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Kyoto University
Long-time behavior of solutions of Hamilton-Jacobi equations with convex and coercive Hamiltonians*

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1 Introduction and Preliminaries.

This paper is concerned with the Cauchy problem for the Hamilton-Jacobi equation

\[
\begin{cases}
  u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\
  u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^n,
\end{cases}
\]

(1)

where the Hamiltonian $H$ satisfies the following conditions:

\begin{enumerate}
  \item[(A1)] $H \in \text{BUC}(\mathbb{R}^n \times B(0, R))$ for all $R > 0$, where $B(0, R) := \{x \in \mathbb{R}^n \mid |x| \leq R\}$,
  \item[(A2)] $\inf\{H(x, p) \mid x \in \mathbb{R}^n, |p| \geq R\} \to +\infty$ as $R \to +\infty$,
  \item[(A3)] $H(x, p)$ is convex with respect to $p$ for every $x \in \mathbb{R}^n$.
\end{enumerate}

Note that the solvability of (1) in the sense of viscosity solution is well known. (See for instance Appendix A of [14] for the proof. See also [1, 7, 19] for the general theory of viscosity solutions.)

**Theorem 1.1.** Assume (A1)-(A3). Then, for any $T > 0$ and $u_0 \in \text{UC}(\mathbb{R}^n)$, there exists a viscosity solution $u \in \text{UC}(\mathbb{R}^n \times (0, T))$ of $u_t + H(x, Du) = 0$ in $\mathbb{R}^n \times (0, T)$ satisfying $u(\cdot, 0) = u_0$ on $\mathbb{R}^n$. Moreover, the solution is unique in the class $\text{UC}(\mathbb{R}^n \times [0, T])$ for every $T > 0$.

The objective of this paper is to investigate the long-time behavior of the viscosity solution to (1). More precisely, we prove the convergence of the form

\[
u(x, t) + at - \phi(x) \to 0 \quad \text{in } C(\mathbb{R}^n) \quad \text{as } t \to \infty
\]

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for some $a \in \mathbb{R}$ and $\phi \in C(\mathbb{R}^n)$, where $C(\mathbb{R}^n)$ is equipped with the topology of locally uniform convergence. Note that the function $\phi(x) - at$, called the asymptotic solution of (1), enjoys the following time-independent Hamilton-Jacobi equation in the viscosity sense:

$$H(x, D\phi) = a \quad \text{in} \quad \mathbb{R}^n.$$  

(3)

We denote by $S^{-}_{H-a}$ (resp. $S^{+}_{H-a}$ and $S_{H-a}$) the set of continuous viscosity subsolutions (resp. supersolutions and solutions) of (3). Observe here that if there exists an $a \in \mathbb{R}$ such that $\phi_0 \leq u_0 \leq \psi_0$ in $\mathbb{R}^n$ for some $\phi_0 \in S^{-}_{H-a}$ and $\psi_0 \in S^{+}_{H-a}$, then in view of the standard comparison theorem, we see that

$$t^{-1}u(\cdot, t) \rightarrow -a \quad \text{in} \quad C(\mathbb{R}^n) \quad \text{as} \quad t \rightarrow \infty.$$  

(4)

Our interest is, therefore, to investigate asymptotics of the next order.

In this paper, we deal with the case where $a = 0$, namely, we assume that

(A4) there exist $\phi_0 \in S^{-}_{H}$ and $\psi_0 \in S^{+}_{H}$ such that $\phi_0 \leq \psi_0$ in $\mathbb{R}^n$, and prove the convergence $u(\cdot, t) \rightarrow \phi$ in $C(\mathbb{R}^n)$ as $t \rightarrow \infty$ for any given initial function $u_0$ in the class

$$\Phi_0 := \{u_0 \in UC(\mathbb{R}^n) \mid \phi_0 - C \leq u_0 \leq \psi_0 + C \text{ in } \mathbb{R}^n \text{ for some } C > 0\},$$

where $\phi$ may depend on the choice of $u_0$. Notice here that assuming $a = 0$ is not a real restriction. Indeed, once (4) is established, (2) can be reduced to the case where $a = 0$ by considering $H - a$ and $u(x, t) + at$ instead of $H$ and $u(x, t)$, respectively.

The study on asymptotic problems of this type has been developed especially in the last decade. As one of the most typical cases, it was proved that if $H$ satisfies (A1), (A2), and $H(x, p)$ is $\mathbb{Z}^n$-periodic with respect to $x$ and is strictly convex with respect to $p$, then there exists a unique $a \in \mathbb{R}$ such that (2) is valid for every $\mathbb{Z}^n$-periodic initial function $u_0 \in \text{BUC}(\mathbb{R}^n)$. We refer to the literatures [3, 5, 8, 9, 10, 20, 21, 22, 23] and references therein for more details. Remark that [3] deals with non-convex Hamiltonians whereas the others are concerned only with convex ones.

It has also been of interest in recent years on the long-time behavior of viscosity solutions to (1) that are not necessarily spatially periodic. As far as non-periodic solutions are concerned, the above (A1)-(A4) are insufficient to obtain the convergence (2) for every $u_0 \in \Phi_0$ even if we admit strict convexity for $H$ in any sense (see [4, 14]). The papers [2, 12, 14, 17] deal with some situations in which the solution of (1) has indeed the required convergence of the form (2) for suitable $(a, \phi)$.

Motivated by these earlier results, we established in [16], on which this paper is based, general convergence results for the solution of (1) which, on the one hand, cover most of existing results, and, on the other hand, involve a few observations which seem to be new. The first one is concerned with strict convexity for $H$. As pointed out in several literatures, it is necessary in some situations to require a sort of strict convexity for $H$ so that the solution of (1) converges to an asymptotic solution as $t \rightarrow \infty$. In the present paper, we use condition (A5)$_+$ or (A5)$_-$ which guarantees, respectively, strict convexity of $H(x, p)$ in $p$ uniformly in the sets $\{H \geq 0\}$ or $\{H \leq 0\}$ (see Section 2 for their precise definitions). We point out here that
in spite of our convexity assumption (A3), the latter condition is not covered by [3] in which convergence of the type (2) is obtained in the periodic case under fairly weak assumptions on $H$.

The second observation is discussed in connection with our dynamical approach basing on the following classical variational formula:

$$u(x, t) = \inf \left\{ \int_{-t}^{0} L(\eta(s), \dot{\eta}(s)) \, ds + u_0(\eta(-t)) \mid \eta \in C([-t, 0]; x) \right\}, \tag{5}$$

where $L(x, \xi) := \sup_{p \in \mathbb{R}^n} (p : \xi - H(x, p))$ and $C([-t, 0]; x) := \{ \eta \in AC([-t, 0], \mathbb{R}^n) \mid \eta(0) = x \}$, and we denote by $AC([-t, 0], \mathbb{R}^n)$ the set of curves $\eta : [-t, 0] \to \mathbb{R}^n$ being absolutely continuous on $[-s, 0]$ for all $0 < s \leq t$. It is standard to see that the function $u(x, t)$ defined by (5) is indeed the viscosity solution of (1). It will be revealed in Section 3 that, for each $x \in \mathbb{R}^n$, solutions, say $\eta(t)$, of the variational problem in the right-hand side of (5) possess a distinctive behavior as $t \to \infty$ called "switch-back", from which we obtain a new type of convergence result. As far as we know, such a motion in the asymmetric behavior of solutions of (1) was not studied before.

One other novelty of this paper (and thus that of [16]) is related to Hamiltonians and initial data with "weak" periodicity. In Section 4, we give some results which particularly extend [14] studying Hamilton-Jacobi equations with semi-periodic Hamiltonians and semi-almost periodic initial data. See also [13] for some information in this direction.

In the rest of this introductory section, we briefly sketch the procedure for the proof of (2) (see also [14]). Let $(T_t)_{t \geq 0}$ be the nonlinear semigroup on UC$(\mathbb{R}^n)$ defined by $(T_t u_0)(x) := u(x, t)$, where $u(x, t)$ is the solution of the Cauchy problem (1). For a given $u_0 \in \Phi_0$, we set

$$u_0^-(x) := \sup \{ \phi(x) \mid \phi \in S^+_H, \phi \leq u_0 \text{ in } \mathbb{R}^n \}, \quad u_\infty(x) := \inf \{ \psi(x) \mid \psi \in S_H, \psi \geq u_0^- \text{ in } \mathbb{R}^n \}.$$  

Then, it follows that $u_0^- \in S^-_H$ and $u_\infty \in S^+_H$ by standard arguments in the viscosity solution theory. It is also well known (e.g. [8, 11, 17]) that $u_0^-$ can be represented as

$$u_0^-(x) = \inf \{ d_H(x, y) + u_0(y) \mid y \in \mathbb{R}^n \}, \quad x \in \mathbb{R}^n, \tag{6}$$

where $d_H$ is defined by

$$d_H(x, y) := \sup \{ \phi(x) - \phi(y) \mid \phi \in S^-_H \}. \tag{7}$$

Note that $d_H(\cdot, y) \in S^-_H$ for all $y \in \mathbb{R}^n$ and $d_H$ can be written as

$$d_H(x, y) = \inf \left\{ \int_{-t}^{0} L(\eta(s), \dot{\eta}(s)) \, ds \mid \eta(t) = y \right\}. \tag{8}$$

Moreover, we can show the following lemma (see Lemma 4.1 of [14] for the proof).

**Lemma 1.2.** Assume (A1)-(A4). Then, for every $u_0 \in \Phi_0$, one has $u_\infty \in S_H$ and

$$(T_t u_0^-)(x) = \inf_{s \geq t} u(x, s), \quad u_\infty(x) = \lim_{t \to \infty} \inf_{s \geq t} u(x, t).$$

Hence, the problem is reduced to proving the convergence

$$T_t u_0 \to u_\infty \quad \text{in } C(\mathbb{R}^n) \text{ as } t \to \infty. \tag{9}$$
Now, for a fixed $x \in \mathbb{R}^n$, we set $u^+(x) := \limsup_{t \rightarrow \infty} u(x, t)$ and choose any diverging sequence $\{t_j\}_j \subset (0, \infty)$ such that $u^+(x) = \lim_{j \rightarrow \infty} u(x, t_j)$. The rough idea of showing (9) is to find a family of curves $\mu_j \in \mathcal{C}([-t_j, 0]; x)$, $j \in \mathbb{N}$, such that

$$u_\infty(x) \geq \lim_{j \rightarrow \infty} \left( \int_{-t_j}^{0} L(\mu_j(s), \dot{\mu}_j(s)) \, ds + u_0(\mu_j(-t_j)) \right).$$

If (10) is true for some $\{\mu_j\}$, then in view of (5),

$$u^+(x) = \lim_{j \rightarrow \infty} u(x, t_j) \leq \lim_{j \rightarrow \infty} \left( \int_{-t_j}^{0} L(\mu_j(s), \dot{\mu}_j(s)) \, ds + u_0(\mu_j(-t)) \right) \leq u_\infty(x),$$

from which we conclude that $u(x, t) \rightharpoonup u_\infty(x)$ as $t \rightarrow \infty$ for all $x \in \mathbb{R}^n$. We remark here that, under our assumptions (A1)-(A4), the above pointwise convergence yields locally uniform convergence (9) (e.g. [17] for its justification). Observe also that $\mu_j$ can be regarded, up to a small error, as a minimizer of the right-hand side of (5) with $t = t_j$ for each $j \in \mathbb{N}$. In the following sections, we divide our consideration into several situations according to the type of $\{\mu_j\}$.

In any case, the so-called extremal curves play an important role. Recall that for given $x \in \mathbb{R}^n$ and $\phi \in \mathcal{S}_H$, a curve $\gamma \in \mathcal{C}((-\infty, 0]; x)$ is said an extremal curve for $\phi$ at $x$ if it satisfies

$$\phi(x) = \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s)) \, ds + \phi(\gamma(-t)) \quad \text{for all} \quad t > 0.$$ 

The existence of such curves is guaranteed by Lemma 3.3 of [14]. We denote by $\mathcal{E}_x(\phi)$ the set of all extremal curves for $\phi$ at $x$. We often use the notation $\mathcal{E}_x := \mathcal{E}_x(u_\infty)$ for simplicity of notation.

This paper is organized as follows. In the next section, we establish a theorem which covers, as particular cases, some results of Barles-Roquejoffre [2] and Ishii [17]. At the end of Section 2, we also discuss the relationship between the long-time behavior of extremal curves and ideal boundaries studied in Ishii-Mitake [18]. In Sections 3, we treat a class of Hamiltonians that provide switch-back motions for $\mu_j$. Section 4 is devoted to establishing some results concerning the long-time behavior of viscosity solutions of Hamilton-Jacobi equations with weak periodicity. Several examples are given in the final section.

