r^{n-1}|U_r|^{p-2}U_r\right)_r+r^{n-1}|U|^{q-2}U=0, \tag{3}$$

or

$$r\left(|U_r|^{p-2}U_r\right)_r+(n-1)|U_r|^{p-2}U_r+r|U|^{q-2}U=0. \tag{4}$$

The aim of this report is to extend the results for (2) to the radial p Laplacian (3) by a Briot-Bouquet type theorem of two variables.

Remark 0.1. R. Gérard and H. Tahara studied $t\frac{\partial u}{\partial t} = \Phi(t, x, u, \frac{\partial u}{\partial x})$ and generalized it of many variables and of higher order case in [3]. Since our version of a Briot-Bouquet type theorem of two (or several) variables is not covered by theirs, a proof is given, inspired by their work.

2 A Briot-Bouquet type theorem

We recall a classical Briot-Bouquet type theorem of one variable in complex domain. We assume

- $\Phi(t, h)$ is holomorphic near $(0, 0) \in \boldsymbol{C}^2$,
- $\Phi(0, 0) = 0$,
- $\dfrac{\partial \Phi}{\partial h}(0, 0)$ is not any positive integers.

Theorem 1 (Briot-Bouquet).

$$t\frac{dh}{dt} = \Phi(t, h)$$

has a unique holomorphic solution near $t = 0$, *satisfying* $h(0) = 0$.

If $\Phi(t, h)$ *is real analytic, so is* $h(t)$, *too.*

Proof. Set $a_{\alpha,i} = \dfrac{1}{\alpha! i!} \dfrac{\partial^{\alpha+i}\Phi}{\partial t^\alpha \partial h^i}(0, 0)$. Notice we can rewrite the equation by

$$t\frac{dh}{dt} - a_{0,1}h = a_{1,0}t + \sum_{2 \le \alpha+i} a_{\alpha,i} t^\alpha h^i.$$

Moreover, the left hand side satisfies the condition that there exists $\delta > 0$ such that

$$|\alpha - a_{0,1}| \ge \delta$$

for all $\alpha \in \boldsymbol{N}^* = \{1, 2, 3, \cdots\}$.

Formal solution: Let $\hat{h}(t) = \sum_{\alpha=1}^{\infty} h_\alpha t^\alpha$ be a formal solution . Then, we have

$$(1 - a_{0,1})h_1 = a_{1,0},$$
$$(\alpha - a_{0,1})h_\alpha = Q_\alpha(a_{\alpha',i}, h_{\alpha''}; \alpha' + i \le \alpha, \alpha'' \le \alpha - 1)$$

for all $\alpha \geq 2$, where Q_α is a polynomial with nonnegative integer coefficients.

Convergence: Then, we prove convergence of $\hat{h}(t)$ through the implicit function theorem.
An auxiary equation of $H(t)$ is given by

$$\delta H = |a_{1,0}|t + \sum_{2\leq\alpha+i} |a_{\alpha,i}|t^\alpha H^i.$$

with $H(0) = 0$.
There exists a unique convergent series function $H(t) = \sum_{\alpha=1}^\infty H_\alpha t^\alpha$ by the implicit function theorem. Since $\delta H_\alpha = Q_\alpha(|a_{\alpha',i}|, H_{\alpha''}; \alpha' + i \leq \alpha, \alpha'' \leq \alpha - 1)$,

$$\begin{aligned} |h_1| &= |a_{1,0}|/|1 - a_{0,1}| \\ &\leq |a_{1,0}|/\delta = H_1, \\ |h_\alpha| &= |Q_\alpha(a_{\alpha'}, h_{\alpha''})/(\alpha - a_{0,1})| \\ &\leq Q_\alpha(|a_{\alpha'}|, H_{\alpha''})/\delta = H_\alpha \end{aligned}$$

for $\alpha \geq 2$ by induction.

□

We will make use of a Briot-Bouquet type theorem of two variables for our main results. We state it in a slightly more general form for convenience.

