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A Note on Automorphisms of Cellular Automata

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Abstract

For the set of cellular automata CA=$(\mathbb{Z}^d, Q, f, \nu)$ with local function $f : Q^n \to Q$ and neighborhood ν of size n, we define an *automorphism* which naturally induces a classification of CA: Two CA A and B are called *automorphic*, if and only if there is a pair of permutations (π, φ) of ν and Q, respectively, such that $(f_B, \nu_B) = (\varphi^{-1} f_A^{\pi} \varphi, \nu_A^{\pi})$. The set of the pairs of permutations (π, φ) is seen isomorphic to the direct product of two symmetric groups S_n and S_q. The set $\mathcal{P}_{n,q}$ of local functions $f : Q^n \to Q$ is classified by use of this automorphism group. Particularly, $\mathcal{P}_{3,2}$ which corresponds to 256 ECA (1-dimensional 3-neighbors 2-states CA) is classified into 46 automorphism classes, which is compared with the historical classification into 88 classes. The classification also refers to surjectivity, injecitivity and reversibility of CA.

1 Introduction

In recent years we have been dealing with *local structures* of cellular automata (CA for short) [1, 2, 3]. As a continuation, we study here *automorphisms* of CA. CA are called automorphic if and only if their global behaviors are the same when changing (permuting) the neighborhoods and renaming the cell states. Formally two CA $A = (\mathbb{Z}^d, Q, f_A, \nu_A)$ and $B = (\mathbb{Z}^d, Q, f_B, \nu_B)$ are called *automorphic*, if and only if there is a pair of permutations (π, φ) of ν_A and Q respectively, such that $(f_B, \nu_B) = (\varphi^{-1} f_A^{\pi} \varphi, \nu_A^{\pi})$.

For fixed n and q, the set of the pairs of permutations (π, φ) is seen isomorphic to the direct product of symmetric groups S_n and S_q, which will be called an *automorphism group* and denoted $Aut(n, q)$. By use of the automorphism group, we can classify all CA into automorphism classes. Generally speaking, a classification is a well-worn but useful method in searching for *good* CA like reversible CA.

In Appendix, Table 1, we give the complete classification of 256 ECA ($n = 3, q = 2$) into 46 automorphism classes, which is compared with the historical classification of ECA into 88 classes as shown in [4] and [5]. Typically our classification shows that 6 ECA including rule 110 become computation universal when the neighborhood and the state set are *appropriately permuted.* Table 1. also shows a computer test based on the Sutner-Tarjan algorithm about surjectivity, injecitivity and reversibility of CA; Two classes (6 functions) are surjective and injective (reversible), 7 classes are surjective but not injective and the rests are neither surjective nor injective.

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2 Preliminaries

2.1 CA and local structures

A cellular automaton is defined by a 4-tuple $(\mathbb{Z}^d, Q, f, \nu)$, where $\mathbb{Z}^d$ is a d-dimensional Euclidean space, Q is a finite set of *cell states*, $f : Q^n \to Q$ is a *local function* and ν is a *neighborhood*.

- **[neighborhood]**: *A neighborhood* is a mapping $\nu : \mathbb{N}_n \to \mathbb{Z}^d$, where $\mathbb{N}_n = \{1, 2, \ldots, n\}$ and $n \in \mathbb{N}$. This can equivalently be seen as a list ν with n components $(\nu_1, \ldots, \nu_n)$, where $\nu_i = \nu(i), 1 \leq i \leq n$, is called the i-th *neighbor*.
- **[local structure]**: *A pair* (f, ν) is called a *local structure* of CA. We call n the *arity* of the local structure. A CA is often identified with its local structure.
- **[global function]**:A local structure uniquely induces a *global function* $F : Q^{\mathbb{Z}^d} \to Q^{\mathbb{Z}^d}$, which is defined by
$$F(c)(p) = f(c(p + \nu_1), c(p + \nu_2), ..., c(p + \nu_n)),$$
for any $c \in Q^{\mathbb{Z}^d}$ a *global configuration*, where $c(p)$ is the state of cell $p \in \mathbb{Z}^d$ in c.

2.2 Permutation equivalence of local structures

Definition 1 [equivalence] *Two local structures* (f, ν) *and* (f', ν') *are called equivalent, if and only if they induce the same global function. In that case we write* $(f, \nu) \approx (f', \nu')$.