### 2 First convergence result.

Let $H$ satisfy (A1)-(A4) and let $u_0 \in \Phi_0$. We begin this section with a few simple lemmas.

**Lemma 2.1.** Suppose that for every $x \in \mathbb{R}^n$, there exists a $\gamma \in \mathcal{E}_x$ such that

$$\lim_{t \rightarrow \infty} (u_0 - u_\infty)(\gamma(-t)) = 0.$$ 

Then, the convergence (9) holds.
Proof. Let \( \gamma \in \mathcal{E}_x \) satisfy (12). By the definition of extremal curves and the variational formula (5), we see that
\[
u(x, t) \leq \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s)) \, ds + u_0(\gamma(-t)) = u_\infty(x) - u_\infty(\gamma(-t)) + u_0(\gamma(-t))
\]
for all \( t > 0 \). In view of (12) and Lemma 1.2, we conclude that
\[
\limsup_{t \to \infty} \nu(x, t) \leq u_\infty(x) + \lim_{t \to \infty}(u_0 - u_\infty)(\gamma(-t)) = u_\infty(x) = \liminf_{t \to \infty} \nu(x, t),
\]
which implies (9).

We next prove that if \( H \) satisfies a sort of strict convexity, then (12) is not necessarily needed for extremal curves \( \gamma = \{ \gamma(-t) \mid t > 0 \} \) bounded in \( \mathbb{R}^n \). We set \( Q := \{(x, p) \in \mathbb{R}^{2n} \mid H(x, p) = 0 \} \) and
\[
S := \{(x, \xi) \in \mathbb{R}^{2n} \mid (x, p) \in Q, \ \xi \in D_2^{-}H(x, p) \text{ for some } p \in \mathbb{R}^n \},
\]
where \( D_2^{-}H(x, p) \) stands for the subdifferential of \( H \) with respect to the \( p \)-variable. In what follows, we use the following assumption:

\[\text{(A5)}_+ \quad \text{(resp. (A5)_-)} \quad \text{There exists a modulus } \omega \text{ satisfying } \omega(r) > 0 \text{ for } r > 0 \text{ such that for all } (x, p) \in Q, \ \xi \in D_2^{-}H(x, p) \text{ and } q \in \mathbb{R}^n, \]
\[
H(x, p + q) \geq \xi \cdot q + \omega((\xi \cdot q)_{+}) \quad \text{(resp. } \geq \xi \cdot q + \omega((\xi \cdot q)_{-})).
\]

Roughly speaking, \((A5)_+ \) (resp. \((A5)_- \)) means that \( H(x, \cdot) \) is strictly convex on the set \( \{ p \in \mathbb{R}^n \mid H(x, p) \geq 0 \} \) (resp. \( \{ p \in \mathbb{R}^n \mid H(x, p) \leq 0 \} \) uniformly in \( x \in \mathbb{R}^n \). Notice here that condition \((A5)_- \) has been discussed in [15] when \( n = 1 \). This strict convexity yields the following property for \( L \).

Lemma 2.2. Let \( H \) satisfy (A1)-(A4) and \((A5)_+ \) (resp. \((A5)_- \)). Then, there exists a constant \( \delta_1 > 0 \) and a modulus \( \omega_1 \) such that for any \( \epsilon \in [0, \delta_1] \) (resp. \( \epsilon \in [-\delta_1, 0] \)) and \((x, \xi) \in S\),
\[
L(x, (1 + \epsilon)p) \leq (1 + \epsilon)L(x, \xi) + |\epsilon|\omega_1(|\epsilon|).
\]

Proof. The proof of (14) under \((A5)_+ \) is exactly the same as that of Lemma 3.2 in [14]. Moreover, by a careful review of its proof, we see that (14) is also true under \((A5)_- \).

Remark 2.3. The estimate of this type was proved first by [8] when \( H(x, \cdot) \) is strictly convex.

Proposition 2.4. Let \( H \) satisfy (A1)-(A4) and one of \((A5)_+ \) or \((A5)_- \). Let \( u_0 \in \Phi_0, x \in \mathbb{R}^n \) and \( \gamma \in \mathcal{E}_x, \) and suppose that \( u^+(x) = \lim_{j \to \infty} u(x, t_j) \) and \( \sup_{j} |\gamma(-t_j)| < \infty \) for some diverging sequence \( \{t_j\} \subset (0, \infty) \). Then, \( u^+(x) \leq u_\infty(x) \).

Proof. Fix any \( \delta > 0 \) and set \( x_j := \gamma(-t_j) \) for \( j \in \mathbb{N} \). By taking a subsequence if necessary, we may assume that \( x_j \to y \) as \( j \to \infty \) for some \( y \in \mathbb{R}^n \).

In view of coercivity (A2), we see that \( \{u(\cdot, t) \mid t > 0\} \) is equi-continuous on \( \mathbb{R}^n \) and \( u_0^- \) and \( u_\infty \) are Lipschitz continuous on \( \mathbb{R}^n \). In particular, there exists an \( \epsilon > 0 \) such that \( |x - x'| < \epsilon \) implies
\[
|u(x, t) - u(x', t)| + |u_0^-(x) - u_0^-(x')| + |u_\infty(x) - u_\infty(x')| < \delta
\]
(15)
for every \( t > 0 \). In what follows, we fix such \( \epsilon > 0 \) and assume that \( |x_j - y| < \epsilon \) for all \( j \in \mathbb{N} \).

We first assume \((A5)_+\) and show that \( u^+(x) \leq u_{\infty}(x) \). Fix a \( \tau > 0 \) so that \( u_0^-(y) + \delta > u(y, \tau) \). For each \( j \in \mathbb{N} \), we set \( \epsilon_j := (t_j - \tau)^{-1} \tau \) and define \( \gamma_j(s) := \gamma((1 + \epsilon_j)s) \). Then, from (5), (14) and the fact that \((\gamma(s), \dot{\gamma}(s)) \in S \) for a.e. \( s \in (-\infty, 0) \), we have

\[
\begin{align*}
    u(x, t_j) &\leq \int_{-t_j + \tau}^{0} L(\gamma_j(s), \dot{\gamma}_j(s)) \, ds + u(x_j, \tau) + \delta \\
    &< u_{\infty}(x) - u_{\infty}(x_j) + t_j \epsilon_j \omega_1(\epsilon_j) + u(y, \tau) + \delta\\
    &\leq u_{\infty}(x) - u_{\infty}(y) + t_j \epsilon_j \omega_1(\epsilon_j) + u_{0}^-(y) + 3\delta.
\end{align*}
\]

By letting \( j \to \infty \) and then \( \delta \to 0 \), we obtain \( u^+(x) \leq u_{\infty}(x) \).

We next assume \((A5)_-\). Observe from (5) and (15) that

\[
\begin{align*}
    u(x, t_j) &\leq \int_{-t_1}^{0} L(\gamma(s), \dot{\gamma}(s)) \, ds + u(x_1, t_j - t_1) \\
    &< u_{\infty}(x) - u_{\infty}(x_1) + u(x_2, t_j - t_1) + 2\delta < u_{\infty}(x) - u_{\infty}(y) + u(x_2, t_j - t_1) + 3\delta.
\end{align*}
\]

By renumbering \( \{t_j\} \) if necessary, we may assume that \( t_2 > t_1 + \tau \). For each \( j \in \mathbb{N} \), we set

\[
    \epsilon_j = \frac{t_2 - t_1 - \tau}{t_j - t_1 - \tau}, \quad \gamma_j(s) = \gamma(-t_2 + (1 - \epsilon_j)s), \quad s \leq 0.
\]

Since \( \epsilon_j \to 0 \) as \( j \to 0 \), we may assume that \( \epsilon_j \in (0, \delta_1) \) for all \( j \in \mathbb{N} \), where \( \delta_1 \) is the constant taken from Lemma 2.2. Then, in view of (15) and the fact that \( u_0^-(y) + \delta > u(y, \tau) \), we see that

\[
\begin{align*}
    u(x_2, t_j - t_1) &\leq \int_{-t_1 + \tau}^{0} L(\gamma_j(s), \dot{\gamma}_j(s)) \, ds + u(x_j, \tau) \\
    &< u_{\infty}(x_2) - u_{\infty}(x_j) + t_j \epsilon_j \omega_1(\epsilon_j) + u(y, \tau) + \delta < t_j \epsilon_j \omega_1(\epsilon_j) + u_0^-(y) + 4\delta.
\end{align*}
\]

Thus, we have

\[
\begin{align*}
    u(x, t_j) &< u_{\infty}(x) - u_{\infty}(y) + u(x_2, t_j - t_1) + 3\delta \\
    &< u_{\infty}(x) - u_{\infty}(y) + t_j \epsilon_j \omega_1(\epsilon_j) + u_0^-(y) + 7\delta < u_{\infty}(x) + t_j \epsilon_j \omega_1(\epsilon_j) + 7\delta.
\end{align*}
\]

By letting \( j \to \infty \) and then \( \delta \to 0 \), we get \( u^+(x) \leq u_{\infty}(x) \). \( \square \)

We are now in position to state the main theorem of this section. For a given \( \phi \in S_H \), we define the set \( \Lambda(\phi) \) by

\[
\Lambda(\phi) := \{ \{\gamma(-t_j)\}_j \subset \mathbb{R}^n \mid \gamma \in \mathcal{E}_x(\phi) \text{ and } |\gamma(-t_j)| \to \infty \text{ as } j \to \infty \}.
\]

(16)

In what follows, we set \( \Lambda := \Lambda(u_{\infty}) \) if there is no confusion.

**Theorem 2.5.** Let \( H \) satisfy \((A1)-(A4)\) and one of \((A5)_+\) or \((A5)_-\), and let \( u_0 \in \Phi_0 \). Then, the convergence (9) holds provided that

\[
\lim_{j \to \infty} (u_0 - u_{\infty})(x_j) = 0 \quad \text{for all } \{x_j\} \in \Lambda.
\]

(17)
Proof. Fix any $x \in \mathbb{R}^n$ and any diverging $\{t_j\}$ such that $u^+(x) = \lim_{j \to \infty} u(x, t_j)$. We take an arbitrary $\gamma \in E_x$ and set $x_j = \gamma(-t_j)$ for $j \in \mathbb{N}$. If $\lim_{j \to \infty} |x_j| = \infty$, then we get $u^+(x) \leq u_\infty(x)$ by Lemma 2.1 and (17). On the other hand, if $\lim \inf_{j \to \infty} |x_j| < \infty$, then by taking a subsequence if necessary, we may assume that $\sup_{j \in \mathbb{N}} |x_j| < \infty$. Thus, we can apply Proposition 2.4 to get the same inequality.

As an easy consequence of Theorem 2.5, we obtain the following convergence result which covers, as typical cases, Theorem 4.2 of [2] and (a version of) Theorem 1.3 in [17] (see also Remark 2.10 below).

**Theorem 2.6.** Let $H$ satisfy (A1)-(A4) and $u_0 \in \Phi_0$. Let $\psi \in \text{Lip}(\mathbb{R}^n)$ and $\sigma \in C(\mathbb{R}^n)$ be such that

$$H(x, D\psi(x)) \leq -\sigma(x) \quad \text{a.e. } x \in \mathbb{R}^n.$$  

Then, one has the convergence (9) provided one of the following (a) or (b) holds:

(a) $\sigma(x) > 0$ for all $x \in \mathbb{R}^n$ and condition (17),

(b) $(A5)_+$ or $(A5)_-$, and

$$\sigma \geq 0 \text{ in } \mathbb{R}^n \setminus B(0, R) \text{ for some } R > 0 \text{ and } \lim_{|x| \to \infty} (\psi_0 - \psi)(x) = \infty.$$

**Remark 2.7.** Let $A_H \subset \mathbb{R}^n$ be the Aubry set for $H$, i.e., $A_H := \{y \in \mathbb{R}^n | d_H(\cdot, y) \in S_H\}$. Then, we see that condition (a) yields $A_H = \emptyset$. On the other hand, condition (b) implies that $A_H$ is non-empty and compact.

Before proving Theorem 2.6, we point out the following facts.

**Lemma 2.8.** Let $H$ satisfy (A1)-(A4) and $u_0 \in \Phi_0$. Let $D \subset \mathbb{R}^n$ be an open set and suppose that there exist $\delta > 0$ and $\psi \in S_H^c$ such that $\sup_{D} |\psi - \phi_0| < \infty$ and

$$H(x, D\psi(x)) \leq -\delta \quad \text{a.e. } x \in D.$$  

Then, for any $\varepsilon > 0$, $x \in D$ and $\gamma \in E_x$, there exists a $\tau > 0$ such that $\gamma(-t) \not\in D_\varepsilon$ for all $t \geq \tau$, where $D_\varepsilon := \{x \in D | \text{dist}(x, D^c) > \varepsilon\}$.