Let $\boldsymbol{N} = \{0, 1, 2, \cdots\}$. $B = \{\beta\}$ is a fixed finite subset of $\boldsymbol{N}^d$, where $\beta = (\beta_1, \cdots, \beta_d)$ is a d-dimensional multi-index with $|\beta| \geq 1$. Let $(t, h, \rho_B) = (t_1, \cdots, t_d, h, \{\rho_\beta; \beta \in B\}) \in \boldsymbol{C}^{d+1+|B|}$ be local variables near the origin, where $|B|$ is the number of the elements in B. Let $\xi = (\xi_1, \cdots, \xi_d) \in \boldsymbol{C}^d$ be global variables. $\alpha = (\alpha_1, \cdots, \alpha_d), \beta$ and γ denote d-dimensional power indices in $\boldsymbol{N}^d$.

Theorem 2. *We assume that a holomorphic function $\phi(t, h, \rho_B)$ and a polynomial*

$$L(\xi) = \sum_{0\leq|\gamma|\leq r} l_\gamma \xi^\gamma$$

satisfy
(i) *$\phi(t, \rho_B)$ has a power series expansion near $(0, 0)$ without linear*

parts:

$$\phi(t,\rho_B) = \sum_{2\le|\alpha|+|i_B|} a_{\alpha,i_B} t^\alpha \rho_B^{i_B} = \sum_{2\le|\alpha|+|i_B|} a_{\alpha,i_B} t^\alpha \prod_{\beta\in B} \rho_\beta^{i_\beta},$$

where $|i_B| = \sum_{\beta\in B} i_\beta$ *for a multi-index* $i_B = (i_\beta)_{\beta\in B}$ *and*
(ii) *there exists a positive constant* δ *such that for all d-dimensional multi indices* α *with* $|\alpha| \ge 1$,

$$|\sum_{0\le|\gamma|\le r} l_\gamma \alpha^\gamma| \ge \delta \max\{1, \alpha^\beta; \beta\in B\}, \tag{5}$$

where α^β *denotes the coefficient of* $\left(t\frac{\partial}{\partial t}\right)^\beta t^\alpha$.

Then, a nonlinear equation

$$\sum_{0\le|\gamma|\le r} l_\gamma \left(t\frac{\partial}{\partial t}\right)^\gamma h(t) = a\cdot t + \sum_{2\le|\alpha|+|i_B|} a_{\alpha,i_B} t^\alpha \cdot \prod_{\beta\in B}\left(\left(t\frac{\partial}{\partial t}\right)^\beta h(t)\right)^{i_\beta} \tag{6}$$

has a unique holomorphic solution $h(t)$ *near the origin with* $h(0) = 0$.

Proof. We will follow the previous proof.

Construction of $\hat{h}(t)$: We set

$$\hat{h}(t) = \sum_{|\alpha|\ge 1} h_\alpha t^\alpha. \tag{7}$$

Substituting (7) into (6), we have

$$L(\alpha)h_\alpha = a_\alpha \quad \text{when } |\alpha| = 1,$$

and

$$L(\alpha)h_\alpha = Q_\alpha(a_{\alpha',i_B}, h_{\alpha''}, (\alpha_\beta)^\beta h_{\alpha_\beta}; \beta \in B$$
$$\text{the indices at least satisfy}$$
$$|\alpha'| + |i_B| \leq |\alpha|, |\alpha''| \leq |\alpha| - 1$$
$$|\alpha_\beta| \leq |\alpha| - 1),$$

where $\alpha', \alpha'', \alpha_\beta$ are copies of α. Thus, h_α are determined succesively.

Convergence of $\hat{h}$: An auxiliary analytic equation (cf. Gérard-Tahara [3])

$$\delta H = |a_1|t_1 + |a_2|t_2 + \sum_{2\leq|\alpha|+|i_B|} |a_{\alpha,i_B}| t^\alpha (H(t))^{|i_B|}.$$

Solving this equation of H by the implicit function theorem, we have a unique holomorphic solution near the origin $t = 0$ with $H(0) = 0$. We claim

$$H(t) = \sum_\alpha H_\alpha t^\alpha >> \hat{h}(t).$$

More strongly we claim,

$$H_\alpha \geq \max\{|h_\alpha|, |\alpha^\beta h_\alpha|; \beta \in B\}.$$

We notice

$$H_\alpha = \frac{1}{\delta} Q_\alpha(|a_{\alpha',i_B}|, H_{\alpha''}, H_{\alpha_\beta}; \text{the indices satisfy at least } |\alpha'| + |i_B| \leq |\alpha|, |\alpha''| \leq |\alpha| - 1, |\alpha_\beta| \leq |\alpha| - 1, \beta \in B). \tag{8}$$