Definition 2 [reduced local structure] *A local structure is called reduced, if and only if the following conditions are fulfilled:*

- f *depends on all arguments.*
- ν *is injective, i.e.* $\nu_i \neq \nu_j$ *for* $i \neq j$ *in the list of neighborhood* ν.

Lemma 1 *For each local structure* (f, ν) *there is an equivalent reduced local structure* (f', ν').

Remark 1 *In this note we assume that every local structure is reduced, though most results hold for non-reduced local structures, see [3].*

Definition 3 [permutation of local structures] *Let* π *denote a permutation of the numbers in* $\mathbb{N}_n$. *The set of all permutations of the numbers from* $\mathbb{N}_n$ *constitutes a symmetric group* S_n.

- *For a neighborhood* ν, *denote by* ν^π *the neighborhood defined by* $\nu^\pi_{\pi(i)} = \nu_i$.
- *For an n-tuple* $\ell \in Q^n$, *denote by* ℓ^π *the permutation of* ℓ *such that* $\ell^\pi(i) = \ell(\pi(i))$ *for* $1 \leq i \leq n$.

  For a local function $f : Q^n \to Q$, *denote by* f^π *the local function* $f^\pi : Q^n \to Q$ *such that* $f^\pi(\ell) = f(\ell^\pi)$ *for all* ℓ.

  When a local function is expressed by a polynomial $f(x_1, ..., x_n)$, *then* $f^\pi = f(x_{\pi(1)}, ..., x_{\pi(n)})$, *see Section 2.3.*

Example 1 *Below is shown the set of 6 permutations of the elementary neighborhood (ENB)* $(-1, 0, 1)$, *which is isomorphic to the symmetric group* S_3 *of degree3.*

$$ENB^{\pi_0} = (-1, 0, 1), ENB^{\pi_1} = (-1, 1, 0), ENB^{\pi_2} = (0, -1, 1),$$
$$ENB^{\pi_3} = (0, 1, -1), ENB^{\pi_4} = (1, -1, 0), ENB^{\pi_5} = (1, 0, -1).$$

Lemma 2 *(f, ν) and (f^π, ν^π) are equivalent for any permutation π.*

Lemma 3 *If (f, ν) and (f', ν') are two reduced local structures which are equivalent, then there is a permutation π such that $\nu^\pi = \nu'$.*

Theorem 1 [permutation-equivalence of local structures]
If (f, ν) and (f', ν') are two reduced local structures which are equivalent, then there is a permutation π such that $(f^\pi, \nu^\pi) = (f', \nu')$.

We give here a lemma that equivalence of local structures is conserved when changing the position of neighborhoods. It means that we only need to consider the permutations of ENB $(-1, 0, 1)$ as listed in Example 1, as far as we are concerned with equivalence/automorphism of 3-neighbors CA in $\mathbb{Z}^d$, $d \geq 1$.

Consider an injective map $r : \mathbb{Z}^d \to \mathbb{Z}^d$ which is used to change the positions of neighbors. To neighborhood $\nu = (\nu_1, \ldots, \nu_n)$, r is applied componentwise. For the resulting neighborhood we write $r\nu$. That is $(\forall i)(r\nu)_i = r(\nu_i)$.

Lemma 4 [equivalence is preserved from changing neighborhoods]
If $(f, \nu) \approx (f', \nu')$ for two, possibly non-reduced, local structures, then $(f, r\nu) \approx (f', r\nu')$.

2.3 Polynomials over finite fields

In the following, Q is considered to be a finite field $GF(q)$, q a prime power p^k. Then a function $f : Q^n \to Q$ is uniquely expressed by a polynomial over $GF(q)$ in n indeterminates $x_1, ..., x_n$ of degree less than q in each indeterminate. In other words, the set of all such polynomials is a polynomial ring over $GF(q) \mod (x_1^q - x_1) \cdots (x_n^q - x_n)$, which will be denoted by $\mathcal{P}_{n,q}$, $n \geq 1, q \geq 2$. Obviously $\#\mathcal{P}_{n,q} = q^{q^n}$. For this expression by polynomials over finite fields, we refer to [6] and page 386, Notes to Chapter 7 of [7]. For small n and q, f is written as follows.

