<table>
<thead>
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<th>Grobner bases on projective bimodules and the Hochschild cohomology (Languages, Computations, and Algorithms in Algebraic Systems)</th>
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<tbody>
<tr>
<td>Author(s)</td>
<td>KOBAYASHI, YUJI</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2009), 1655: 132-139</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2009-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/140849">http://hdl.handle.net/2433/140849</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Gröbner bases on projective bimodules
and the Hochschild cohomology *

Part IV. (Co)homology

Yuji Kobayashi

Department of Information Science, Toho University
Funabashi 274-8510, Japan

This is a continuation of the previous papers [3], [4] and [5]. We develop the
theory of Gröbner bases on projective modules over an algebra based on a well-ordered
semigroup. We construct resolutions of modules admitting Gröbner bases. This gives
an effective way to compute the (co)homology of such modules.

14 Suitable orders

Let $S = B \cup \{0\}$ be a well-ordered reflexive semigroup with 0 and $K$ be a
commutative ring with 1. Let $F = K \cdot B$ be the $K$-algebra based on $B$ and let
$I$ be a (two-sided) ideal of $F$. Let $A = F/I$ be the quotient algebra of $F$ by $I$
and $\rho : F \rightarrow A$ be the natural surjection. We fix a (reduced) Gröbner basis $G$
of $I$.

Let $X$ be an left edged set and $F \cdot X$ be the projective left $F$-module generated
by $X$. Assume that a left compatible well-order $>$ on $B \cdot X$ is given and it is
extended to a partial order $\triangleright$ on $F \cdot X$ in a natural way. The leading term of
$f \in F \cdot X$ with respect to $\triangleright$ is denoted by $\mathrm{lt}(f)$.

Let $H$ be a set of monic left uniform elements of $F \cdot X$, which is considered
to be a left edged set. Let $F \cdot H$ be the projective left $F$-module generated by $H$.
For $h \in H$, $[h]$ denotes the formal generator of $F \cdot H$ corresponding to $h \in H$.

We define a (strict) partial order $\triangleright'$ on $B \cdot X$ as follows. For $x[h], x'[h'] \in
B \cdot H$, such that $x \cdot \mathrm{lt}(h) \neq 0$ and $x' \cdot (h') \neq 0$, define $x[h] \triangleright' x'[h']$ if and only if

(i) $x \cdot \mathrm{lt}(h) > x' \cdot \mathrm{lt}(h')$, or
(ii) $x \cdot \mathrm{lt}(h) = x' \cdot \mathrm{lt}(h')$ and $x > x'$.

Clearly, this partial order is well founded. Let $L'(H)$ (resp. $L''(H)$) be the
$K$-subspace of $F \cdot H$ spanned by

$\{x[h] \in B \cdot X | x \cdot \mathrm{lt}(h) \neq 0\}$ (resp. $\{x[h] \in B \cdot X | x \cdot \mathrm{lt}(h) = 0\}$).

*This is a preliminary report and the details appear elsewhere.
Easily we see that $L''(H)$ is an $F$-submodule of $F \cdot H$ and

$$F \cdot H = L'(H) \oplus L''(H)$$

holds.

The partial order $>'$ is total on $\{x[h] \in B \cdot X \mid x \cdot \text{lt}(h) \neq 0\}$, and is extended to a partial order $>\prime$ on $L'(H)$ in the same way as we did on $F \cdot X$. The partial order $>\prime$ satisfies the following weak compatibility. For $f, g \in L'(H)$ and $a, b \in B$

(1) $f > g, axw \neq 0 \Rightarrow (af)'> (ag)'$, and
(2) $a > b, axw \neq 0 \Rightarrow (af)' > (bf)'$,

where $(af)', (ag)'$ and $(bf)'$ are the projections of $af, ag$ and $bf$ to $L'(H)$ respectively.