**Proof.** Fix any $\varepsilon > 0$, $x \in D$ and $\gamma \in E_x$. Observe that $\sup_{t > \tau} |(u_\infty - \phi_0)(\gamma(-t))| < \infty$. Indeed, for every $t > s \geq 0$, we have

$$\phi_0(\gamma(-s)) - \phi_0(\gamma(-t)) \leq \int_{-t}^{-s} L(\gamma(r), \dot{\gamma}(r)) \, dr = u_\infty(\gamma(-s)) - u_\infty(\gamma(-t)),$$

which implies that the function $t \mapsto (u_\infty - \phi_0)(\gamma(-t))$ is non-increasing on $[0, \infty)$. Since $\inf_{\mathbb{R}^n} (u_\infty - \phi_0) > -\infty$, we conclude that $\sup_{t > \tau} |(u_\infty - \phi_0)(\gamma(-t))| < \infty$.

Next, we claim that for any $s > 0$, there exists a $t > s$ such that $\gamma(-t) \not\in D$. Indeed, suppose that $\gamma(-t) \in D$ for all $t > s$. Then, in view of (19), for every $t > s$,

$$\psi(\gamma(-s)) - \psi(\gamma(-t)) + \int_{-t}^{-s} \delta \, dr \leq \int_{-t}^{s} L(\gamma(r), \dot{\gamma}(r)) \, dr = u_\infty(\gamma(-s)) - u_\infty(\gamma(-t)).$$

Since $\sup_{D} |\psi - \phi_0| < \infty$ by assumption, we have

$$\delta(t - s) \leq 2 \sup_{r > 0} |(u_\infty - \phi_0)(\gamma(-r))| + 2 \sup_{y \in D} |(\phi_0 - \psi)(y)| \quad \text{for all } t > s.$$
By letting \( t \to \infty \), we get the contradiction. Thus, we can choose a diverging \( \{ t_j^+ \} \subset (0, \infty) \) such that \( \gamma(-t_j^+) \notin D \) for all \( j \in \mathbb{N} \).

We now show that there exists a \( \tau > 0 \) such that \( \gamma(-t) \notin D_\varepsilon \) for all \( t \geq \tau \). We argue by contradiction. Suppose that there exists a diverging \( \{ t_j^- \} \subset (0, \infty) \) such that \( \gamma(-t_j^-) \in D_\varepsilon \) for all \( j \in \mathbb{N} \). By renumbering \( \{ t_j^+ \} \) and \( \{ t_j^- \} \) if necessary, we may assume that \( t_j^- < t_j^+ < t_{j+1}^- \) for all \( j \in \mathbb{N} \).

We take any \( A > 0 \). Then, there exists a \( C_A > 0 \) such that
\[
L(x, \xi) - q \cdot \xi \geq A|\xi| - C_A \quad \text{for all } (x, \xi) \in \mathbb{R}^{2n} \text{ and } q \in B(O, A).
\] (20)
Indeed, by setting \( C_A := \sup\{|H(x,p)||x \in \mathbb{R}^n, p \in B(O,2A)\} \), we have
\[
L(x, \xi) = \sup_{p \in \mathbb{R}^n} \{\xi \cdot p - H(x,p)\} \geq \xi \cdot (q + A|\xi|^{-1}\xi) - H(x, q + A|\xi|^{-1}\xi) \geq q \cdot \xi + A|\xi| - C_A
\] for every \( x \in \mathbb{R}^n, \xi \neq 0 \) and \( q \in B(O, A) \).

We observe that
\[
\psi(\gamma(-s)) - \psi(\gamma(-t)) = \int_{-t}^{-s} q(r) \cdot \dot{\gamma}(r) dr \quad \text{for all } t > s \geq 0
\] (21)
for some \( q \in L^\infty(-\infty,0;\mathbb{R}^n) \) satisfying \( q \in \partial_c \psi(\gamma(r)) \) for a.e. \( r \in (-\infty,0] \), where \( \partial_c \psi(z) \) stands for the Clarke differential of \( \psi \) at \( z \in \mathbb{R}^n \), namely,
\[
\partial_c \psi(z) := \bigcap_{r>0} \overline{co} \{ D\psi(y) | y \in B(z,r), \phi \text{ is differentiable at } y \}.
\]
In view of (20) and (21), we obtain
\[
\int_{-t}^{-s} (A|\dot{\gamma}(r)| - C_A) dr \leq \int_{-t}^{-s} L(\gamma(r), \dot{\gamma}(r)) dr - (\psi(\gamma(-s)) - \psi(\gamma(-t)))
\]
\[
= (u_\infty - \psi)(\gamma(-s)) - (u_\infty - \psi)(\gamma(-t)).
\]
Now, for each \( j \in \mathbb{N} \), we set \( \tau_j^- := \inf\{t > t_j^- | \gamma(-t) \notin D\} \), \( \tau_j^+ := \sup\{t < t_{j+1}^- | \gamma(-t) \notin D\} \), and choose any \( a, b > 0 \) such that \( (a, b) \subset (-\tau_j^-, -t_j^-) \) or \( (a, b) \subset (-t_{j+1}^-, -\tau_j^+) \) for some \( j \in \mathbb{N} \). Since \( \gamma((a,b)) \subset D \), we see that
\[
\int_a^b |\dot{\gamma}(s)| ds \leq A^{-1}C_A(b-a) + 2A^{-1} \sup_{D} |u_\infty - \psi|.
\]
Fix an \( A > 0 \) so large that \( 2A^{-1} \sup_{D} |u_\infty - \psi| < \varepsilon/2 \). Then, we see that for all \( j \in \mathbb{N} \),
\[
\varepsilon \leq \int_{-t_j^-}^{t_j^+} |\dot{\gamma}(s)| ds \leq \frac{\varepsilon}{2} + A^{-1}C_A(t_j^- - t_j^+), \quad \varepsilon \leq \int_{-t_j^-}^{t_{j+1}^-} |\dot{\gamma}(s)| ds \leq \frac{\varepsilon}{2} + A^{-1}C_A(t_{j+1}^- - t_j^+).
\]
From these estimates, for any \( N \in \mathbb{N} \), we have
\[
2 \sup_{D} |u_\infty - \psi| \geq (u_\infty - \psi)(\gamma(-t_1^-)) + (u_\infty - \psi)(\gamma(-t_{N+1}^-))
\]
\[
\geq \sum_{j=1}^{N} \left( \int_{-t_j^-}^{t_j^+} + \int_{-t_{j+1}^-}^{t_j^+} \right) \delta ds \geq \delta A C_A^{-1} \varepsilon N.
\]
By letting \( N \to \infty \), we get the contradiction. Hence, we conclude that \( \gamma(-t) \notin D_\varepsilon \) for all \( t \geq \tau \) for some \( \tau > 0 \). 
\( \square \)
Lemma 2.9. Assume (A1)-(A4) and let $u_0 \in \Phi_0$. Assume also (b) in Theorem 2.6. Then, the set $\{\gamma(-t) \mid t > 0\}$ is bounded in $\mathbb{R}^n$ for every $\gamma \in \mathcal{E}_x$.

Proof. Observe first that $u_{\infty} \geq \phi_0 - C$ in $\mathbb{R}^n$ for some $C > 0$. Then, in view of (18), we see that for every $t > 0$,

$$\psi(x) - \psi(\gamma(-t)) + \int_{-t}^{0} \sigma(\gamma(s)) \, ds \leq \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s)) \, ds \leq u_{\infty}(x) - \phi_0(\gamma(-t)) + C.$$ 

Thus,

$$(\phi_0 - \psi)(\gamma(-t)) + \int_{-t}^{0} \sigma(\gamma(s)) \, ds \leq (u_{\infty} - \psi)(x) + C \quad \text{for all } t > 0.$$ 

From this and property (b), we conclude that the set $\{\gamma(-t) \mid t > 0\}$ is bounded.

Proof of Theorem 2.6. We assume (a). Notice from Lemma 2.8 that $|\gamma(-t)| \to \infty$ as $t \to \infty$ for every $\gamma \in \mathcal{E}_x$. Thus, in view of (17) and Lemma 2.1, we get the convergence (9).

Assume next that (b) holds. Then, by Lemma 2.9, $\sup_{t>0} |\gamma(-t)| < \infty$ for any $\gamma \in \mathcal{E}_x$. Thus, we can apply Proposition 2.4 to obtain the convergence (9).

Remark 2.10. Theorem 2.6 with (a) generalizes Theorem 4.2 of Barles-Roquejoffre [2]. In our context, their assumption is equivalent to say that the function $\sigma$ in (18) satisfies $\sigma \geq \delta$ in $\mathbb{R}^n$ for some $\delta > 0$ and

$$\lim_{|x| \to \infty} (u_0 - u_{\infty})(x) = 0. \quad (22)$$

Remark that (22) is strictly stronger than (17). We discuss this point in Example 5.1.

Another remark is that Theorem 2.6 with (b) is a version of Theorem 1.3 of [17] in which the following condition is imposed in addition to the whole strict convexity of $H$:

There exist $\phi_i \in C^{0+1}(\mathbb{R}^n)$ and $\sigma_i \in C(\mathbb{R}^n)$ with $i=0,1$ such that for $i=0,1$,

$$H(x,D\phi_i(x)) \leq -\sigma_i(x) \quad \text{a.e. } x, \quad \lim_{|x| \to \infty} \sigma_i(x) = \infty, \quad \lim_{|x| \to \infty} (\phi_0 - \phi_1)(x) = \infty. \quad (23)$$

Notice here that the second condition in (23) can be replaced with $\sigma_i \geq 0$ in $\mathbb{R}^n$ once we have shown $t^{-1}u(x,t) \to 0$ as $t \to \infty$.

Remark 2.11. In Theorem 2.6, the family of minimizing curves $\{\mu_j\}$ in the right-hand side of (5) with $t = t_j$ for each $j \in \mathbb{N}$ can be constructed as follows. We first consider (a). In this case, it suffices to set $\mu_j(s) = \gamma(s)$, $s \in [-t_j,0]$, for each $j \in \mathbb{N}$. In particular, we find that $|\mu_j(-t_j)| = |\gamma(-t_j)| \to \infty$ as $j \to \infty$.

We next consider (b). For simplicity, we only deal with the case where (A5)$_+$ holds. For $j \in \mathbb{N}$, we choose $\eta_j \in C([-\tau,0];x_j)$ such that

$$u(x_j,\tau) + \delta > \int_{-\tau}^{0} L(\eta_j(s), \dot{\eta}_j(s)) \, ds + u_0(\eta_j(-\tau)),$$

where $\tau > 0$ is the number taken in Theorem 2.4. Then, the curve $\mu_j \in C([-t_j,0];x)$ can be constructed as

$$\mu_j(s) = \begin{cases} \gamma((1 + \varepsilon_j)s) & \text{if } s \in [-t_j + \tau,0], \\ \eta_j(s + t_j - \tau) & \text{if } s \in [-t_j, -t_j + \tau], \end{cases} \quad (24)$$

where $\varepsilon_j := (t_j - \tau)^{-1}\tau$. From this and the boundedness of $\{\gamma(-t) \mid t > 0\}$, we easily see that there exists an $R > 0$ such that $\{\mu_j(s) \mid s \in [-t_j,0]\} \subset B(0,R)$ for all $j \in \mathbb{N}$. 
Before closing this section, we discuss the relationship between the set $\Lambda$ and the ideal boundary in the sense of Ishii-Mitake [18]. For this purpose, we recall the notation used in Sections 4 and 5 of [18].

We denote by $A_H$ the Aubry set for $H$ and set $\Omega_0 := \mathbb{R}^n \setminus A_H$. Let $\pi : \phi \mapsto \{ \phi + c \mid c \in \mathbb{R} \}$ be the projection from $C(\mathbb{R}^n)$ to the quotient space $C(\mathbb{R}^n)/\mathbb{R}$, and let $d^\pi : \Omega_0 \rightarrow C(\mathbb{R}^n)/\mathbb{R}$ be the mapping defined by $d^\pi(y) := \pi(d_H(\cdot, y))$. We set $\mathcal{D}_0 := d^\pi(\Omega_0)$. Note that $d^\pi$ is bijective in view of Lemma 4.2 of [18] and the definition of $\mathcal{D}_0$.

We fix a standard complete metric $\rho$ on $C(\mathbb{R}^n)$ which defines the topology of locally uniform convergence. We denote by $\rho^\pi$ the induced metric on $C(\mathbb{R}^n)/\mathbb{R}$, that is,
\[ \rho^\pi(\xi_1, \xi_2) := \inf \{ \rho(\phi_1, \phi_2) \mid \phi_1 \in \xi_1, \phi_2 \in \xi_2 \}, \quad \xi_1, \xi_2 \in C(\mathbb{R}^n)/\mathbb{R}. \]

Then, we can define the metric $\rho_0$ on $\Omega_0$ by $\rho_0(x, y) := \rho^\pi(d^\pi(x), d^\pi(y))$. Observe from Proposition 4.3 of [18] that the identity map $x \mapsto x$ is a homeomorphism from $(\Omega_0, \rho_0)$ to $(\Omega_0, \rho_E)$, where $\rho_E$ stands for the Euclidean distance.