We start with $|\alpha| = 1$:

$$|h_\alpha| = |a_\alpha / L(\alpha)| \leq |a_\alpha|/\delta = H_\alpha.$$

Then, by induction, we have

$$\max\{1, \alpha^\beta;\ \beta \in B\} \cdot |h_\alpha| \leq \frac{1}{\delta} Q_\alpha(|a_{\alpha',i_B}|, |h_{\alpha''}|, |(\alpha_\beta)^\beta h_{\alpha_\beta}|;$$
$$\text{indices satisfy at least}$$
$$|\alpha'| + |i_B| \leq |\alpha|, |\alpha''| \leq |\alpha| - 1,$$
$$|\alpha_\beta| \leq |\alpha| - 1)$$

$$\leq \frac{1}{\delta} Q_\alpha(|a_{\alpha',i,i_B}|, H_{\alpha'}, H_{\alpha_\beta};$$

the indices satisfy at least

$|\alpha'| + |i_B| \leq |\alpha|, |\alpha''| \leq |\alpha| - 1,$

$|\alpha_\beta| \leq |\alpha| - 1)$

Therefore, we have obtained $\max\{|h_\alpha|, |\alpha^\beta h_\alpha|\} \leq H_\alpha$. □

We need this Briot-Bouquet type theorem in case where $d = 2$, $r = 2$, $\max\{|\beta|; \beta \in B\} = 1$.

3 Local uniqueness

To assure that weak solutions in $W^{1,p}(I)$ are filled locally by our analytical method, we need local uniqueness of solutions to the Cauchy problem to (3). We adapt proofs of uniqueness in J. Benedikt [1] and P. Drábek and M. Ôtani [2] for forth order p-elliptic equations, completing them with an energy inequality.

Let $I = [a, b]$ be a compact interval in $(0, \infty)$.

Proposition 1 (Local uniqueness). *Let r_0 be an arbitrary positive constant in I. Local solutions on I are uniquely determined near r_0 by initial data $U(r_0)$ and $U_r(r_0)$.*

Proof. Set $V(r) = r^{n-1}|U_r(r)|^{p-2}U_r(r)$, $p' = p/(p-1)$ and $f_p(X) = |X|^{p-2}X$. Notice $f_{p,X}(X) = (p-1)|X|^{p-2}$. Then, we have from the equation,

$$\begin{cases} V_r(r) &= -r^{n-1}|U(r)|^{q-2}U(r) = -r^{n-1}f_q(U(r)), \\ U_r(r) &= r^{\frac{1-n}{p-1}}|V(r)|^{p'-2}V(r) = f_{p'}(r^{1-n}V(r)). \end{cases} \tag{9}$$

Suppose

$$V_1(r_0) = V_2(r_0), \quad U_1(r_0) = U_2(r_0)$$
$$|U_0| + |V_0| > 0.$$

We will show that there exists a positive constant ϵ such that $U_1(r) = U_2(r)$ on $J(\epsilon) = [r_0 - \epsilon, r_0 + \epsilon]$, as proved in Benedikt [1] in 4th order p elliptic ordinary differential equation.

Case (i): $1 < p \leq 2$ and $2 \leq q$.

$$\begin{aligned} V_1(r) - V_2(r) &= r^{n-1}\{f_p(U_{1,r}(r)) - f_p(U_{2,r}(r))\} \\ &= r^{n-1}\int_{U_{2,r}(r)}^{U_{1,r}(r)} f_{p,X}(\tau)d\tau \end{aligned}$$

We set $K_1 = \max\{|U_{i,r}(r)|; r \in I,\, i = 1,2\}$. Noticing $f_{p,X}(X)$ is positive decreasing on $(0,\infty)$, when $1 < p < 2$, we have

$$\begin{aligned} &|r^{n-1}\{f_p(U_{1,r}(r)) - f_p(U_{2,r}(r))\}| \\ &\qquad \geq (r_0 - \epsilon)^{n-1}(p-1)K_1^{p-2}|U_{1,r}(r) - U_{2,r}(r)|. \end{aligned}$$