A left compatible well-order $> \text{ on } B \cdot H = \{x[h] \mid x \in B, h \in H\}$ is suitable, if

(i) it extends the partial order $> \text{ on } L'(H)$, and
(ii) $x[h] > \int(xt)$ for any $x \in B$ and $h = w\xi - t \in H$,

where $>$ is the partial order on $F \cdot H$ naturally extended from $>$. So, a well-order $> \text{ on } F \cdot H$ is suitable, if for any $x, x', a, b \in B$ and $h = w\xi - t, h' = w'\xi' - t' \in H$

with $w, w' \in B, \xi, \xi' \in X$ and $t, t' \in F \cdot X$, the following conditions are satisfied:

(iii) $x[h] > x'[h'], yx \neq 0, yx' \neq 0 \Rightarrow yx[h] > yx'[h'],$
(iv) $a > b, ax \neq 0, bx \neq 0 \Rightarrow ax[h] > bx[h].$
(v) $xw \neq 0, x^w' \neq 0, xw > x^w' \text{ or } (xw = x^w', x > x') \Rightarrow x[h] > x'[h'],$

and (ii) above.

Remark that if $xw \neq 0$, the inequality $x[h] > \int(xt)$ in (ii) follows from (iii).

If the base semigroup $S$ is coherent, that is, $xy \neq 0$ for any $x, y \in B$ with $\tau(x) = \sigma(y)$, then $F \cdot H = L'(H)$ and $>'$ is a total order on $B \cdot H$, and hence $>'$ itself is suitable. We do not know the general condition for the existence of a suitable well-order. In the next section we assume that $>'$ is a suitable well-order on $B \cdot H$, and it is extended to a partial order $> \text{ on } F \cdot H$. For a nonzero $f \in LF \cdot H$, $\text{lt}(f)$ denotes the maximal term of $f$ with respect to $>$, and set $rt(f) = f - \text{lt}(f)$.

15 Gröbner basis made from critical pairs and critical z-elements

Lct $h = w\xi - t, h' = w'\xi - t' \in H$ and $x, x' \in B$ such that $xw = x^w' \neq 0$, the appearance $(x, w)$ is at the immediate right of $(x', w')$ in $xw$ and $x$ and $x'$ are left coprime, then we have the critical pair of the first kind and the element

$$c_1 = x[h] - x'[h'] + \int(x \cdot t) - \int(x't')$$

in (11.1) ([5]). Since $(x, w\xi) > (x', w'\xi), x[h] > x'[h']$ by (iii) above. Moreover,

$x[h] > \int(xt)$ and $x'[h'] > \int(x't')$ by (ii) (or (iii)). Thus, $\text{lt}(c_1) = z[h]$.
Let \( u - v \in G \) and \( x, y, y' \in B \) such that \( xw = yuy' \neq 0 \), \( (x, w\xi) \) is rightmost in \( xw\xi \). \((y, u, y')\) is rightmost in \( xw \). and \( x \) and \( y \) are coprime, then we have the cortical pair of the second kind and the element

\[
c_2 = x[h] + \int (x \cdot t) - \int (yuy'\xi)
\]

in (11.2). We have \( x[h] \succ \int(xt) \) and \( x[h] \succ \int(yuy'\xi) \) because \( xw\xi \succ xt \) and \( xw\xi \succ yuy'\xi \). Thus, \( \text{lt}(c_2) = z[h] \).

Let \( z \in B \) be such that \( xw = 0 \), then we have a \( z \)-element \( zt \). This situation is critical, if there is no nonidempotent left factor \( y \) of \( z \); \( z = yz' \) such that \( y'w = 0 \). In this case we call \( zt \) a critical \( z \)-element, and we have the element

\[
c_3 = z[h] + \int(z \cdot t)
\]

in (11.3) made from a critical \( z \)-element. We see \( \text{lt}(c_3) = z[h] \) by (ii).

Let \( C \) be the set of the elements \( c_1, c_2 \) made from critical pairs together with the elements \( c_3 \) made form critical \( z \)-elements.

Let \( \delta : F \cdot H \to F \cdot X \) be the morphism of left \( F \)-modules defined by \( \partial_1([h]) = h \) for \( h \in H \), and let \( \rho : F \cdot X \to A \cdot X \) be the canonical surjection. Let \( \mathcal{K} = \text{Ker}(\delta \circ \rho) \).

**Theorem 15.1.** If \( H \) is a Gröbner basis on \( F \cdot X \) and \( \succ \) is a suitable well-order on \( B \cdot H \), then the set \( C \) is a Gröbner basis on \( F \cdot H \) of the kernel \( \mathcal{K} \) modulo \( G \).

Under the existence of a suitable order we can strengthen Theorem 11.3 in [5] as follows. Remark that the set \( C \) here excludes \( z \)-elements that are not critical.