Let $(\hat{\Omega}_0, \rho_0)$ be the completion of $(\Omega_0, \rho_0)$. Since $d^\pi : (\Omega_0, \rho_0) \rightarrow (\mathcal{D}_0/\mathbb{R}, \rho^\pi)$ is isometric by the definition of $\rho_0$, $d^\pi$ can be extended to the isomorphism $(\hat{\Omega}_0, \rho_0) \rightarrow (\overline{\mathcal{D}_0/\mathbb{R}}, \rho^\pi)$, where $\overline{\mathcal{D}_0/\mathbb{R}}$ denotes the closure of $\mathcal{D}_0/\mathbb{R}$ in $C(\mathbb{R}^n)/\mathbb{R}$ with respect to $\rho^\pi$. Following the paper [18], we call the set $\Delta_0 := \hat{\Omega}_0 \setminus \Omega_0$ the ideal boundary of $\Omega_0$. We also denote by $\Delta_0^*$ the totality of points $y \in \Delta_0$ such that for some sequence $\{y_j\} \subset \Omega_0$,
\[ \phi(y_j) + d_H(\cdot, y_j) \rightarrow \phi \quad \text{in} \quad C(\mathbb{R}^n) \quad \text{as} \quad j \rightarrow \infty \quad \text{for all} \quad \phi \in d^\pi(y). \] (25)

Now, let $\{x_j\} \in \Lambda(\psi)$ for a given $\psi \in S_H$, where $\Lambda(\psi)$ is defined by (16). Then, by mimicking the arguments in Section 5 of [18], we easily see that there exist a subsequence $\{y_j\} \subset \{x_j\}$ and a $y \in \Delta_0$ such that $\rho_0(y_j, y) \rightarrow 0$ as $j \rightarrow \infty$ and (25) holds. In particular, $y \in \Delta_0^*$. We set
\[ \Lambda_0(\psi) := \{ y \in \Delta_0^* \mid \lim_{j \rightarrow \infty} \rho_0(x_j, y) = 0 \quad \text{for some} \quad \{x_j\} \subset \Lambda(\psi) \}. \] (26)

Then by definition, $\Lambda_0(\psi) \subset \Delta_0^* \setminus A_H$ for all $\psi \in S_H$. In what follows, we use the notation $\Lambda_0 := \Lambda_0(u_\infty)$.

Similarly as in [18], for given $u \in \text{UC}(\mathbb{R}^n)$ and $y \in \Delta_0^*$, we define the function $g(u, y) : \mathbb{R}^n \rightarrow (-\infty, \infty]$ by
\[ g(u, y)(x) := \phi(x) + \lim_{r \rightarrow 0} \sup \{(u - \phi)(\xi) \mid \xi \in \Omega_0, \rho_0(\xi, y) < r \}, \]
where $\phi$ is any element of $d^\pi(y)$ and remark that $g(u, y)(x)$ does not depend on the choice of $\phi \in d^\pi(y)$. If $g(u, y) = g(v, y)$ for some $y \in \Delta_0^*$ and $u, v \in \text{UC}(\mathbb{R}^n)$, then $\lim_{j \rightarrow \infty} (u - v)(x_j) = 0$ for every $\{x_j\} \subset \mathbb{R}^n$ such that $\lim_{j \rightarrow \infty} \rho_0(x_j, y) = 0$.

Taking into account these observations, we reformulate Theorem 2.5 as follows.

**Theorem 2.12.** Let $H$ satisfy $(A1)$-$(A4)$ and one of $(A5)_+$ or $(A5)_-$. Let $u_0 \in \Phi_0$. Then, the convergence (9) holds provided that
\[ g(u_\infty, y) = g(u_0, y) \quad \text{in} \quad \mathbb{R}^n \quad \text{for all} \quad y \in \Lambda_0. \]

We next try to obtain a representation formula for $u_\infty$ in terms of the ideal boundary. For $u \in \text{UC}(\mathbb{R}^n)$ and $y \in A_H$, we set $g(u, y) := d_H(\cdot, y) + u(y)$. Recall first the following theorem.
**Theorem 2.13** (Theorem 5.4 of [18]). Let $u \in S_{H}$. Then,

$$u(x) = \inf \{g(u, y)(x) \mid y \in \Delta_{0}^{*} \cup \mathcal{A}_{H} \}. \quad (27)$$

By using this theorem, we have the following representation formula for $u_{\infty}$ which is a natural generalization of the usual ones (e.g. Theorem 5.7 of [8] and Theorem 8.1 of [17]).

**Proposition 2.14.** Let $H$ satisfy (A1)-(A4) and let $u_{0} \in \Phi_{0}$. Then,

$$u_{\infty}(x) = \inf \{g(u_{0}^{-}, y)(x) \mid y \in \Lambda_{0} \cup \mathcal{A}_{H} \}. \quad (29)$$

To show this proposition, we use the following lemma.

**Lemma 2.15.** Let $H$ satisfy (A1)-(A4) and let $u_{0} \in \Phi_{0}$. Then, for every $x \in \mathbb{R}^{n}$ and $\gamma \in \mathcal{E}_{x}$,

$$\lim_{t \to \infty} (u_{\infty} - u_{0}^{-})(\gamma(-t)) = 0. \quad (28)$$

**Proof.** Let $(T_{t})_{t \geq 0}$ be the semigroup defined in Section 1. Then, from the variational formula (5) with $u_{0}^{-}$ in place of $u_{0}$, we observe that for every $t > 0$,

$$(T_{t} u_{0}^{-})(x) \leq \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s)) ds + u_{0}^{-}(\gamma(-t)) = u_{\infty}(x) - u_{\infty}(\gamma(-t)) + u_{0}^{-}(\gamma(-t)).$$

Since $(T_{t} u_{0}^{-})(x) \to u_{\infty}(x)$ as $t \to \infty$ by Lemma 1.2, we have $\limsup_{t \to \infty} (u_{\infty} - u_{0}^{-})(\gamma(-t)) \leq 0$. Noting that $u_{\infty} \geq u_{0}^{-}$ in $\mathbb{R}^{n}$ by definition, we obtain (28).

**Proof of Proposition 2.14.** Remark first that, by a careful review of the original proof of Theorem 5.4 in [18], the representation formula (27) can be rewritten as

$$u(x) = \inf \{g(u, y)(x) \mid y \in \Lambda_{0}(u) \cup \mathcal{A}_{H} \}. \quad (29)$$

We also observe from Lemma 2.15 and the definition of $g(u, y)$ that $g(u_{\infty}, y) = g(u_{0}^{-}, y)$ for all $y \in \Lambda_{0} \cup \mathcal{A}_{H}$. Hence, the proof is complete by setting $u = u_{\infty}$ in (29).

3 **Second convergence result.**

In this section, we deal with Hamiltonians that provide another type of motions for $\{\mu_{j}\}$ which we call in this paper "switch-back". In order to explain the meaning of this word, we begin with a simple example.

Let $n = 1$ and consider the Cauchy problem

\[
\begin{aligned}
    u_{t} + |Du| - e^{-|x|} &= 0 & \text{in } \mathbb{R} \times (0, +\infty), \\
    u(\cdot, 0) &= \min\{|x| - 2, 0\} & \text{on } \mathbb{R}.
\end{aligned}
\]

Clearly, the Hamiltonian $H(x, p) := |p| - e^{-|x|}$ satisfies (A1)-(A3). Since $e^{-|x|} \in S_{H}$, $H$ enjoys (A4) with $\phi_{0} = \psi_{0} = e^{-|x|}$, and the initial function $u_{0}(x) := \min\{|x| - 2, 0\}$ belongs to $\Phi_{0} = \text{BUC}(\mathbb{R})$. We see moreover that $u_{0}^{-}(x) = -e^{-|x|} - 1$ and $u_{\infty}(x) = e^{-|x|} - 1$. 


Let $L(x,\xi)$ be the Lagrangian associated with $H$, that is, $L(x,\xi)=\chi_{[-1,1]}(\xi)+e^{-|x|}$, where $\chi_{[-1,1]}(\xi) := 0$ for $|\xi| \leq 1$ and $\chi_{[-1,1]}(\xi) := +\infty$ for $|\xi| > 1$. For a given $x \in \mathbb{R}$, we define $\gamma \in C((-\infty, 0]; x)$ by $\gamma(s) := x - \text{sgn}(x) s$ for $s \in (-\infty, 0]$, where we have set $\text{sgn}(x) := 1$ for $x \geq 0$ and $\text{sgn}(x) = -1$ for $x < 0$. Then, it is easy to see that $\gamma \in \mathcal{E}_x$ and $|\gamma(t)| \to \infty$ as $t \to \infty$. We choose a diverging $\{t_j\} \subset (0, \infty)$ such that $u^*(x) = \lim_{t \to \infty} u(x,t_j)$ and $|x| < t_j$ for all $j \in \mathbb{N}$.

We next define $\mu_j \in C([-t_j, 0]; x)$, $j \in \mathbb{N}$, by

$$
\mu_j(s) := \begin{cases} 
\gamma(s) & \text{for } -\frac{t_j - |x|}{2} \leq s \leq 0, \\
\text{sgn}(x)(s+t_j) & \text{for } -t_j \leq s \leq -\frac{t_j - |x|}{2}.
\end{cases}
$$

Note that $u_0(\mu_j(-t_j)) = u_0(0) = -2$ for all $j \in \mathbb{N}$. Then, we see that

$$
u(x,t_j) \leq \int_{-t_j}^{0} L(\mu_j(s), \dot{\mu}_j(s)) \, ds + u_0(\mu_j(-t_j)) = e^{-|x|} - 2e^{-t_j + |x|/2} \to_{j \to \infty} u_\infty(x).
$$

Thus, (9) is valid. We remark here that if $t_j$ is sufficiently large, then $\mu_j(-t)$ goes toward $\infty$ or $-\infty$ along the curve $\gamma$ up to the time $t = (t_j - |x|)/2$ and then it turns back to the origin. This motion explains well the word “switch-back”.

It is also worth mentioning that the condition (17) in Theorem 2.5 does not hold in this case. Indeed, since $\lim_{t \to \infty} |\gamma(t)| = \infty$, we have $\lim_{t \to \infty} (u_0 - u_\infty)(\gamma(-t)) = 1 > 0$.

We now consider a more general situation. In the rest of this section, we assume the following:

(A6) $H(x,0) \leq 0$ for all $x \in \mathbb{R}^n$ and there exists a $\lambda \geq 1$ such that

$$
H(x,-\lambda p) \geq H(x,p) \text{ for all } (x,p) \in \mathbb{R}^{2n}.
$$

Note that (A6) implies

$$
L(x,-\lambda^{-1}x) \leq L(x,\xi) \text{ for all } (x,\xi) \in \mathbb{R}^{2n}.
$$

Theorem 3.1. Let $H$ satisfy (A1)-(A3), (A4) with $\phi_0 = 0$ and (A6). Then, the convergence (9) holds for every $u_0 \in \Phi_0$.

Remark 3.2. Assumption (A6) can be relaxed as

(A6)' There exists a $\lambda \geq 1$ such that for every $(x,p) \in Q$, $\xi \in D^-_2 H(x,p)$, $q \in \mathbb{R}^n$ and $q' \in \partial_c \phi_0(x)$,

$$
H(x,q'-\lambda q) \geq \xi \cdot (q' + q - p),
$$

where $\phi_0 \in S^-_H$ is taken from (A4) and $\partial_c \phi_0(x)$ denotes the Clarke derivative of $\phi_0$ at $x \in \mathbb{R}^n$.

Assumption (A6) is a particular case where $\phi_0 = 0$ in (A6)'. See [16] for details.

Proof of Theorem 3.1. Fix any $u_0 \in \Phi_0$, $x \in \mathbb{R}^n$ and $\gamma \in \mathcal{E}_x$. Since $\phi_0 = 0$ by assumption, we see that $u_\infty \geq -C$ in $\mathbb{R}^n$ for some $C > 0$. We also observe that $L \geq 0$ in $\mathbb{R}^{2n}$ in view of the assumption $H(\cdot,0) \leq 0$ in $\mathbb{R}^n$. In particular, the function $t \mapsto \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s)) \, ds$ is non-decreasing and

$$
0 \leq \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s)) \, ds = u_\infty(x) - u_\infty(\gamma(-t)) \leq u_\infty(x) + C \text{ for all } t \geq 0.
$$
Fix an arbitrary $\varepsilon > 0$. Then, there exists a $t_0 > 0$ such that
\[
\int_{-t_0}^{-t_0-\theta} L(\gamma(s), \dot{\gamma}(s)) \, ds < \varepsilon \quad \text{for all } \theta > 0.
\]  \hfill (33)

We next choose a $\tau > 0$ such that
\[
u_0^- (\gamma(-t_0)) + \varepsilon > u(\gamma(-t_0), \tau).
\]  \hfill (34)

Now, we fix any diverging $\{t_j\} \subset (0, \infty)$ so that $u^+(x) = \lim_{j \to \infty} u(x, t_j)$ and then take $\{\theta_j\} \subset (0, \infty)$ such that $t_j = t_0 + (1 + \lambda)\theta_j + \tau$ for all $j \in \mathbb{N}$, where $\lambda \geq 1$ is the constant taken from (A6). Note that $\theta_j \to \infty$ as $j \to \infty$.