We set $K_0 = \max\{|U_i(r)|; r \in I,\, i = 1,2\}$.

$$V_1(r) - V_2(r) = -\int_{r_0}^{r} \tau^{n-1}\{f_q(U_1(\tau)) - f_q(U_2(\tau))\}d\tau.$$

On the other hand, we have

$$\begin{aligned} |f_q(U_1(\tau)) - f_q(U_2(\tau))| &= \left|\int_{U_2(\tau)}^{U_1(\tau)} f_{q,X}(\sigma)d\sigma\right| \\ &\leq (q-1)K_2^{q-2}|U_1(\tau) - U_2(\tau)| \\ &\leq (q-1)K_2^{q-2}\left|\int_{r_0}^{\tau}\{U_{1,r}(\sigma) - U_{2,r}(\sigma)\}d\sigma\right| \\ &\leq (q-1)K_2^{q-2}\epsilon\|U_{1,r} - U_{2,r}\|_{J(\epsilon)}, \end{aligned}$$

where $\| U \|_{J(\epsilon)} = \max_{|r-r_0|\leq\epsilon}|U(r)|$.

Choosing ϵ sufficiently small, we conclude

$$\|U_{1,r} - U_{2,r}\|_{J(\epsilon)} = 0.$$

Hence, $V_1 = V_2$ on $J(\epsilon)$, therefore, $V_{1,r} = V_{2,r}$, which gives $U_1(r) = U_2(r)$ on $J(\epsilon)$

Since we can proceed the rest as in [1], we show only classification of cases.

Case (ii): $1 < p \leq 2$ and $1 < q < 2$

Subcase (ii-1): We assume also $U_0 \neq 0$.

Subcase (ii-2): $U_0 = 0$, $V_0 \neq 0$.

Case (iii): $2 < p$, and $2 \le q$.
Subcase (iii-1): $V_1(r_0) = V_2(r_0) \ne 0$.
Subcase (iii-2): $V_1(r_0) = V_2(r_0) = 0$ and $U_1(r_0) = U_2(r_0) \ne 0$.
Case (iv): $2 < p$ and $1 < q < 2$.
Subcase (iv-1): $U_0 = 0$, and $V_0 \ne 0$.
Subcase (iv-2): $U_0 \ne 0$, and $V_0 = 0$.

We need different arguments, when $U_0 = V_0 = 0$. We assume $U_0 = V_0 = 0$.

To complete the proof, we use an energy inequality.

Proposition 2. (i) *Every nonzero $W^{1,p}(I)$ solution U on I has $C^1(\bar{I})$ regularity.*
(ii) *Then, it satisfies the energy equality*

$$\begin{aligned} &\frac{p-1}{p}|U_r(r)|^p + \frac{1}{q}|U(r)|^q \\ &= \frac{p-1}{p}|U_r(c)|^p + \frac{1}{q}|U(c)|^q \\ &- (n-1)\int_c^r \frac{1}{\sigma}|U_r(\sigma)|^p d\sigma \end{aligned} \tag{10}$$

for all $r, c \in I$.
(iii) *If $U(r_0) = U_r(r_0) = 0$ for some $r_0 \in I$, then, $U(r) = 0$ on I.*

Remark 2.1. (i) is due to M. Ôtani [6].
When $n = 1$, (ii) is the energy equality.
When $n = 1$, (iii) is trivial (for any $p, q > 1$) in virtue of the energy equality. When $n > 1$, this completes the proof of local uniqueness.

Proof of (iii). $U(r) = 0$ for all $r \in [r_0, b]$ in virtue of the energy inequality.

For all $r \in [a, r_0]$ we have

$$\frac{p-1}{p}|U_r(r)|^p + \frac{1}{q}|U(r)|^q = (n-1)\int_r^{r_0} \frac{1}{\sigma}|U_r(\sigma)|^p d\sigma,$$

therefore,

$$\frac{p-1}{p}|U_r(r)|^p \le (n-1)\int_r^{r_0} \frac{1}{\sigma}|U_r(\sigma)|^p d\sigma.$$

By Gronwall's lemma, we have $U_r(r) = 0$ on I. Since $U(r_0) = 0$, it implies $U(r) = 0$ on I. □

Remark 2.2. We note a different proof, when $q \geq p$ as in [2].