**Corollary 15.2.** If \( H \) is a Gröbner basis and \( \succ \) is a suitable order on \( F \cdot H \), then \( C \) generates \( \mathcal{K} \) modulo \( G \).

### 16 Projective resolutions

Let \( M \) be a left \( A \)-module defined by a Gröbner basis \( H \) on the projective left \( A \)-module \( A \cdot X \) generated by a left edged set \( X \), that is, \( M \cong F \cdot X / L^\ell(H, G) \), where \( L^\ell(H, G) \) is the submodule of \( F \cdot X \) generated by \( H \) modulo \( G \). We assume that there is a suitable order \( \succ \) on \( B \cdot H \).

Let \( C \) be the Gröbner basis on \( F \cdot H \) made from critical pairs and critical \( z \)-elements in the previous section. Considering \( C \) to be a left edged set, we have the projective left \( A \)-module \( A \cdot C \). Let \( \partial' : A \cdot C \to A \cdot H \) be the morphism of left \( A \)-modules defined by

\[
\partial'([c]) = c
\]

for \( c \in C \). Let \( \eta : A \cdot X \to M \) be the canonical surjection. Since \( H \) generates \( L^\ell(H, G) \) and \( C \) generates the kernel \( \text{Ker}(\rho \circ \delta) \) modulo \( G \), we have
Theorem 16.1. The sequence

\[ A \cdot C \xrightarrow{\partial'} A \cdot H \xrightarrow{\partial} A \cdot X \xrightarrow{\eta} M \to 0 \]

is exact.

Suppose that a suitable well-order can be defined on the projective left \( F \)-module \( F \cdot C \), then we have the Gröbner basis \( D \) on \( F \cdot C \) made from critical pairs and critical \( z \)-elements with respect to \( C \) and \( G \) and a morphism \( \partial'' : A \cdot D \to A \cdot C \) defined by \( \partial''([d]) = d \). If we can repeat this construction (that is, if a suitable well-order exists at every step), then we can construct a projective resolution of \( M \).

Corollary 16.2. Let \( M \) be a left \( A \)-module defined by a Gröbner basis \( X_1 \) on the projective left \( A \)-module \( A \cdot X_0 \) generated by a left edged set \( X_0 \). If at every step above, a suitable well-order exists, we have a projective resolution of \( M \):

\[ \cdots \to A \cdot X_n \xrightarrow{\delta_n} A \cdot X_{n-1} \to \cdots \to A \cdot X_1 \xrightarrow{\delta_1} A \cdot X_0 \xrightarrow{\eta} M \to 0. \]

Suppose that \( F \) has an identity element 1 and \( A \) is supplemented with a morphism \( \epsilon : A \to K \). Let \( X \) be a generating set of nonidemtents of \( B \), then \( \{ a - \epsilon(\rho(a)) \cdot 1 \mid a \in X \} \) forms a Gröbner basis for \( \text{Ker}(\epsilon) \) modulo \( G \). Starting with this Gröbner basis, we can construct a projective resolution of \( K \) and we can compute the (co)homology of the algebra \( A \) (or the semigroup \( S \)).

17 Bimodules and the Hochschild cohomology

The enveloping semigroup \( S^e = (B \times B) \cup \{0\} \) of \( S = B \cup \{0\} \) is a well-ordered reflexive semigroup, in which the product and the order are given as

\[(x, y) \cdot (x', y') = (xx', yy'),\]

and

\[(x, y) \succ (x', y') \iff x \succ x' \text{ or } (x = x' \text{ and } y \succ y')\]

for \( x, y, x', y' \in B \), respectively. The enveloping algebra \( A^e = A \otimes_K A^o \) of \( A = F/I \) is isomorphic to the quotient \( F^e/I^e \), where \( I^e = I \otimes F + F \otimes I \), and the set

\[ G^e = \{ g \otimes 1, 1 \otimes g \mid g \in G \}. \]

is a Gröbner basis of the ideal \( I^e \). An \( F \)-bimodule (resp. \( A \)-bimodule) is naturally a left \( F^e \)-module (resp. left \( A^e \)-module).

Let \( X \) be an edged set and

\[ F \cdot X \cdot F = \bigoplus_{\xi \in X} F\sigma(\xi) \times \tau(\xi)F \]

be the projective \( F \)-bimodule generated by \( X \) and let \( H \) be a set of monic uniform elements of \( F \cdot X \cdot F \). We have three kinds of critical pairs with respect
to $H$ modulo $G$. Let $h = w\xi z - t, h' = w'\xi z' - t' \in H$, $u - v \in G$ and $x, y, x', y' \in B$.