For each $j \in \mathbb{N}$, we set $t_{1j} := t_0 + \theta_j$ and $t_{2j} := t_{1j} + \lambda\theta_j$, and we define $\gamma_j \in C([-t_{2j}, 0]; x)$ by
\[
\gamma_j(s) := \begin{cases} 
\gamma(s) & \text{if } s \in [-t_{1j}, 0], \\
\gamma(-\lambda^{-1}s - (1 + \lambda^{-1})t_{1j}) & \text{if } s \in [-t_{2j}, -t_{1j}]. 
\end{cases}
\]  \hfill (35)

Note that $\gamma_j(-t_0) = \gamma_j(-t_{2j}) = \gamma(-t_0)$. Then, in view of (31) and (33), we see that
\[
\int_{-t_{2j}}^{-t_{1j}} L(\gamma_j(s), \dot{\gamma}_j(s)) \, ds = \lambda \int_{-t_{1j}}^{-t_0} L(\gamma(s), -\lambda^{-1}\dot{\gamma}(s)) \, ds \leq \lambda \int_{-t_0}^{-t_0-\theta_j} L(\gamma(s), \dot{\gamma}(s)) \, ds < \lambda \varepsilon.
\]

On the other hand, in view of (34) and the inequality $u_\infty \geq u_0^-$ in $\mathbb{R}^n$,
\[
u_\infty(x) = \int_{-t_0}^{0} L(\gamma(s), \dot{\gamma}(s)) \, ds + u_\infty(\gamma(-t_0)) \geq \int_{-t_0}^{0} L(\gamma(s), \dot{\gamma}(s)) \, ds + u(\gamma(-t_0), \tau) - \varepsilon.
\]

In combination with these estimates, we obtain
\[
\nu_\infty(x) + (2 + \lambda)\varepsilon > \int_{-t_0}^{0} L(\gamma, \dot{\gamma}) \, ds + \int_{-t_0}^{-t_{1j}} L(\gamma, \dot{\gamma}) \, ds + \int_{-t_{2j}}^{-t_0} L(\gamma_j, \dot{\gamma}_j) \, ds + u(\gamma(-t_0), \tau)
\]
\[
= \int_{-t_{2j}}^{0} L(\gamma_j(s), \dot{\gamma}_j(s)) \, ds + u(\gamma_j(-t_2), \tau) \geq u(x, t_j).
\]

By letting $j \to \infty$, we have $u^+(x) = \lim_{j \to \infty} u(x, t_j) \leq u_\infty(x) + (2 + \lambda)\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain $u^+(x) \leq u_\infty(x)$.

We give in Example 5.2 a more concrete example which satisfies (A6).

**Remark 3.3.** Suppose in addition to (A6) that $H(x, 0) < 0$ for all $x \in \mathbb{R}^n$. Then, in view of Lemma 2.8, we have $|\gamma(-t)| \to \infty$ as $t \to \infty$ for any $\gamma \in E_x$. We now fix a diverging $\{t_j\} \subset (0, \infty)$ such that $u^+(x) = \lim_{j \to \infty} u(x, t_j)$ and choose $\eta \in C([-\tau, 0]; \gamma(-t_0))$ such that
\[
\int_{-\tau}^{0} L(\eta(s), \dot{\eta}(s)) \, ds + u_0(\eta(-\tau))
\]
\[
= u(\gamma(-t_0), \tau) + \varepsilon > \int_{-\tau}^{0} L(\eta(s), \dot{\eta}(s)) \, ds + u_0(\eta(-\tau)).
\]

If we define $\mu_j \in C([-t_j, 0]; x)$, $j \in \mathbb{N}$, by
\[
\mu_j(s) := \begin{cases} 
\gamma_j(s) & \text{if } s \in [-t_{2j}, 0], \\
\eta(s + t_{2j}) & \text{if } s \in [-t_j, -t_{2j}], 
\end{cases}
\]
then we observe the switch-back of $\mu_j$ as in the previous example. In particular, we have neither (a) $\mu_j = \gamma$ for all $j \in \mathbb{N}$, nor (b) $\mu_j$ is bounded uniformly in $j \in \mathbb{N}$. In this sense, the switch-back motion presents a striking contrast to the curves in Section 2.
4 Third convergence result.

This section is concerned with the Cauchy problem (1) with Hamiltonian and initial function having “weak” periodicity. In this case, one other type of motions for \( \{\mu_j\} \) takes place. In the rest of this section, we always assume that \( H \) satisfies (A1)-(A3), (A4) with \( \phi_0 = \psi_0 = \phi \) for some fixed \( \phi \in \mathcal{S}_H \). The class of initial data \( \Phi_0 \) is, therefore, written as

\[
\Phi_0 = \{u_0 \in \text{UC}(\mathbb{R}^n) \mid \phi - C \leq u_0 \leq \phi + C \text{ in } \mathbb{R}^n \text{ for some } C > 0\}.
\]

Fix an arbitrary \( u_0 \in \Phi_0 \). Then, there exists a \( C > 0 \) such that

\[
u_0 - 2C \leq \phi - C \leq u_0 \leq \phi + C \leq u_0 + 2C \text{ in } \mathbb{R}^n.
\]

Let \( \{y_j\} \subset \mathbb{R}^n \) be any sequence. By taking a subsequence if necessary, we may assume in view of (A1) and the Ascoli-Arzela theorem that

\[
\begin{align*}
H(\cdot + y_j, \cdot) \longrightarrow G \quad &\text{in } C(\mathbb{R}^{2n}) \text{ as } j \to \infty, \\
u_0(\cdot + y_j) - u_0(y_j) \longrightarrow v_0 \quad &\text{in } C(\mathbb{R}^n) \text{ as } j \to \infty,
\end{align*}
\]

for some \( G \in C(\mathbb{R}^{2n}) \) and \( v_0 \in \text{UC}(\mathbb{R}^n) \). Note that \( G \) satisfies (A1)-(A3) with \( G \) in place of \( H \). We denote by \( \mathcal{S}_{\overline{G}} \) (resp. \( \mathcal{S}_{G}^{+}, \mathcal{S}_{G}^{-} \)) the set of all continuous viscosity subsolutions (resp. supersolutions, solutions) of

\[
G(x, D\phi) = 0 \quad \text{in } \mathbb{R}^n.
\]

Since the family \( \{u_\infty(\cdot + y_j) - u_0(y_j)\}_j \) is uniformly bounded and equi-continuous on any compact subset of \( \mathbb{R}^n \), there exist a function \( \overline{u}_\infty \in C(\mathbb{R}^n) \) and a subsequence of \( \{y_j\} \), which we denote by the same \( \{y_j\} \), such that

\[
u_\infty(\cdot + y_j) - u_0(y_j) \longrightarrow \overline{u}_\infty \quad \text{in } C(\mathbb{R}^n) \text{ as } j \to \infty.
\]

Remark that \( \overline{u}_\infty \in \mathcal{S}_{G} \) by virtue of the stability property of viscosity solutions. We see moreover that \( v_0 - 2C \leq \overline{u}_\infty \leq u_0 + 2C \) in \( \mathbb{R}^n \). Thus, the functions

\[
\begin{align*}
v_\overline{0}(x) := \sup \{ \phi(x) \mid \phi \in \mathcal{S}_{G}^{-}, \phi \leq v_0 \text{ in } \mathbb{R}^n \} \in \mathcal{S}_{G}^{-}, \\
v_\infty(x) := \inf \{ \psi(x) \mid \psi \in \mathcal{S}_{G}, \psi \geq v_\overline{0} \text{ in } \mathbb{R}^n \} \in \mathcal{S}_{G}
\end{align*}
\]

are well-defined and satisfy

\[
v_0 - 4C \leq v_\overline{0} \leq v_\infty \leq v_0 + 4C \quad \text{in } \mathbb{R}^n.
\]

We next consider the Cauchy problem

\[
\begin{cases}
v_t + G(x, Dv) = 0 \quad &\text{in } \mathbb{R}^n \times (0, +\infty), \\
v(\cdot, 0) = v_0 \quad &\text{on } \mathbb{R}^n,
\end{cases}
\]

and let \( v(x, t) \) be the solution of (41). Remark here that \( \liminf_{t\to \infty} v(x, t) = v_\infty(x) \) in view of Lemma 1.2. Moreover, by (36), (37) and the stability property for viscosity solutions of (41), we observe that \( u(\cdot + y_j, \cdot) - u_0(y_j) \longrightarrow v \) in \( C(\mathbb{R}^{2n}) \) as \( j \to \infty \). Taking into account these observations, we claim the following.
Theorem 4.1. Let $H$ satisfy (A1)-(A9), (A4) with $\psi_0 = \phi_0 = \phi$ for some $\phi \in S_H$, and (A5)$_+$. Let $u_0 \in \Phi_0$. Then, the convergence (9) holds provided that for any sequence $\{y_j\} \subset \mathbb{R}^n$ satisfying (37) for some $v_0 \in \text{UC}(\mathbb{R}^n)$, there exists a subsequence, which we denote by the same $\{y_j\}$, such that

$$
\limsup_{j \to \infty} (u_\infty(y_j) - u_0(y_j)) \geq v_\infty(0). \tag{42}
$$

Moreover, condition (A5)$_+$ can be replaced by (A5)$_-$ if the following holds true in addition to (42):

$$
u(y_j, \cdot) - u_0(y_j) \longrightarrow v(0, \cdot) \quad \text{uniformly in } [0, \infty) \text{ as } j \to \infty. \tag{43}
$$

Proof. Fix any $x \in \mathbb{R}^n$ and any diverging sequence $\{t_j\} \subset (0, \infty)$ such that $u^+(x) = \lim_{j \to \infty} u(x, t_j)$. We also fix a $\gamma \in S_x$ and set $y_j := \gamma(-t_j)$ for $j \in \mathbb{N}$. Then, there exists a subsequence of $\{y_j\}$ such that (36) and (37) hold for some $G \in C(\mathbb{R}^{2n})$ and $v_0 \in \text{UC}(\mathbb{R}^n)$, respectively. In what follows, we fix an arbitrary $\delta > 0$ and choose a $\tau > 0$ so that $v(0, \tau) - v_\infty(0) < \delta$, where $v$ is the unique viscosity solution of (41).

We first assume (A5)$_+$ and (42). For each $j \in \mathbb{N}$, we set $\epsilon_j := (t_j - \tau)^{-1}\tau$ and define $\gamma_j \in C([-t_j + \tau, 0]; \mathbb{R}^n)$ by $\gamma_j(s) = \gamma((1 + \epsilon_j)s)$. Note that $\gamma_j(-t_j + \tau) = y_j$ for all $j \in \mathbb{N}$. By renumbering $j \in \mathbb{N}$, we may assume that $\epsilon_j \in (0, \delta_1)$ for all $j \in \mathbb{N}$, where $\delta_1$ is the constant taken from Lemma 2.2. Then, in view of (14), we see that

$$
u(x, t_j) \leq \int_{-t_j}^{0} L(\gamma_j, \gamma_j') \, ds + u(\gamma_j(-t_j + \tau), \tau)
\leq \int_{-t_j}^{0} L(\gamma, \gamma') \, ds + t_j \epsilon_j \omega_1(\epsilon_j) + u(y_j, \tau).
$$

Since $v(0, \tau) - v_\infty(0) < \delta$ and $u(y_j, \tau) - u_0(y_j) \longrightarrow v(0, \tau)$ as $j \to \infty$, we conclude in combination with (42) that

$$
u^+(x) - u_\infty(x) \leq -\limsup_{j \to \infty} (u_\infty(y_j) - u_0(y_j)) + \lim_{j \to \infty} (u(y_j, \tau) - u_0(y_j))
\leq -v_\infty(0) + v(0, \tau) < \delta.
$$

Hence, letting $\delta \to 0$ yields $\nu^+(x) \leq u_\infty(x)$.