- If $f \in \mathcal{P}_{3,q}$,

$$\begin{aligned} f(x_1, x_2, x_3) = u_0 + u_1x_1 + u_2x_2 + \cdots + u_i x_1^h x_2^j x_3^k + \cdots \\ + u_{q^3-2} x_1^{q-1} x_2^{q-1} x_3^{q-2} + u_{q^3-1} x_1^{q-1} x_2^{q-1} x_3^{q-1}, \\ \text{where } u_i \in GF(q),\ 0 \leq i \leq q^3 - 1. \quad (1) \end{aligned}$$

- If $f \in \mathcal{P}_{3,2}$ (Boolean function),

$$\begin{aligned} f(x_1, x_2, x_3) = u_0 + u_1x_1 + u_2x_2 + u_3x_3 \\ + u_4x_1x_2 + u_5x_1x_3 + u_6x_2x_3 + u_7x_1x_2x_3, \\ \text{where } u_i \in GF(2) = \{0, 1\}, 0 \leq i \leq 7. \quad (2) \end{aligned}$$

Note that $a \vee b$ (Boolean) $= a + b + ab$ (polynomial) and $a \wedge b$(Boolean) $= ab$(polynomial).

3 Automorphism of CA

Assume that $A = (\mathbb{Z}^d, Q, f_A, \nu_A)$ and $B = (\mathbb{Z}^d, Q, f_B, \nu_B)$ are two CA having the same arity of local structures. Then we consider a *pair of permutations* (π, φ), where π and φ are permutations of ν and Q, respectively. Note that φ naturally extends to $\varphi : Q^{\mathbb{Z}^d} \to Q^{\mathbb{Z}^d}$.

Definition 4 *A and B are called automorphic under* (π, φ), *if and only if there is a pair* (π, φ), *such that*

$$(f_B, \nu_B) = (\varphi^{-1} f_A^{\pi} \varphi, \nu_A^{\pi}). \tag{3}$$

In this case, (π, φ) *is called an automorphism of CA. Symbolically* $A \underset{(\pi,\varphi)}{\cong} B$.

Example 2 $(f_{15}, ENB) \underset{(\pi_0,\varphi_0)}{\ncong} (f_{240}, ENB)$, *but* $(f_{15}, ENB) \underset{(\pi_0,\varphi_1)}{\cong} (f_{240}, ENB)$.

3.1 Automorphism group of CA

We see that the sets of all permutations π of ν and φ of Q are isomorphic to *symmetric groups* S_n and S_q, respectively. Indeed, we have

Lemma 5

$$Aut(n, q) \stackrel{\Delta}{\equiv} \{(\pi, \varphi) | \pi \in S_n, \varphi \in S_q\} = S_n \times S_q. \tag{4}$$

Proof: Since π and φ permute ν and Q *independently*, we see that if $A \underset{(\pi,\varphi)}{\cong} B$ and $B \underset{(\pi',\varphi')}{\cong} C$ for some $\pi, \pi' \in S_n$ and $\varphi, \varphi' \in S_q$, then $A \underset{(\pi'\pi,\varphi'\varphi)}{\cong} C$. ■

Definition 5 $Aut(n, q)$ *will be called an automorphism group of CA. Since symmetric groups are generally nonabelian,* $Aut(n, q)$ *is nonabelian.*

3.2 Automorphism classification of CA

We define a classification of CA utilizing the above defined automorphism, so that all functions contained by a class may have the same global functions (global behaviors), say, surjectivity, injectivity and reversibility, when the neighborhoods and the states are *appropriately permuted.*

Definition 6 $Aut(n, q)$ *naturally induces an automorphism classification* $\mathcal{NW}$ *of* $\mathcal{P}_{n,q}$*:*
$\mathcal{NW} = \{[f_1], [f_2], ..., [f_{m(n,q)}]\}$, *where* f_i *is a representative of class* $[f_i]$, $1 \leq i \leq m(n, q)$. *That is,* $f' \in [f]$, *if and only if there is a* $(\varphi, \pi) \in Aut(n, q)$ *such that* $(f', \nu') = (\varphi^{-1} f^{\pi} \varphi, \nu^{\pi})$. $m(n, q)$ *is the number of the classes.*

Remark 2 *In the terminology of the theory of finite groups, the automorphism classification* $\mathcal{NW}$ *is considered to be the conjugacy classification of* $\mathcal{P}_{n,q}$ *by* $Aut(n, q)$, *where elements of* $Aut(n, q)$ *act on the polynomial ring* $\mathcal{P}_{n,q}$. *The automorphism class is the conjugacy class :* $[f] = \{\varphi^{-1} f^{\pi} \varphi | \pi \in S_n, \varphi \in S_q\}$ *and the size is the number of conjugacy classes:* $m(n, q) = \#(\mathcal{P}_{n,q}/Aut(n, q))$.