First suppose that $xw = x'w' \neq 0$ and $zy = z'y' \neq 0$, $x$ and $x'$ are left coprime, $y$ and $y'$ are right coprime, and the appearance of $w\xi z$ in the context $(x, y)$ is immediate right of the appearance of $w'\xi z'$ in the context $(x', y')$. Then we have a critical pair $(xty, x't'y')$ of the first kind and the element

$$c_1 = x[h]y - x'[h']y' + \int(xty) + \int(x't'y').$$

of the projective $F$-bimodule $F \cdot H \cdot F$ generated by $H$. Next suppose that $xw = yuy' \neq 0$, $u$ is rightmost in $xw$, $w\xi$ is rightmost in $xw\xi$ and $x$ and $y$ are left coprime. Then, we have a critical pair $(xty, yvy'\xi w')$ of the second kind, and an element

$$c_2 = x[h] - \int(yv\xi y\xi z) + \int(xt)$$

of $F \cdot H \cdot F$. Dually suppose that $zx = y'u' \neq 0$, $u$ is leftmost in $zx$, $\xi z$ is leftmost in $xw\xi$, and $x$ and $y$ are right coprime. Then, we have a critical pair $(txu, w\xi y'vy)$ of the third kind, and an element

$$c_3 = [h][x] - \int(w\xi y'vy) + \int(tx)$$

of $F \cdot H \cdot F$. If $xw = 0$ but $xt \neq 0$ and there is no nonidempotent left factor $y$ of $x$; $x = yuy'$ such that $x'w = 0$, we have a critical $z$-element $xt$ and an element

$$c_4 = x[h] + \int(xt).$$

If $zx = 0$ but $tx \neq 0$ and there is no nonidempotent right factor $y$ of $x$; $x = x'y'$ such that $zx' = 0$, we have a critical $z$-element $tx$ and an element

$$c_5 = [h][x] + \int(tx).$$

Let $C$ be the collection of all elements $c_1, c_2, c_3, c_4$ and $c_5$ above, and let $A \cdot C \cdot A$ be the projective $A$-module generated by $C$.

Let $\delta : F \cdot H \cdot F \to F \cdot X \cdot F$ be the morphisms of left $F$-bimodules defined by $\delta([h]) = h$ for $h \in H$, and let $\rho : F \cdot X \cdot F \to A \cdot X \cdot A$ be the canonical surjection. Let $M$ be the $A$-bimodule defined by $H$ modulo $G$, that is, $M = A \cdot X \cdot A/L(M, G)$, where $L(M, G)$ is the submodule of $A \cdot X \cdot A$ generated by $\rho(M)$. Let $\partial : A \cdot H \cdot A \to A \cdot X \cdot A$ and $\partial' : A \cdot C \cdot A \to A \cdot H \cdot A$ be the morphisms of $A$-bimodules defined by $\partial([h]) = h$ and $\partial'([c]) = c$.

**Theorem 17.1.** If $H$ is a Gröbner basis on $F \cdot X \cdot F$ and $>$ is a suitable well-order on $B \cdot H \cdot B$, then the set $C$ is a Gröbner basis on $F \cdot H \cdot F$ of the kernel of $\rho \circ \delta$ modulo $G$. Moreover we have an exact sequence of $A$-bimodules:

$$A \cdot C \cdot A \xrightarrow{\partial'} A \cdot H \cdot A \xrightarrow{\partial} A \cdot X \cdot A \xrightarrow{\eta} M \to 0$$

**Corollary 17.2.** Let $M$ be an $A$-bimodule defined by a Gröbner basis $X_1$ on the projective left $F$-bimodule $F \cdot X_0 \cdot F$ generated by a left edcaged set $X_0$. If at every step above, a suitable well-order exists, we have a projective $A$-bimodule resolution of $M$:

$$\cdots \to A \cdot X_n \cdot A \xrightarrow{\partial_n} A \cdot X_{n-1} \cdot A \to \cdots \to A \cdot X_1 \cdot A \xrightarrow{\partial_1} A \cdot X_0 \cdot A \xrightarrow{\eta} M \to 0.$$
Let $E$ be the set of all idempotents in $B$, and let $X$ be a generating set of nonidempotents of $B$. Considering them as edged sets we have projective $F$-bimodules $F \cdot E \cdot F, F \cdot X \cdot F$ and $A$-bimodules $A \cdot E \cdot A, A \cdot X \cdot A$ generated by them. We have an augmentation map $\epsilon : F \cdot E \cdot F \to F$ and $\bar{\epsilon} : A \cdot E \cdot A \to A$ defined by $\epsilon([e]) = e$ and $\bar{\epsilon}([e]) = e$ for $e \in E$.