We next assume (A5)$_-$, (42) and (43). In view of (39) and (43), and by renumbering $\{t_j\}$ if necessary, we may assume that for every $j \in \mathbb{N}$ and $t > 0$,

$$
|u(y_j, t) - u_0(y_j) - v(0, t)| + |u_\infty(y_j) - u_0(y_j) - \overline{u}_\infty(0)| < \delta. \tag{44}
$$

Hereafter, we always use the same $\{t_j\}$ to denote its subsequence. Then, we observe that

$$
u(x, t_j) \leq \int_{-t_j}^{0} L(\gamma(s), \gamma'(s)) \, ds + u(y_1, t_j - t_1) = u_\infty(x) - u_\infty(y_1) + u(y_1, t_j - t_1)
< u_\infty(x) - \overline{u}_\infty(0) + u(y_2, t_j - t_1) - u_0(y_2) + 3\delta.
$$

We may assume without loss of generality that $t_2 > t_1 + \tau$. For each $j \geq 2$, we set

$$
\epsilon_j = \frac{t_2 - t_1 - \tau}{t_j - t_1 - \tau}, \quad \gamma_j(s) = \gamma(-t_2 + (1 - \epsilon_j)s), \quad s \leq 0.
$$
Note that $\varepsilon_j \to 0$ as $j \to \infty$ and $\gamma_j((1-\varepsilon_j)(-t_j + t_1 + \tau)) = \gamma(-t_j) = y_j$ for all $j \geq 2$. Then, in view of (14) and (44),
\[
u(y_2, t_j - t_1) \leq \int_{-t_j + t_1 + \tau}^{0} L(\gamma_j(s), \gamma_j(s)) \, ds + \nu(y_j, \tau)
< \nu_{\infty}(y_2) - \nu_{\infty}(y_j) + t_j \varepsilon_j \omega_1(\varepsilon_j) + v(0, \tau) + \nu_0(y_j) + \delta.
\]
Thus, we have
\[
u(x, t_j) - \nu_{\infty}(x) < \nu_{\infty}(y_2) - \nu_0(y_2) - \overline{u}_{\infty}(0) + t_j \varepsilon_j \omega_1(\varepsilon_j) + v(0, \tau) - \nu_{\infty}(y_j) + \nu_0(y_j) + 4\delta
< \nu_{\infty}(0) - \nu_{\infty}(y_j) + \nu_0(y_j) + t_j \varepsilon_j \omega_1(\varepsilon_j) + 6\delta.
\]
Taking into account (42) and letting $j \to \infty$ and then $\delta \to 0$, we get $u^+(x) \leq u_{\infty}(x).$ 

**Corollary 4.2.** Let $H$ satisfy (A1)-(A3), (A4) with $\phi_0 = \psi_0 = \phi$ for some $\phi \in S_H$, and (A5+) \hspace{1pt}. Let $u_0 \in \Phi_0$. Then, the convergence (9) holds provided that for any sequence $\{y_j\} \subset \mathbb{R}^n$ satisfying (37) for some $v_0 \in UC(\mathbb{R}^n)$, there exists a subsequence such that
\[
u_0(\cdot + y_j) - \nu_0(y_j) \longrightarrow v_0^- \quad \text{in} \quad C(\mathbb{R}^n) \quad \text{as} \quad j \to \infty.
\]

**Proof.** It suffices to check (42). Observe first that
\[
u_{\infty}(\cdot + y_j) - \nu_0(y_j) \geq \nu_0^- (\cdot + y_j) - \nu_0(y_j) \quad \text{in} \quad \mathbb{R}^n \quad \text{for all} \quad j \in \mathbb{N}.
\]
In view of (39) and (45), for a suitable subsequence of $\{y_j\}$, we see that
\[
u_{\infty}(x) = \lim_{j \to \infty} \nu_{\infty}(x + y_j) - \nu_0(y_j) \geq v_0^-(x) \quad \text{for all} \quad x \in \mathbb{R}^n.
\]
Since $\overline{u}_{\infty} \in S_G$, we have $\overline{u}_{\infty}(x) \geq v_{\infty}(x) \geq v_0^- (x)$ for all $x \in \mathbb{R}^n$. Thus, (42) is valid by setting $x = 0.$

We point out here that Theorem 4.1 covers, as a particular case, Theorem 2.2 of [14] dealing with upper semi-periodic Hamiltonians and obliquely lower semi-almost periodic initial data. Here, we recall that $H$ is upper (resp. lower) semi-periodic if for any sequence $\{y'_j\} \subset \mathbb{R}^n$, there exist a subsequence $\{y_j\} \subset \{y'_j\}$, a function $G \in C(\mathbb{R}^{2n})$ and a sequence $\{\xi_j\} \subset \mathbb{R}^n$ converging to zero as $j \to \infty$ such that $H(\cdot + y_j, \cdot) \geq G$ in $C(\mathbb{R}^{2n})$ as $j \to \infty$ and
\[
u(x, t_j) - \nu_0(y_j) \leq G \quad \text{resp.} \geq G \quad \text{in} \quad \mathbb{R}^{2n} \quad \text{for all} \quad j \in \mathbb{N}.
\]
We say that $u_0 \in UC(\mathbb{R}^n)$ is obliquely lower (resp. upper) semi-almost periodic if for any $\varepsilon > 0$ and any sequence $\{y'_j\} \subset \mathbb{R}^n$, there exist a subsequence $\{y_j\} \subset \{y'_j\}$ and a function $v_0 \in UC(\mathbb{R}^n)$ such that $u_0(\cdot + y_j) - u_0(y_j) \geq v_0$ in $C(\mathbb{R}^n)$ as $j \to \infty$ and
\[
u_0(\cdot + y_j) - \nu_0(y_j) - v_0(\cdot) > -\varepsilon \quad \text{resp.} < \varepsilon \quad \text{in} \quad \mathbb{R}^n \quad \text{for all} \quad j \in \mathbb{N}.
\]
If $u_0$ is both obliquely lower and upper semi-almost periodic, we say that $u_0$ is obliquely almost periodic.
Theorem 4.3 (cf. Theorem 2.2 of [14]). Let $H$ satisfy (A1)-(A3), (A4) with $\phi_0 = \psi_0 = \phi$ for some $\phi \in \mathcal{S}_H$, and (A5). Let $u_0 \in \Phi_0$ and assume that $H$ and $u_0$ are, respectively, upper semi-periodic and obliquely lower semi-almost periodic. Then, the convergence (9) holds.

Proof. We check (45) in Corollary 4.2. Since the family $\{u_0^{-}(\cdot + y_j) - u_0(y_j) | j \in \mathbb{N}\}$ is pre-compact in $C(\mathbb{R}^n)$, we can extract a subsequence of $\{y_j\}$, which we denote by $\{y_j\}$ again, such that $u_0^{-}(\cdot + y_j) - u_0(y_j) \rightarrow w$ in $C(\mathbb{R}^n)$ as $j \rightarrow \infty$ for some $w \in UC(\mathbb{R}^n)$. It suffices to show that $w = v_0^{-}$ in $\mathbb{R}^n$. Note that $w \in \mathcal{S}_G^-$ in view of the stability of viscosity property.

Observe first that upper semi-periodicity (46) together with the Lipschitz continuity of $d_H(\cdot, \cdot)$ in both variables ensure that for any $\varepsilon > 0$ and $x \in \mathbb{R}^n$, there exists a $j_0 \in \mathbb{N}$ such that

$$d_H(x + y_j, \cdot + y_j) \geq d_G(x, \cdot) - \varepsilon \quad \text{in } \mathbb{R}^n \quad \text{for all } j \geq j_0. \quad (48)$$

From this and obliquely lower semi-almost periodicity (47), we obtain

$$u_0^{-}(x + y_j) - u_0(y_j) = \inf_{x \in \mathbb{R}^n} (d_H(x + y_j, x + y_j) + u_0(x + y_j) - u_0(y_j))$$

$$> \inf_{x \in \mathbb{R}^n} (d_G(x, x) + v_0(x)) - 2\varepsilon = v_0^{-}(x) - 2\varepsilon.$$

On the other hand, since $u_0^{-} \leq u_0$ in $\mathbb{R}^n$, we have

$$u_0^{-}(\cdot + y_j) - u_0(y_j) \leq u_0(\cdot + y_j) - u_0(y_j) \quad \text{in } \mathbb{R}^n.$$ By taking the limit $j \rightarrow \infty$ in the last two inequalities and then letting $\varepsilon \rightarrow 0$, we get $v_0^{-} \leq w \leq v_0$ in $\mathbb{R}^n$. Hence, we conclude that $w = v_0^{-}$ in $\mathbb{R}^n$. \hfill \Box

Remark 4.4. If $H(x, p)$ is $\mathbb{Z}^n$-periodic with respect to $x$ for all $p \in \mathbb{R}^n$, then (48) is obvious from the identity $d_H(\cdot + k, \cdot + k) = d_H$ in $\mathbb{R}^{2n}$ for all $k \in \mathbb{Z}^n$. Notice here that Theorem 4.1 does not require, a priori, any periodicity for $H$ and $u_0$. We give in Section 5 an example having neither upper semi-periodicity for $H$ nor obliquely lower semi-almost periodicity for $u_0$, but enjoying the conditions required in Theorem 4.1.

Concerning the latter part of Theorem 4.1, we have the following result.

Theorem 4.5. Let $H$ satisfy (A1)-(A3), (A4) with $\phi_0 = \psi_0 = \phi$ for some $\phi \in \mathcal{S}_H$, and (A5). Let $u_0 \in \Phi_0$ and assume that $H(x, p)$ is $\mathbb{Z}^n$-periodic with respect to $x$ for all $p \in \mathbb{R}^n$ and $u_0$ is obliquely almost periodic. Then, the convergence (9) holds.

Proof. It suffices to check (43). Let $\{y_j\} \subset \mathbb{R}^n$ be any sequence. We first observe from the obliquely almost periodicity for $u_0$ that along a subsequence of $\{y_j\}$,

$$u_0(\cdot + y_j) - u_0(y_j) \rightarrow v_0 \quad \text{uniformly in } \mathbb{R}^n \quad \text{as } j \rightarrow \infty. \quad (49)$$

Observe also from the $\mathbb{Z}^n$-periodicity for $H$ that there exists a bounded $\{\xi_j\} \subset \mathbb{R}^n$ converging to some $\xi \in \mathbb{R}^n$ as $j \rightarrow \infty$ such that $H(x + y_j, p) = H(x + \xi_j, p)$ for all $(x, p) \in \mathbb{R}^{2n}$ and $j \in \mathbb{N}$, and $H(x + \xi_j, p) \rightarrow H(x + \xi, p)$ uniformly in $\mathbb{R}^n \times B(0, R)$ as $j \rightarrow \infty$ for all $R > 0$.

We now set $G(x, p) := H(x + \xi, p)$ and let $v_j(x, t) \in C(\mathbb{R}^n \times [0, \infty))$, $j \in \mathbb{N}$, be the solution of

$$v_j + G(x, Dv) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \quad (50)$$
satisfying \( v_j(\cdot, 0) = u_0(\cdot + y_j) - u_0(y_j) \) in \( \mathbb{R}^n \). Note that by uniqueness,
\[
 u(x + y_j, t) - u_0(y_j) = v_j(x + \xi_j - \xi, t) \quad \text{for all} \quad (x, t) \in \mathbb{R}^n \times [0, \infty) \quad \text{and} \quad j \in \mathbb{N}.
\]

Then, by using the nonexpansive property for solutions of (50) and the equi-continuity on \( \mathbb{R}^n \) for \( \{v_j(\cdot, t) \mid t > 0, j \in \mathbb{N}\} \), we have
\[
|u(x + y_j, t) - u_0(y_j) - v(x, t)| \leq |v_j(x + \xi_j - \xi, t) - v_j(x, t)| + |v_j(x, t) - v(x, t)|
\]
\[
\leq \omega(|\xi_j - \xi|) + \omega(|x + y_j - u_0(y_j) - v_0(x)|),
\]

where \( \omega \) is a modulus. Thus, in view of (49) and letting \( j \to \infty \), we obtain (43).

**Remark 4.6.** We now discuss the construction of \( \{\mu_j\} \) corresponding to Theorem 4.1. For simplicity, we only consider the case where \( (A5)_+ \) holds. Let \( \tau > 0 \) be the number taken in the proof of Theorem 4.1. For each \( j \in \mathbb{N} \), we choose an \( \eta_j \in C([-\tau, 0]; y_j) \) such that
\[
 u(y_j, \tau) + \delta > \int_{-\tau}^{0} L(\eta_j(s), \dot{\eta}_j(s)) \, ds + u_0(\eta_j(-\tau)).
\]

We then define \( \mu_j \in C([-t_j, 0]; x), j \in \mathbb{N} \), by
\[
\mu_j(s) = \begin{cases} 
\gamma_j(s) & \text{if } s \in [-t_j + \tau, 0], \\
\eta_j(s + t_j - \tau) & \text{if } s \in [-t_j, -t_j + \tau].
\end{cases}
\]

Suppose that \( \sup_{t>0}|\gamma(-t)| < \infty \). Then, \( \{\mu_j\} \) is nothing but the one discussed in Remark 2.11. On the contrary, if \( \{\gamma(-t) \mid t > 0\} \) is unbounded, then we have one other type of motions for \( \{\mu_j\} \) which ensures the convergence (9). Notice here that condition (17) does not hold in general.