We note

$$r^{n-1} f_p(U_r(r)) = V(r) = \int_{r_0}^{r} V_r(\tau) d\tau = -\int_{r_0}^{r} \tau^{n-1} f_q(U(\tau)) d\tau.$$

We have $r_0^{n-1} \parallel U_r \parallel_{J(\epsilon)}^{p-1} \leq \epsilon \parallel V_r \parallel_{J(\epsilon)} \leq \epsilon(r_0+\epsilon)^{n-1} \parallel U \parallel^{q-1} \leq \epsilon^q (r_0+\epsilon)^{n-1} \parallel U_r \parallel_{J(\epsilon)}^{q-1}$. If we assume $\parallel U \parallel_{J(\epsilon)} > 0$, we have $r_0^{n-1} \leq \epsilon^q \parallel U_r \parallel_{J(\epsilon)}^{q-p}$ for any small positive ϵ. This gives contradiction, hence $\parallel U \parallel_{J(\epsilon)} = 0$.

4 Analytic singularities

We shall now describe local analytic singularities of the solution $U(r)$ to (4).

When $n = 1$, a classical Briot-Bouquet type theorem of one variable was sufficient to obtain the unique existence of analytic solution to the nonlinear ordinary differential equation ([8]).

When $U(r_0) \neq 0$ and $U_r(r_0) \neq 0$, r_0 is an analytic point of solution $U(r)$. Hence, we consider two types of singularities:

- $r_0 = \sigma$ where $U(\sigma) = 0$ and $U_r(\sigma) = A \neq 0$,
- $r_0 = \tau$ where $U(\tau) = A \neq 0$ and $U_r(\tau) = 0$.

CASE 1. σ where $U(\sigma) = 0$ and $U_r(\sigma) = A \neq 0$. We assume $A > 0$ without loss of generality. We treat the case where $\sigma > 0$.

Theorem 3. *For $1 < p, q < \infty$, there exists a unique analytic function $F(t,s)$ in a neigborhood of the origin such that we have near $r = \sigma$*

$$U(r) = (r-\sigma) F(r-\sigma, |r-\sigma|^q). \tag{11}$$

$F(t,s)$ is a holomorphic solution to

$$\begin{aligned} &(p-1)(\sigma+t)\{F(t,s) + tF_t(r,s) + qsF_s(t,s)\}^{p-2} \\ &\{tF_t(t,s) + qsF_s(t,s) \\ &+ t(tF_t(t,s))_t + 2qtsF_{t,s}(t,s) + q^2 s(sF_s(t,s))_s\} \\ &+ (\sigma+t)s(F(t,s))^{q-1} \quad \text{(continued)} \end{aligned} \tag{12}$$

$$+ (n-1)t\{F(t,s) + tF_t(t,s) + qsF_s(t,s)\}^{p-1} \\ = 0 \tag{13}$$

with

$$F(0,0) = A. \tag{14}$$

Consequently, we can compute the expansion of $U(r)$ *at* $r = \sigma$:

$$U(r) = (r-\sigma)\left\{A - \frac{n-1}{2\sigma(p-1)}A(r-\sigma) \right. \\ \left. - \frac{A^{q-p+1}}{(p-1)q(q+1)}|r-\sigma|^q + \cdots \right\}. \tag{15}$$

Proof. We reduce equation (13) by change of the unknown function

$$F(t,s) = A + h(t,s)$$

into

$$L\left(t\frac{\partial}{\partial t}, s\frac{\partial}{\partial s}\right) h(t,s) = -\frac{1}{(p-1)(\sigma+t)} \\ \times \{A + h(t,s) + th_t(t,s) + qsh_s(t,s)\}^{2-p} \\ \times \left[(\sigma+t)s(A+h(t,s))^{q-1} + (n-1)t \right. \\ \left. \{A + h(t,s) + th_t(t,s) + qsh_s(t,s)\}^{p-1}\right] \tag{16}$$

$$= a_{1,0}t + a_{0,1}s \\ + \sum_{2 \le p+q+i+j+k} a_{p,q,i,j,k} t^p s^q \\ \times (h(t,s))^i \left(t\frac{\partial h}{\partial t}(t,s)\right)^j \left(s\frac{\partial h}{\partial s}(t,s)\right)^k, \tag{17}$$

where

$$a_{1,0} = -\frac{(n-1)A}{(p-1)\sigma} \\ a_{0,1} = -\frac{1}{p-1}A^{q-p+1}. \tag{18}$$

We have $L(\alpha,\beta) = \alpha + q\beta + \alpha^2 + 2q\alpha\beta + q^2\beta^2$.