Let $$H = \{ a[\tau(a)] - [\sigma(a)]a \mid a \in X \}.$$ Then, $H$ is a Gröbner basis on $F \cdot E \cdot F$ for Ker$\epsilon$. In this way we have an exact sequence

$$A \cdot X \cdot A \to A \cdot E \cdot A \to M \to 0,$$

where the morphism $\partial$ is defined by $\partial([a]) = a[\tau(a)] - [\sigma(a)]a$ ($a \in X$). Thus, if under the existence of suitable order in every step, we can construct a projective $A$-bimodule resolution of $A$. This gives a way to compute the Hochschild cohomology of the algebra $A$.

## 18 Examples

Since the free monoid $\Sigma^*$ is well-ordered and coherent, its submonoids are well-ordered and coherent. So, the existence of suitable order is guaranteed in every step of construction. In this section we pick up some easy submonoids of $\Sigma^*$ and compute the (co)homology (other examples can be found in [1], [2]).

**Example 18.1.** Let $B$ be the submonoid of $\{a\}^*$ generated by $X = \{a^2, a^3\}$. $B$ is isomorphic to the additive monoid $\mathbb{N} \setminus \{1\}$ of natural numbers excluding 1. Let $F = K \cdot B$ be the algebra based on $B \cup \{0\}$. We have an augmentation map $\epsilon : F \cdot [] \cdot F \to F$ given by $\epsilon([]) = 1$, and a Gröbner basis

$$\{ \alpha_1 = a^2[,] - [,]a^2, \beta_1 = a^3[,] - [,]a^3 \}$$

of Ker$\epsilon$. Let $X = \{\alpha, \beta\}$ and define a morphism $\partial_1 : F \cdot X \cdot F \to F \cdot [] \cdot F$ by $\partial_1([\alpha]) = \alpha_1$, and $\partial_1([\beta]) = \beta_1$.

From the equation $a^3 \cdot a^2 = a^2 \cdot a^3$ we have a critical pair of the first kind $(a^3[,]a^2, a^2[,]a^3)$ and an element

$$\alpha_2 = a^3[\alpha] - [\alpha]a^3 - a^2[\beta] + [\beta]a^2$$

of $F \cdot X \cdot F$. From the equation $(a^2)^2 \cdot a^2 = a^3 \cdot a^3$ we have another critical pair of first kind $(a^4[,]a^2, a^3[,]a^3)$ and an element

$$\beta_2 = a^4[\alpha] + a^2[\alpha]a^2 + [\alpha]a^4 - a^3[\beta] - [\beta]a^3$$

of $F \cdot X \cdot F$. There is no critical pairs of the other kinds because the Gröbner basis $G$ on $F$ is empty. There is no $\varepsilon$-element either because $S$ is coherent. Hence, these two elements form a Gröbner basis of Ker$\partial_1$. We have a morphism $\partial_2 : F \cdot X \cdot F \to F \cdot X \cdot F$ given by $\partial_2([\alpha]) = \alpha_2$ and $\partial_2([\beta]) = \beta_2$. Note that \text{lt}(\alpha_2) = a^3[\alpha]$ and \text{lt}(\beta_2) = a^4[\alpha].
From the equation \(a^3 \cdot a^3 = a^2 \cdot a^4\) we have an element
\[
\alpha_3 = a^3[\alpha] + [\alpha]a^3 - a^2[\beta] + [\beta]a^2,
\]
and from the equation \((a^2)^2 \cdot a^3 = a^3 \cdot a^4\) we have an element
\[
\beta_3 = a^4[\alpha] + a^2[\alpha]a^2 + [\alpha]a^4 - a^3[\beta] + [\beta]a^3.
\]
They form a Gröbner basis of \(\text{Ker}(\partial_2)\). Continuing this calculation we can construct a free bimodule resolution of \(F\):
\[
\rightarrow A \cdot X \cdot A \xrightarrow{\partial_3} A \cdot X \cdot A \rightarrow \cdots \rightarrow A \cdot X \cdot A \xrightarrow{\partial_1} A \cdot [\cdot] \cdot A \xrightarrow{\eta} F,
\]
where \(\partial_n\) is given by
\[
\partial_1([\alpha]) = a^2[\cdot] - [\cdot]a^2, \quad \partial_1([\beta]) = a^3[\cdot] - [\cdot]a^3, \\
\partial_n([\alpha]) = a^3[\alpha] + (-1)^{n-1}[\alpha]a^3 - a^2[\beta] + [\beta]a^2
\]
and
\[
\partial_n([\beta]) = a^4[\alpha] + a^2[\alpha]a^2 + [\alpha]a^4 - a^3[\beta] + (-1)^{n-1}[\beta]a^3
\]
for \(n \geq 2\).