## 5 Examples.

We begin with an example concerning condition (a) of Theorem 2.6.

**Example 5.1.** Fix any \( p_0 \in \mathbb{R}^n \) such that \( |p_0| < 1 \) and define \( H \) by \( H = H(p) := |p - p_0| - 1 \) for \( p \in \mathbb{R}^n \). Note that the corresponding Lagrangian is \( L(\xi) = p_0 \cdot \xi + 1 + \chi_{B(0,1)}(\xi) \), where \( \chi_{B(0,1)}(\xi) := 0 \) on \( B(0, 1) \) and \( \chi_{B(0,1)}(\xi) = \infty \) on \( \mathbb{R}^n \backslash B(0, 1) \). It is easy to check that \( H \) enjoys (A1)-(A3) as well as the first part of condition (a) in Theorem 2.6. We also see by Lemma 2.8 that any extremal curve \( \gamma \) is diverging, namely, \( |\gamma(-t)| \to \infty \) as \( t \to \infty \).

We first identify the ideal boundary \( \Delta_0 \) for \( H \). Let \( d_H \) be the function defined by (7). Observe in view of (7) or (8) that \( d_H(x, y) = |x - y| + p_0 \cdot (x - y), x, y \in \mathbb{R}^n \). We take any diverging sequence \( \{y_j\} \subset \mathbb{R}^n \). Since
\[
d_H(x, y_j) - d_H(0, y_j) = |x - y_j| - |y_j| + p_0 \cdot x = \frac{|x|^2 - 2y_j \cdot x}{|x - y_j|^2 + |y_j|^2} + p_0 \cdot x
\]
for all \( j \in \mathbb{N} \), we see that \( \{d_H(\cdot, y_j) - d_H(0, y_j)\}_j \) converges in \( C(\mathbb{R}^n) \) to some function if and only if \( \frac{y_j}{|y_j|} \to \hat{y} \) as \( j \to \infty \) for some \( \hat{y} \in \partial B(0, 1) \) in which case we have
\[
d_H(x, y_j) - d_H(0, y_j) \to -\hat{y} \cdot x + p_0 \cdot x = (p_0 - \hat{y}) \cdot x \quad \text{as} \quad j \to \infty.
\]
This implies that the sequence \( \{d^n(y_j)\} \) converges in \( (C(\mathbb{R}^n)/\mathbb{R}, \rho^n) \) to \( \pi((p_0 - \hat{y}) \cdot x) \) as \( j \to \infty \). Thus, in view of the fact that \( \mathcal{A}_H = \emptyset \), we may identify \( \Delta_0 \) with \( \partial B(0,1) \) through the mapping

\[
\partial B(0,1) \ni \hat{y} \mapsto \pi((p_0 - \hat{y}) \cdot x) \in \Delta_0 = (\overline{D_0}/\mathbb{R}) \setminus (D_0/\mathbb{R}).
\]

We now fix any \( q_0 \in \partial B(0,1) \) and set \( \phi(x) := (p_0 + q_0) \cdot x \) for \( x \in \mathbb{R}^n \). Note that \( \phi \in S_H \). We try to identify the set \( \Lambda_0(\phi) \) defined by (26). Observe first that \( \gamma \) is an extremal curve for \( \phi \) at some \( x \in \mathbb{R}^n \) if and only if

\[
\phi(x) - \phi(\gamma(-t)) = \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s)) \, ds = d_H(x, \gamma(-t)) \quad \text{for all } t > 0.
\]

From this and the explicit forms of \( \phi \), \( L \) and \( d_H \), we see that

\[
(p_0 + q_0) \cdot (x - \gamma(-t)) = p_0 \cdot (x - \gamma(-t)) + t = |x - \gamma(-t)| + p_0 \cdot (x - \gamma(-t)),
\]

from which we deduce after some computations that \( \gamma(-t) = x - tq_0 \) for all \( t \geq 0 \). Let \( \{t_j\} \subset (0, \infty) \) be any diverging sequence and set \( y_j := \gamma(-t_j) \). Then as \( j \to \infty \),

\[
\frac{y_j}{|y_j|} = \frac{x - t_jq_0}{|x - t_jq_0|} \to -\frac{q_0}{|q_0|} =: -q_0 \in \partial B(0,1),
\]

from which we conclude that \( \Lambda_0(\phi) = \{-q_0\} \).

We now set \( \phi_0(x) := \min\{(p_0 + q_0) \cdot x, 0\}, x \in \mathbb{R}^n \). Notice that \( \phi_0 \in \mathcal{S}_H^0 \) in view of (A3), and that (A4) is valid with the above \( \phi_0 \) and \( \psi_0(x) := \phi(x) = (p_0 + q_0) \cdot x \in \mathcal{S}_H \). Let \( u_0 \in \Phi_0 \) be any initial function satisfying

\[
\lim_{\lambda \to \infty} (u_0 - \phi_0)(x - \lambda q_0) = 0 \quad \text{for all } x \in \mathbb{R}^n.
\]

Then, we can see that \( u_\infty(x) = \phi(x) \) for \( x \in \mathbb{R}^n \), and therefore \( \Lambda_0 = \{-q_0\} \) and (17) holds. Hence, by Theorem 2.5, we have the convergence (9). We remark here that if we choose \( u_0 := \phi_0 \), then, \( \lim_{j \to \infty} (u_0 - u_\infty)(x_j) = -\infty \) for any \( \{x_j\} \) such that \( \lim_{j \to \infty} u_\infty(x_j) = \infty \). This example shows that (22) is strictly stronger than (17).

On the other hand, if we set \( \phi(x) := \inf\{(p_0 + q) \cdot x \mid q \in \partial B(0,1)\}, x \in \mathbb{R}^n \), then \( \phi \in \mathcal{S}_H \) in view of (A3). Since \( \phi = -d_H(0, \cdot) \) in \( \mathbb{R}^n \), we observe that \( \gamma \in \mathcal{E}_x(\phi) \) for \( x \neq 0 \) if and only if

\[
\gamma(-t) = x + t \frac{x}{|x|} \quad \text{for all } t \geq 0.
\]

We conclude in particular that \( \Lambda_0(\phi) = \partial B(0,1) \). Hence, \( \{x_j\} \in \Lambda(\phi) \) if and only if \( \lim_{j \to \infty}|x_j| = \infty \).

We now choose \( \phi_0 = \psi_0 = \phi \) in (A4) and let \( u_0 \in \Phi_0 \) be any initial function such that \( \lim_{|x| \to \infty} (u_0 - \phi)(x) = 0 \). Then, we easily see that \( u_\infty = \phi \) in \( \mathbb{R}^n \). Thus, two conditions (17) and (22) are equivalent in this case.

The next example is concerned with Theorem 3.1.

**Example 5.2.** Let \( H \) satisfy (A1)-(A3) and \( H(x,0) \leq 0 \) for all \( x \in \mathbb{R}^n \). By setting \( H_0 := H - H(\cdot,0) \) and \( \sigma := -H(\cdot,0) \), \( H \) can be written as

\[
H(x,p) = H_0(x,p) - \sigma(x), \quad \sigma(x) \geq 0, \quad (x,p) \in \mathbb{R}^{2n}.
\]
Note that $H_0(x,0) = 0$ for all $x \in \mathbb{R}^n$.

We assume here that there exist $\alpha > 0$, $\beta \geq 1$, $\gamma > 1$ and $C_0 > 0$ such that

$$\alpha|p|^{\beta} \leq H_0(x,p) \leq \alpha^{-1}|p|^{\beta} \quad \sigma(x) \leq C_0(1 + |x|)^{-\beta \gamma}, \quad \text{for all } (x,p) \in \mathbb{R}^{2n}. \quad (51)$$

Next, we define $\psi_0 \in \text{Lip}(\mathbb{R}^n)$ by $\psi_0(x) := -\alpha^{-1}C_0 \int_0^{|x|}(1 + r)^{-\gamma}dr + C_1$, $x \in \mathbb{R}^n$, where $C_1 > 0$ is taken so that $\psi_0 \geq 0$ in $\mathbb{R}^n$. Then, for $x \neq 0$,

$$H(x,D\psi_0(x)) \geq \alpha|D\psi_0(x)|^{\beta} - \sigma(x) = C_0(1 + |x|)^{-\beta \gamma} - \sigma(x) \geq 0,$$

which implies that $\psi_0 \in S_{H}^+$. In particular, $H$ satisfies (A4) with $\phi_0 = 0$ and the above $\psi_0$.

We now claim that $H$ satisfies property (A6). Let $\lambda > 0$ be a constant which will be specified later. Observe that

$$H_0(x,-\lambda p) \geq \alpha|\lambda p|^{\beta} \geq \alpha^2 \lambda^2 \cdot \alpha^{-1}|p|^{\beta} = \alpha^2 \lambda^2 H_0(x,p) \quad \text{for all } (x,p) \in \mathbb{R}^{2n}.$$ 

Since $H_0 \geq 0$ in $\mathbb{R}^{2n}$ in view of the first condition of (51), by choosing $\lambda$ so that $\alpha^2 \lambda^2 \geq 1$, we get $H(x,-\lambda p) \geq H(x,p)$ for all $(x,p) \in \mathbb{R}^{2n}$. Hence, $H$ satisfies (A6). In this case, we have $\Phi_0 = \text{BUC}(\mathbb{R}^n)$.

We give here an example of Theorem 4.1.

**Example 5.3.** Let $n = 1$, and let $f \in \text{BUC}(\mathbb{R})$ be any function such that $f \geq 0$ in $\mathbb{R}$. We set $F(x) := \int_0^x f(y) dy$ for $x \in \mathbb{R}$ and define $H \in \text{C}(\mathbb{R}^2)$ and $\phi \in \text{UC}(\mathbb{R})$ by

$$H(x,p) := p^2 - f(x)^2, \quad \phi(x) := \min\{F(x), -F(x)\}, \quad (x,p) \in \mathbb{R}^2.$$ 

Note that $H$ satisfies (A1)-(A3) and (A5)$_\phi$. Moreover, since $F$, $-F \in S_{H}$, we see in view of convexity (A3) that $\phi \in S_{H}$. Thus, assumption (A4) is also fulfilled with $\phi_0 = \psi_0 = \phi$.

Now, let $p_0 \in \text{BUC}(\mathbb{R})$ be any function satisfying the following property: for any $\varepsilon > 0$, there exists an $l > 0$ such that

$$\min_{|y| \leq l} p_0(x+y) < \inf_{\mathbb{R}} p_0 + \varepsilon \quad \text{for all } x \in \mathbb{R}. \quad (52)$$

Remark that (52) is valid for any (lower semi-)almost periodic function.

We set $u_0 := \phi + p_0 \in \Phi_0$ and let $u(x,t)$ be the solution of the Cauchy problem (1) with $H$ and $u_0$ defined above. What we prove is the following convergence:

$$u(\cdot,t) \longrightarrow \phi + \inf_{\mathbb{R}} (u_0 - \phi) \quad \text{in } C(\mathbb{R}) \quad \text{as } t \rightarrow \infty. \quad (53)$$

In what follows, we only consider the case where $\inf_{\mathbb{R}} (u_0 - \phi) = \inf_{\mathbb{R}} p_0 = 0$ (which does not lose any generality). In this case, we have $u_\infty = \phi$ in $\mathbb{R}$. Note also that condition (17) of Theorem 2.5 does not hold in general.

To show the convergence (53), we check (42) in Theorem 4.1. Notice that Theorem 2.2 of [14] cannot be applied to this example since both $H$ and $u_0$ do not satisfy semi- or semi-almost periodicity assumptions. Fix any $x \in \mathbb{R}$, $\gamma \in \mathcal{E}_x$, and choose any diverging $\{t_j\} \subset (0,\infty)$ such that $u^+(x) = \lim_{j \rightarrow \infty} u(x,t_j)$. We set $y_j := \gamma(-t_j)$ for $j \in \mathbb{N}$. By taking a subsequence of $\{y_j\}$ if necessary, we have either $\sup_j |y_j| < \infty$ or $\lim_{j \rightarrow \infty} |y_j| = \infty$. Since the former case can be
reduced to Theorem 2.5, it suffices to consider the latter case. In what follows, we assume that
\( \lim_{j \to \infty} y_j = \infty \) (the case where \( \lim_{j \to \infty} y_j = -\infty \) can be treated in a similar way), and any
subsequence of \( \{y_j\} \) will be denoted by the same \( \{y_j\} \).