Since $L(1,0) = 2$ and $L(0,1) = q(q+1)$, $F_t(0,0)$ and $F_s(0,0)$ are determined and so on. Thus, the unique existence of the solution is obtained by the B-B type theorem of two variables.

Next, $(r-\sigma)F(r-\sigma, |r-\sigma|^q)$ is a C^2 function near σ. It satisfies (1) with the prescribed Cauchy data. By Proposition 1, it is equal to the unique solution $U(r)$ with the same Cauchy data. □

Corollary 1 ([4], [7],[8]). (i) *When q is an even integer more than 1, the solution $U(r)$ is real analytic near σ.*
(ii) *When q is not an even integer, the solution $U(r)$ is of class $C^{<q>}$ at σ, where $<x>$ is the least integer greater than or equal to x.*

CASE 2. τ where $U(\tau) = A$ and $U_r(\tau) = 0$.
As in the case 1, we can assume without loss of generality that $A > 0$.

Theorem 4. *For any p and q satisfying $1 < p, q < \infty$, there exists a unique analytic function $G(t,s)$ in a neigborhood of the origin such that we have near $r = \tau$*

$$U(r) = A + |r-\tau|^{\frac{p}{p-1}} G\left(r-\tau, |r-\tau|^{\frac{p}{p-1}}\right), \tag{19}$$

where $G(t,s)$ is a holomorphic solution to the nonlinear equation:

$$\begin{aligned} &(p-1)(t+\tau)\{-(\frac{p}{p-1}G(t,s) \\ &+ tG_t(t,s) + \frac{p}{p-1}sG_s(t,s))\}^{p-2} \\ &\times \left\{ \frac{p}{(p-1)^2}G(t,s) + \frac{p+1}{p-1}tG_t(t,s) \right. \\ &+ \frac{p(p+1)}{(p-1)^2}sG_s(t,s) + t(tG_t)_t(t,s) \end{aligned} \tag{20}$$

$$\begin{aligned} &\left. + \frac{2p}{p-1}tsG_{t,s}(t,s) + \frac{p^2}{(p-1)^2}s(sG_s)_s(t,s) \right\} \\ &+ (\tau+t)(A+sG(t,s))^{q-1} \\ &- (n-1)t\left\{ -(\frac{p}{p-1}G(t,s) \right. \\ &\left. +tG_t(t,s) + \frac{p}{p-1}sG_s(t,s)) \right\}^{p-1} = 0 \end{aligned} \tag{21}$$

with

$$G(0,0) = -\frac{p-1}{p} A^{\frac{q-1}{p-1}}.$$

Consequently, we have a convergent expansion near $r = \tau$:

$$\begin{aligned} U(r) &= A + B|r-\tau|^{\frac{p}{p-1}} + C(r-\tau)|r-\tau|^{\frac{p}{p-1}} \\ &+ D|r-\tau|^{\frac{2p}{p-1}} + \cdots , \end{aligned} \tag{22}$$

where

$$B = -\frac{p-1}{p} A^{\frac{q-1}{p-1}} \text{ and} \tag{23}$$

$$C = \frac{(n-1)}{2(2p-1)} A^{\frac{q-1}{p-1}} \tag{24}$$

$$D = \frac{q-1}{2(2p-1)} \left(\frac{p-1}{p}\right)^2 A^{1+\frac{2(q-p)}{p-1}}. \tag{25}$$

Proof. We show, at first, unique existence of the solution $G(t,s)$.