Remark 3 *Generally, if* G' *is a subgroup of* G, *then the number of automorphism classes induced by* G' *is greater than or equal to that of* G. *For instance the historical classification of ECA into 88 classes is induced by subgroup* $\{(\pi_0, \varphi_0), (\pi_0, \varphi_1), (\pi_5, \varphi_0), (\pi_5, \varphi_1)\}$ *of* $Aut(3, 2)$.

3.3 Automorphism classification of ECA

The complete automorphism classification of 256 ECA ($\mathcal{P}_{2,3}$) into 46 classes is given in Appendix, Table 1. The computer test of surjectivity/injectivity is shown. Table 2. is its taxonomy. We show here some examples which will serve to explain the method.

We first note that in the theory of ECA, where the neighborhood is ENB *without permutation*, f and f' are usually called *conjugate*, if $f' = \varphi_1^{-1} f \varphi_1$, where $\varphi_1(0) = 1$ and $\varphi_1(1) = 0$. In the polynomial expression $f'(x_1, ..., x_n) = 1 + f(1 + x_1, ..., 1 + x_n)$. When $f = f'$, then f is called *self-conjugate.*

NW 9 f_{10} Wolfram number

$\varphi \setminus \pi$	π_0	π_1	π_2	π_3	π_4	π_5
φ_0	f_{10}	f_{12}	f_{34}	f_{68}	f_{48}	f_{80}
φ_1	f_{175}	f_{207}	f_{187}	f_{221}	f_{243}	f_{245}

NW 9 $f_{10} = x_3 + x_1x_3$ polynomial, neither surjective nor injective

$\varphi \setminus \pi$	π_0	π_1	π_2	π_3	π_4	π_5
φ_0	$x_3 + x_1x_3$	$x_2 + x_1x_2$	$x_3 + x_2x_3$	$x_2 + x_2x_3$	$x_1 + x_1x_2$	$x_1 + x_1x_3$
φ_1	$1 + x_1 + x_1x_3$	$1 + x_1 + x_1x_2$	$1 + x_2 + x_2x_3$	$1 + x_3 + x_2x_3$	$1 + x_2 + x_1x_2$	$1 + x_3 + x_1x_3$

NW 32 $f_{110} = x_1x_2x_3 + x_2x_3 + x_2 + x_3$ universal, neither surjective nor injective
Conjugate $f'_{110} = f_{137} = x_1x_2x_3 + x_1x_2 + x_1x_3 + x_1 + x_2 + x_3 + 1$.

$\varphi \setminus \pi$	π_0, π_1	π_2, π_4	π_3, π_5
φ_0	f_{110}	f_{122}	f_{124}
φ_1	f_{137}	f_{161}	f_{193}

NW 19 $f_{30} = x_1 + x_2 + x_3 + x_2x_3$ surjective and not injective
Conjugate $f'_{30} = f_{135} = 1 + x_1 + x_2x_3$.

$\varphi \setminus \pi$	π_0, π_1	π_2, π_4	π_3, π_5
φ_0	f_{30}	f_{86}	f_{54}
φ_1	f_{135}	f_{149}	f_{147}

NW 30* $f_{105} = x_1 + x_2 + x_3 + 1$ surjective and not injective

$\varphi \setminus \pi$	$\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5$
φ_0	f_{105}
φ_1	f_{105}

4 Concluding remarks and Acknowledgements

We defined the automorphism of CA and discussed the classification of CA induced by automorphism groups/subgroups, and particularly gave the automorphism classification of 256 ECA into 45 classes.