Since \( \{f(\cdot + y_j)\}_{j}, \{p_0(\cdot + y_j)\}_{j} \) and \( \{u_0(\cdot + y_j) - u_0(y_j)\}_{j} \) are pre-compact in \( C(\mathbb{R}) \), there
exist \( f_+, q_0 \in \text{BUC}(\mathbb{R}) \) and \( v_0 \in \text{UC}(\mathbb{R}) \) such that
\[
f(\cdot + y_j) \longrightarrow f_+ \quad \text{and} \quad p_0(\cdot + y_j) \longrightarrow q_0 \quad \text{in} \ C(\mathbb{R}) \quad \text{as} \quad j \to \infty \quad (54)
\]
and \( u_0(\cdot + y_j) - u_0(y_j) \longrightarrow v_0 \) in \( C(\mathbb{R}) \) as \( j \to \infty \). Remark here that \( q_0 \) inherits property (52).
Indeed, fix any \( \varepsilon > 0 \) and choose an \( l > 0 \) so that (52) holds. Observe that \( \inf \mathbb{R} q_0 = 0 \) by the
second convergence in (54) and the fact that \( \inf \mathbb{R} p_0 = \inf \mathbb{R} (u_0 - \phi) = 0 \). For each \( j \in \mathbb{N} \), we
choose a \( z_j \in (-l, l) \) such that \( p_0(x + y_j + z_j) \geq \min_{|y| \leq l} p_0(x + y_j + y) < \varepsilon \). Since \( \sup_j |z_j| \leq l \),
we may assume that \( \lim_{j \to \infty} z_j = z \) for some \( z \in (-l, l) \). Thus,
\[
\min_{|y| \leq l} q_0(x + y) \leq q_0(x + z) = \lim_{j \to \infty} p_0(x + y_j + z_j) < \varepsilon,
\]
which shows that (52) is valid with \( q_0 \) in place of \( p_0 \).

We now set \( F_+(x) := \int_{x}^{\infty} f_+(y) \, dy \) for \( x \in \mathbb{R} \). Then, we see that
\[
\phi(\cdot + y_j) - \phi(y_j) \longrightarrow -F_+ \quad \text{in} \ C(\mathbb{R}) \quad \text{as} \quad j \to \infty. \quad (55)
\]
It is also not difficult to check that \( v_0 = -F_+ + q_0 - q_0(0) \) in \( \mathbb{R} \). We set \( G(x, p) := p^2 - f_+(x)^2 \)
and define \( d_G \in C(\mathbb{R}^2) \) by (7) with \( G \) instead of \( H \). Observe that
\[
d_G(x, y) = \max \{ F_+(x) - F_+(y), F_+(y) - F_+(x) \}, \quad x, y \in \mathbb{R}.
\]
Since \( F_+ \) is non-decreasing on \( \mathbb{R} \), we have
\[
v_0^{-}(x) \leq \inf_{y \geq x} \{d_G(x, y) + v_0(y)\} = \inf_{y \geq x} \{ F_+(y) - F_+(x) - F_+(y) + q_0(y) - q_0(0) \}
= -F_+(x) - q_0(0) + \inf_{y \geq x} q_0(y).
\]
In view of property (52) for \( q_0 \), we obtain \( v_0^{-} \leq -F_+ - q_0(0) \) in \( \mathbb{R} \). On the other hand, observing
that \( v_0(x) \geq -F_+(x) - q_0(0) \in S_H \), we have \( v_0^{-} \geq -F_+ - q_0(0) \) in \( \mathbb{R} \). Thus, \( v_0^{-} = -F_+ - q_0(0) \)
in \( \mathbb{R} \). This implies that \( v_\infty = v_0^{-} \) in \( \mathbb{R} \). Since \( v_\infty(0) = -F_+(0) - q_0(0) = -q_0(0) \), we find that
\[
\lim_{j \to \infty} \sup_{y \in \mathbb{R}} (u_\infty - u_0)(y_j) = -\lim_{j \to \infty} \inf_{y \in \mathbb{R}} (u_0 - \phi)(y_j) = -q_0(0) = v_\infty(0),
\]
which is (42).

The following can be regarded as a generalization of the previous example to multi-dimensional
cases.

**Example 5.4.** For each \( i = 1, \ldots, n \), let \( f_i \in \text{BUC}(\mathbb{R}^n) \), \( i = 1, \ldots, n \), be such that \( \inf \mathbb{R}^n f_i \geq 0 \)
or \( \sup \mathbb{R}^n f_i \leq 0 \). We set
\[
H(x, p) = \max_{i \leq j \leq n} \{ p_i^2 - f_i(x) p_i \}, \quad x \in \mathbb{R}^n, \quad p = (p_1, \ldots, p_n) \in \mathbb{R}^n.
\]
Clearly, \( H(x, 0) = 0 \) for all \( x \in \mathbb{R}^n \) and \( H \) satisfies (A1)-(A3), (A4) with \( \phi_0 = \psi_0 = 0 \), and (A5)+. Observe here that \( \Phi_0 = \text{BUC}(\mathbb{R}^n) \). We choose any \( u_0 \in \Phi_0 \) satisfying the following property: for any \( \varepsilon > 0 \), there exists an \( l > 0 \) such that

\[
\min_{|y| \leq l} u_0(x + y) < \inf_{\mathbb{R}^n} u_0 + \varepsilon \quad \text{for all } x \in \mathbb{R}^n. \tag{56}
\]

Let \( u(x, t) \) be the solution of the Cauchy problem (1) with \( H \) and \( u_0 \) defined above. We claim here that (53) holds with \( \phi = 0 \), that is,

\[
u(\cdot, t) \longrightarrow \inf_{\mathbb{R}^n} u_0 \quad \text{in } C(\mathbb{R}^n) \quad \text{as } t \to \infty. \tag{57}\]

To prove this, we check (45) in Corollary 4.2. For this purpose, we may assume without loss of generality that \( \inf_{\mathbb{R}^n} u_0 = 0 \). Then, \( u_0 \geq u_0^- \geq 0 \) in \( \mathbb{R}^n \). We also observe from the assumption on \( f_i \), that for any \( \phi \in \mathcal{S}_H \), \( \phi(x) \) is non-decreasing or non-decreasing with respect to the \( k \)-th component of \( x \) for every \( 1 \leq k \leq n \). This and (56) implies that \( u_0^- = 0 \) in \( \mathbb{R}^n \).

Let \( \{y_j\} \subset \mathbb{R}^n \) be any sequence such that \( u_0(\cdot + y_j) \longrightarrow v_0 \) in \( C(\mathbb{R}^n) \) as \( j \to \infty \) for some \( v_0 \in \text{BUC}(\mathbb{R}^n) \). Remark that \( \inf_{\mathbb{R}^n} v_0 = 0 \) and \( v_0 \) inherits property (56). By taking a subsequence of \( \{y_j\} \) if necessary, we may assume that \( f_i(\cdot + y_j) \longrightarrow g_i \) in \( C(\mathbb{R}^n) \) as \( j \to \infty \) for each \( i = 1, \ldots, n \) for some \( g_i \in \text{BUC}(\mathbb{R}^n) \), \( i = 1, \ldots, n \). Then, we have \( \inf_{\mathbb{R}^n} g_i \geq 0 \) or \( \sup_{\mathbb{R}^n} g_i \leq 0 \) according to the sign of \( f_i \) for each \( i = 1, \ldots, n \).

Now, we set \( G(x, p) = \max_{1 \leq i \leq n} \{p_i^2 - g_i(x)p_1\} \), \( x \in \mathbb{R}^n \), \( p = (p_1, \ldots, p_n) \in \mathbb{R}^n \). Then, for any \( \phi \in \mathcal{S}_G \), \( \phi(x) \) is non-increasing or non-decreasing with respect to the \( k \)-th component of \( x \) for every \( 1 \leq k \leq n \). This fact together with property (56) for \( v_0 \) ensure that \( v_0^- = 0 \) in \( \mathbb{R}^n \). Hence, we conclude that (45) is valid.

**Remark 5.5.** The Hamiltonian in Example 5.4 can be generalized in the following way. Let \( H \) satisfy (A1)-(A3), (A5)+ and \( H(x, 0) = 0 \) for all \( x \in \mathbb{R}^n \). We set \( \phi_0 = \psi_0 = 0 \) in (A4) and choose any \( u_0 \in \Phi_0 \) satisfying (56). We set \( K_H(x) = \{p \in \mathbb{R}^n | H(x, p) \leq 0\} \) for \( x \in \mathbb{R}^n \) and denote by \( K_H^*(x) \) the polar cone of \( K_H(x) \), i.e.,

\[
K_H^*(x) := \{\xi \in \mathbb{R}^n | \xi \cdot p \leq 0 \quad \text{for all } p \in K_H(x)\}.
\]

Fix any \( x \in \mathbb{R}^n \), \( \gamma \in E_x \) and any diverging \( \{t_j\} \subset (0, \infty) \) and set \( y_j := \gamma(-t_j) \) for \( j \in \mathbb{N} \). Let \( G \in C(\mathbb{R}^{2n}) \) and \( v_0 \in C(\mathbb{R}^n) \) be the functions satisfying, respectively, \( H(\cdot + y_j, \cdot) \longrightarrow G \) in \( C(\mathbb{R}^{2n}) \) as \( j \to \infty \), and \( u_0(\cdot + y_j) \longrightarrow v_0 \) in \( C(\mathbb{R}^n) \) as \( j \to \infty \). We define \( K_G(x) \) and \( K_G^*(x) \) similarly as \( K_H(x) \) and \( K_H^*(x) \), respectively. Now, we assume the following:

\(\text{(H)} \quad \text{There exists a cone } K \subset \mathbb{R}^n \text{ with vortex 0 such that } \text{Int}(K) \neq \emptyset \text{ and } K \subset K_H^*(x), K_G^*(x) \text{ for all } x \in \mathbb{R}^n.\)

We claim that the convergence (57) still holds under (H). Note that \( H \) in Example 5.4 satisfies property (H).

To check the claim, we first observe that \( d_H(x, y) = 0 \) if \( x - y \in K \). Indeed, for \( \xi \in K \) and \( t > 0 \), there exists a \( q \in L^\infty(0, t; \mathbb{R}^n) \) such that \( q(s) \in (\partial_c d_H(\cdot, \eta))((y + s\xi) \subset K_H(y + s\xi) \) a.e. \( s \in [0, t] \), and

\[
d_H(y + t\xi, y) = \int_0^t q(s) \cdot \xi \, ds \leq 0,
\]
from which we obtain $d_H(y+\xi, y) = 0$ for all $y \in \mathbb{R}^n$ and $\xi \in K$. Similarly, we have $d_G(y+\xi, y) = 0$ for all $y \in \mathbb{R}^n$ and $\xi \in K$.

Now, fix any $x \in \mathbb{R}^n$. Then, in view of (56), for any $\varepsilon > 0$, there exists a sequence $\{z_j\} \subset \mathbb{R}^n$ such that $x - z_j \in K$ and $u_0(z_j) \leq \inf_{\mathbb{R}^n} u_0 + \varepsilon$ for all $j \in \mathbb{N}$. Thus, $u_0^-(x) \leq d_H(x, z_j) + u_0(z_j) \leq \inf_{\mathbb{R}^n} u_0 + \varepsilon$, which implies that $u_0^- = \inf_{\mathbb{R}^n} u_0$ in $\mathbb{R}^n$. Similarly, we see that $v_0^- = \inf_{\mathbb{R}^n} v_0$ in $\mathbb{R}^n$.

We now show that $\inf_{\mathbb{R}^n} u_0 = \inf_{\mathbb{R}^n} v_0$, from which we obviously obtain (45) in Corollary 4.2 and therefore (57). In view of property (56), we can choose a sequence $\{z_j\} \subset \mathbb{R}^n$ such that $u_0(z_j) \leq \inf_{\mathbb{R}^n} u_0 + \varepsilon$ and $|y_j - z_j| \leq l$ for all $j \in \mathbb{N}$. We may assume by taking a subsequence of $\{z_j\}$ that $y_j - z_j \to z$ for some $z \in \mathbb{R}^n$ as $j \to \infty$. Then, $u_0(z_j) = u_0(z_j - y_j + y_j) \to v_0(z)$ as $j \to \infty$, which implies that $\inf_{\mathbb{R}^n} v_0 \leq v_0(z) \leq \inf_{\mathbb{R}^n} u_0 + \varepsilon$. Thus, we have $\inf_{\mathbb{R}^n} v_0 \leq \inf_{\mathbb{R}^n} u_0$. Since the opposite inequality is obvious by definition, we conclude that $\inf_{\mathbb{R}^n} v_0 = \inf_{\mathbb{R}^n} u_0$. Hence, the proof is complete.

参考文献