We reduce the equation by change of unknown function by

$$G(t,s) = B + h(t,s),$$

into

$$\begin{aligned} &\frac{p}{(p-1)^2} h(t,s) + \frac{p+1}{p-1} t h_t(t,s) + \frac{p(p+1)}{(p-1)^2} s h_s(t,s) \\ &+ t(th_t)_t(t,s) + \frac{2p}{p-1} ts h_{t,s}(t,s) \\ &+ \frac{p^2}{(p-1)^2} s(sh_s)_s(t,s) \end{aligned} \tag{26}$$

$$\begin{aligned} &= -\frac{p}{(p-1)^2} B - \frac{1}{(p-1)(t+\tau)} \\ &\times \left\{ -\frac{pB}{p-1} - \frac{p}{p-1} h(t,s) \right. \\ &\left. -th_t(t,s) - \frac{p}{p-1} s h_s(t,s) \right\}^{2-p} \\ &\times \left\{ (\tau+t)(A + sB + sh(t,s))^{q-1} \right. \end{aligned} \tag{27}$$

$$
\begin{aligned}
&-(n-1)t\left(-\frac{p}{p-1}B-\frac{p}{p-1}h(t,s)\right.\\
&\left.-th_t(t,s)-\frac{p}{p-1}sh_s(t,s)\right)^{p-1}\Bigg\}.
\end{aligned}
\tag{28}
$$

Developing the right hand side with respect ot (t, s, h, ρ, θ), where $\rho = sh_s$ and $\theta = sh_s$, we, at first, obtain

$$
B=-\frac{p-1}{p}A^{\frac{q-1}{p-1}}
$$

by the condition that the constant term vanishes:

$$
-\frac{p}{(p-1)^2}B-\frac{1}{p-1}\{-\frac{p}{p-1}B\}^{2-p}A^{q-1}=0.
$$

Then, we have the development of the R.H.S. is

$$
\begin{aligned}
R.H.S. &= a_{1,0,0,0,0}t + a_{0,1,0,0,0}s\\
&+\frac{2p-p^2}{(p-1)^2}h(t,s)+\frac{2-p}{p-1}th_t(t,s)
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&+\frac{2p-p^2}{(p-1)^2}sh_s(t,s)\\
&+\sum_{2\leq\alpha+\beta+i+j+k}a_{\alpha,\beta,i,j,k}t^\alpha s^\beta\\
&\times(h(t,s))^i\left(t\frac{\partial h}{\partial t}(t,s)\right)^j\left(s\frac{\partial h}{\partial s}(t,s)\right)^k.
\end{aligned}
\tag{30}
$$

Set

$$
\begin{aligned}
l_{0,0}&=\frac{p}{(p-1)^2}-\frac{2p-p^2}{(p-1)^2}=\frac{p}{p-1},\\
l_{1,0}&=\frac{p+1}{p-1}-\frac{2-p}{p-1}=\frac{2p-1}{p-1},\\
l_{0,1}&=\frac{p^2+p}{(p-1)^2}-\frac{2p-p^2}{(p-1)^2}=\frac{p(2p-1)}{(p-1)^2},
\end{aligned}
$$

$$
l_{2,0}=1, l_{1,1}=\frac{2p}{p-1}, l_{0,2}=\frac{p^2}{(p-1)^2}.
$$

Thus, we have the reduced equation

$$L\left(t\frac{\partial}{\partial t}, s\frac{\partial}{\partial s}\right) h(t,s)$$

$$= a_{1,0,0,0,0}t + a_{0,1,0,0,0}s \tag{31}$$

$$+ \sum_{2\le \alpha+\beta+i+j+k} a_{\alpha,\beta,i,j,k} t^{\alpha} s^{\beta}$$

$$\times (h(t,s))^i \left(t\frac{\partial h}{\partial t}(t,s)\right)^j \left(s\frac{\partial h}{\partial s}(t,s)\right)^k . \tag{32}$$

Since L satisfies the condition (5), the unique existence of the solution to (21) is obtained by our Briot-Bouquet type theorem.

□

Corollary 2 ([4], [7],[8]). (i) *If* $p/(p-1)$ *is an even integer, i.e.* $p = (2m+2)/(2m+1)$ $(m = 0, 1, 2, \cdots)$, $u(x)$ *is real analytic at* τ. (ii) *If* $p/(p-1)$ *is not an even integer, the solution* $U(r)$ *is of class* $C^{<\frac{2-p}{p-1}>+1}$ *at* τ*, where* $< x >$ *is the least integer greater than or equal to* x*. Especially, when* $1 < p \le 2$, $U(r)$ *is of class* C^2 *at* τ*. When* $2 < p$, $U(r)$ *is not of class* C^2 *at* τ.

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