The automorphism classification turns out to be no less than the group action (X/G), where $X = \mathcal{P}_{n,q}$ and $G = S_n \times S_q$. By definition S_n acts on (permutes) ν and S_q does Q, respectively. Particularly the Orbit-Counting Lemma applies to calculating the number of conjugacy classes $|X/G|$. For the group action, see for example [8]. For actually classifying CA, we will investigate the algebraic (symmetric) structure of polynomials and/or make use of computer programs. We also wish an algebraic study of polynomials may lead to a general theory of injectivity/surjectivity and reversibility of CA.

For the classification and the injectivity/surjectivity test of ECA, we made use of Java programs **cat-est106d** and **ca-simulator** respectively coded by C. Lode and Ch. Scheben from University of Karlsruhe, to whom our thanks are due.

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Appendix

In Table 1-1, -2, f_i, $0 \le i \le 255$, is ELF having Wolfram number i and the automorphism classes are indexed by NW i, i = 1,2,..., 46, where conjugate functions are bracketed. Singletons are self-conjugate functions. The 7 classes indexed by * consist of surjective but not injecitve functions when the neighborhoods are appropriately permuted. The classes NW12** and NW44** are injective and surjective (i.e. reversible). The other classes are neither surjective nor injective classes. Table 2. is a taxonomy of Table 1.

Table 1-1. Automorphism classification of ELF

NW	Automorphism classes
1	$\{f_0, f_{255}\}$
2	$\{f_1, f_{127}\}$
3	$\{f_2, f_{191}\} \cup \{f_{16}, f_{247}\} \cup \{f_4, f_{223}\}$
4	$\{f_3, f_{63}\} \cup \{f_{17}, f_{119}\} \cup \{f_5, f_{95}\}$
5	$\{f_6, f_{159}\} \cup \{f_{20}, f_{215}\} \cup \{f_{18}, f_{183}\}$
6	$\{f_7, f_{31}\} \cup \{f_{21}, f_{87}\} \cup \{f_{19}, f_{55}\}$
7	$\{f_8, f_{239}\} \cup \{f_{64}, f_{253}\} \cup \{f_{32}, f_{251}\}$
8	$\{f_9, f_{111}\} \cup \{f_{65}, f_{125}\} \cup \{f_{33}, f_{123}\}$
9	$\{f_{10}, f_{175}\} \cup \{f_{80}, f_{245}\} \cup \{f_{12}, f_{207}\} \cup \{f_{68}, f_{221}\} \cup \{f_{34}, f_{187}\} \cup \{f_{48}, f_{243}\}$
10	$\{f_{11}, f_{47}\} \cup \{f_{81}, f_{117}\} \cup \{f_{13}, f_{79}\} \cup \{f_{69}, f_{93}\} \cup \{f_{35}, f_{59}\} \cup \{f_{49}, f_{115}\}$
11	$\{f_{14}, f_{143}\} \cup \{f_{84}, f_{213}\} \cup \{f_{50}, f_{179}\}$
12**	$\{f_{15}\} \cup \{f_{51}\} \cup \{f_{85}\}$ (Reversible class)
13	$\{f_{22}, f_{151}\}$
14	$\{f_{23}\}$
15	$\{f_{24}, f_{231}\} \cup \{f_{66}, f_{189}\} \cup \{f_{36}, f_{219}\}$
16	$\{f_{25}, f_{103}\} \cup \{f_{61}, f_{67}\} \cup \{f_{37}, f_{91}\}$
17	$\{f_{26}, f_{167}\} \cup \{f_{82}, f_{181}\} \cup \{f_{28}, f_{199}\} \cup \{f_{70}, f_{157},\} \cup \{f_{38}, f_{155}\} \cup \{f_{52}, f_{211}\}$
18	$\{f_{27}, f_{39}\} \cup \{f_{53}, f_{83}\} \cup \{f_{29}, f_{71}\}$
19*	$\{f_{30}, f_{135}\} \cup \{f_{86}, f_{149}\} \cup \{f_{54}, f_{147}\}$
20	$\{f_{40}, f_{235}\} \cup \{f_{96}, f_{249}\} \cup \{f_{72}, f_{237}\}$
21	$\{f_{41}, f_{107}\} \cup \{f_{97}, f_{121}\} \cup \{f_{73}, f_{109}\}$
22	$\{f_{42}, f_{171}\} \cup \{f_{112}, f_{241}\} \cup \{f_{76}, f_{205}\}$
23	$\{f_{43}\} \cup \{f_{77}\} \cup \{f_{113}\}$
24	$\{f_{44}, f_{203}\} \cup \{f_{100}, f_{217}\} \cup \{f_{56}, f_{227}\} \cup \{f_{98}, f_{185}\} \cup \{f_{74}, f_{173}\} \cup \{f_{88}, f_{229}\}$
25*	$\{f_{45}, f_{75}\} \cup \{f_{101}, f_{89}\} \cup \{f_{57}, f_{99}\}$
26	$\{f_{46}, f_{139}\} \cup \{f_{116}, f_{209}\} \cup \{f_{58}, f_{163}\} \cup \{f_{114},, f_{177}\} \cup \{f_{78}, f_{141}\} \cup \{f_{92}, f_{197}\}$
27*	$\{f_{60}, f_{195}\} \cup \{f_{102}, f_{153},\} \cup \{f_{90}, f_{165}\}$
28	$\{f_{62}, f_{131}\} \cup \{f_{118}, f_{145}\} \cup \{f_{94}, f_{133}\}$
29	$\{f_{104}, f_{233}\}$
30*	$\{f_{105}\}$