From this resolution we can compute the Hochschild cohomology of \(F\) as follows. Here, \(K\) is a field of characteristic \(p\).

\[
H^0(F) = F,
\]
\[
H^1(F) = \begin{cases} F & \text{ if } p = 2 \text{ or } 3 \\ \oplus_{i \geq 2} K \cdot a^i & \text{ otherwise.} \end{cases}
\]

Let \(n \geq 2\). If \(p = 2\),
\[
H^n(F) = K \oplus K \cdot a^2 \oplus K \cdot a^3 \oplus K \cdot a^5.
\]
If \(p = 3\),
\[
H^n(F) = K \oplus K \cdot a^2 \oplus K \cdot a^4,
\]
and if \(p \neq 2, 3\),
\[
H^n(F) = \begin{cases} K \oplus K \cdot a^2 & \text{if } n \text{ is even} \\ K \cdot (2a^2, 3a^3) \oplus K \cdot (2a^3, 3a^4) & \text{if } n \text{ is odd.} \end{cases}
\]

**Example 18.2.** Let \(B\) be the submonoid of \(\{a, b\}^*\) generated by \(X = \{ab, ba, aba\}\), and let \(S = B \cup \{0\}\) and \(F = K \cdot B\) is the algebra based on \(S\). We have an augmentation \(\epsilon : F \cdot [\cdot] \rightarrow F\) given by \(\epsilon([\cdot]) = 1\). We have a Gröbner basis
\[
\]
of \(\text{Ker}(\epsilon)\) and a differential map
\[
\partial_1 : F \cdot X \cdot F \rightarrow A \cdot [\cdot] \cdot A
\]
with
\[ \partial_1([ab]) = ab[ - ]ab, \quad \partial_1([ba]) = ba[ - ]ba, \]
\[ \partial_1([aba]) = aba[ - ]aba. \]

X is not a code because we have a word equation \((aba)ba = ab(aba)\). From this equation we have a critical pair \((aba[ - ]ba, ab[ - ]aba)\), and we obtain a Gröbner basis of \(\text{Ker}(\partial_1)\):
\[
\{ aba[ba] + [aba]ba - ab[aba] - [ab]aba \}.
\]

In this way we get a free bi-module resolution of \(F\):
\[
0 \rightarrow F \cdot \{ababa\} \cdot F \xrightarrow{\partial_2} F \cdot X \cdot F \xrightarrow{\partial_1} F \cdot [ - ] \cdot F \xrightarrow{\epsilon} F,
\]
where
\[
\partial_2([ababa]) = aba[ba] + [aba]ba - ab[aba] - [ab]aba.
\]

\(F\) is supplemented with \(\epsilon : F \rightarrow K\) defined by \(\epsilon(ab) = \epsilon(ba) = \epsilon(aba) = 0\). Tensoring with the \(F\)-module \(K\) on the right, we have a minimal free left resolution of \(K\):
\[
0 \rightarrow F \cdot \{ababa\} \xrightarrow{\delta_2} F \cdot X \xrightarrow{\delta_1} F \xrightarrow{\epsilon} K,
\]
\[
\delta_1([ab]) = ab, \quad \delta_1([ba]) = ba, \quad \delta_1([aba]) = aba,
\]
\[
\delta_2([ababa]) = aba[ba] - ab[aba].
\]

The Betti number \(b_2 = \dim_K(\text{Tor}_2^F(K, K)) = 1\) seems reflect the ambiguity of \(X\); how distant from codes.

**References**