(continued)

Table 1-2. Automorphism classification of ELF

NW	Automorphism classes
31*	$\{f_{106}, f_{169}\} \cup \{f_{120}, f_{225}\} \cup \{f_{108}, f_{201}\}$
32	$\{f_{110}, f_{137}\} \cup \{f_{124}, f_{193}\} \cup \{f_{122}, f_{161}\}$ (Universal class)
33	$\{f_{126}, f_{129}\}$
34	$\{f_{128}, f_{254}\}$
35	$\{f_{130}, f_{190}\} \cup \{f_{144}, f_{246}\} \cup \{f_{132}, f_{222}\}$
36	$\{f_{134}, f_{158}\} \cup \{f_{148}, f_{214}\} \cup \{f_{146}, f_{182}\}$
37	$\{f_{136}, f_{238}\} \cup \{f_{192}, f_{252}\} \cup \{f_{160}, f_{250}\}$
38	$\{f_{138}, f_{174}\} \cup \{f_{208}, f_{244}\} \cup \{f_{140}, f_{206}\} \cup \{f_{196}, f_{220}\} \cup \{f_{162}, f_{186}\} \cup \{f_{176}, f_{242}\}$
39	$\{f_{142}\} \cup \{f_{212}\} \cup \{f_{178}\}$
40*	$\{f_{150}\}$
41	$\{f_{152}, f_{230}\} \cup \{f_{194}, f_{188}, \} \cup \{f_{164}, f_{218}\}$
42*	$\{f_{154}, f_{166}\} \cup \{f_{180}, f_{210}\} \cup \{f_{156}, f_{198}\}$
43	$\{f_{168}, f_{234}\} \cup \{f_{224}, f_{248}\} \cup \{f_{200}, f_{236}\}$
44**	$\{f_{170}\} \cup \{f_{240}\} \cup \{f_{204}\}$ (Reversible class)
45	$\{f_{172}, f_{202}\} \cup \{f_{216}, f_{228} \cup \{f_{184}, f_{226}\}$
46	$\{f_{232}\}$

Remark 4

The representative functions of 7 classes, which are surjective and not injective classes.

NW12 $f_{15} = 1 + x_1.\ f_{15}^{\pi_1} = f_{51}, \ldots$
NW19 $f_{30} = x_1 + x_2 + x_3 + x_2x_3.\ f_{30}^{\pi_1} = f_{30}, f_{30}^{\pi_2} = f_{86}, f_{30}^{\pi_3} = f_{54}, \ldots$
NW25 $f_{45} = 1 + x_1 + x_3 + x_2x_3.$
NW27 $f_{60} = x_1 + x_2.$
NW30 $f_{105} = 1 + x2 + x_3.$
NW31 $f_{106} = x_3 + x_1x_2.$
NW40 $f_{150} = x_1 + x_2 + x_3.$
NW42 $f_{154} = x_1 + x_3 + x_1x_2.$

Table 2: Taxonomy of automorphism classification of ELF

number of functions in NW class	number of NW classes	number of functions
12	6	72
6	26	156
3	4	12
2	6	12
1	4	4
total	46	256

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