

# A new upper bound for the arithmetical rank of monomial ideals

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## 1. INTRODUCTION

Let  $S$  be a polynomial ring over a field  $K$ . Let  $I \subset S$  be a monomial ideal unless otherwise specified, and  $G(I) = \{m_1, \dots, m_\mu\}$  the minimal set of monomial generators of  $I$ . Set  $\mu(I) = \mu$ .

For a monomial ideal  $I \subset S$ , Taylor [15] constructed an explicit graded free resolution of  $S/I$ :

$$T_\bullet : 0 \longrightarrow T_\mu \xrightarrow{d_\mu} T_{\mu-1} \xrightarrow{d_{\mu-1}} \dots \xrightarrow{d_1} T_0 \longrightarrow S/I \longrightarrow 0,$$

where free basis of  $T_s$  are

$$e_{i_1 \dots i_s}, \quad 1 \leq i_1 < \dots < i_s \leq \mu$$

with the degree

$$\deg e_{i_1 \dots i_s} = \deg \text{lcm}(m_{i_1}, \dots, m_{i_s}),$$

and the differential  $d_s$  is given by

$$d_s(e_{i_1 \dots i_s}) = \sum_{j=1}^s (-1)^{j-1} \frac{\text{lcm}(m_{i_1}, \dots, m_{i_s})}{\text{lcm}(m_{i_1}, \dots, \widehat{m_{i_j}}, \dots, m_{i_s})} e_{i_1 \dots \widehat{i_j} \dots i_s}.$$

This resolution is called the *Taylor resolution* of  $I$ . Although the Taylor resolution is explicit, it is far from a minimal graded free resolution of  $I$  in general.

Later, Lyubeznik [10] found a graded free resolution of  $S/I$  as a subcomplex of the Taylor resolution of  $I$ , which is called a *Lyubeznik resolution* of  $I$ . It is generated by all *L-admissible symbols*  $e_{i_1 \dots i_s}$ , where we say a symbol  $e_{i_1 \dots i_s}$  is *L-admissible* if  $m_q$  does not divide  $\text{lcm}(m_{i_t}, m_{i_{t+1}}, \dots, m_{i_s})$  for all  $t < s$  and for all  $q < i_t$ . Note that a Lyubeznik resolution of  $I$  depends on the order of monomial generators of  $I$ , although the Taylor resolution of  $I$  is determined by  $I$  uniquely. The length of a Lyubeznik resolution of  $I$  also depends on the order of monomial generators of  $I$ . We define the *L-length* of  $I$  as the minimum length of Lyubeznik resolutions of  $I$ .

On the other hand, the *arithmetical rank* of  $I$  is defined by

$$\text{ara } I := \min \left\{ r : \text{there exist } a_1, \dots, a_r \in S \text{ such that } \sqrt{(a_1, \dots, a_r)} = \sqrt{I} \right\}.$$

By the definition, we have  $\text{ara } I \leq \mu(I)$ . Note that  $\mu(I)$  is equal to the length of the Taylor resolution of  $I$ . On the other hand, we have the following theorem, which is the main theorem in this report:

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**Theorem 1.1.** *Let  $I \subset S$  be a monomial ideal. Let  $\lambda$  be the  $L$ -length of  $I$ . Then*

$$\text{ara } I \leq \lambda.$$

Moreover, we assume that  $I$  is squarefree. Then by the result of Lyubeznik [9], we have that the arithmetical rank of  $I$  is bounded from below by the projective dimension of  $S/I$ , denoted by  $\text{pd}_S S/I$ . That is,

$$(1.1) \quad \text{height } I \leq \text{pd}_S S/I \leq \text{ara } I \leq \mu(I).$$

Then it is natural to ask when  $\text{ara } I = \text{pd}_S S/I$  holds. If  $I$  is complete intersection (i.e.,  $\mu(I) = \text{height } I$  holds) or  $\mu(I) = \text{pd}_S S/I$  holds, then we have  $\text{ara } I = \text{pd}_S S/I$  immediately by (1.1). Schmitt–Vogel [14] (see also Schenzel–Vogel [13]) proved the equality when  $\text{arithdeg } I = \text{indeg } I$  holds (in this case, the Alexander dual ideal of  $I$  is complete intersection). Barile–Terai [5], Morales [11] proved the equality when  $I$  has a 2-linear resolution. The author proved the equality when  $\mu(I) - \text{height } I \leq 2$  together with Terai and Yoshida; see [7], [8]. On the other hand, we have the following corollary:

**Corollary 1.2.** *Let  $I \subset S$  be a squarefree monomial ideal and  $\lambda$  the  $L$ -length of  $I$ . Suppose  $\lambda = \text{pd}_S S/I$ . Then  $\text{ara } I = \text{pd}_S S/I$  holds.*

*In particular, the Lyubeznik resolution of  $I$  with respect to some order of monomial generators of  $I$  is minimal; then the same assertion holds true.*

Barile [1], [2], [4] and Novik [12] found some classes of squarefree monomial ideals one of whose Lyubeznik resolutions is minimal.

In Section 2, we show the key points of the proof of Theorem 1.1. But we do not state the detailed proof, which can be seen in [6]. In Section 3, we give some examples to explain the limit and the usability of Theorem 1.1.

## 2. OUTLINE OF THE PROOF OF THEOREM 1.1

Let  $I = (m_1, \dots, m_\mu) \subset S$  be a monomial ideal. We may assume that the  $L$ -length of  $I$ , denoted by  $\lambda$ , is equal to the length of the Lyubeznik resolution of  $I$  with respect to this order. We shall find  $a_1, \dots, a_\lambda \in I$  such that  $\sqrt{(a_1, \dots, a_\lambda)} = \sqrt{I}$ . In fact, the following  $\lambda$  elements satisfy this condition:

$$\left\{ \begin{array}{l} a_1 = m_1, \\ a_2 = m_2 + \sum_{\substack{\{i_1, i_2, \dots, i_{\lambda-1}\} \in L_{\lambda-1} \\ i_1 \geq 3}} m_{i_1} m_{i_2} \cdots m_{i_{\lambda-1}}, \\ \vdots \\ a_\ell = m_\ell + \sum_{\substack{\{i_1, i_2, \dots, i_{\lambda-\ell+1}\} \in L_{\lambda-\ell+1} \\ i_1 \geq \ell+1}} m_{i_1} m_{i_2} \cdots m_{i_{\lambda-\ell+1}}, \\ \vdots \\ a_\lambda = m_\lambda + \sum_{\substack{\{i_1\} \in L_1 \\ i_1 \geq \lambda+1}} m_{i_1} = m_\lambda + m_{\lambda+1} + \cdots + m_\mu, \end{array} \right.$$

where

$$L_s := \left\{ [i_1, i_2, \dots, i_s] \in \mathbb{N}^s : \begin{array}{l} 1 \leq i_1 < i_2 < \dots < i_s \leq \mu(I) \\ e_{i_1 i_2 \dots i_s} \text{ is } L\text{-admissible} \end{array} \right\}.$$

The  $L$ -admissibility plays an important role on this taking. First, we give an example to see properties of the  $L$ -admissibility.

**Example 2.1.** Let  $I$  be the squarefree monomial ideal generated by the following 5 elements:

$$m_1 = x_1 x_2 x_4, \quad m_2 = x_1 x_2 x_3, \quad m_3 = x_1 x_5, \quad m_4 = x_2 x_3 x_6, \quad m_5 = x_4 x_6.$$

Then, is  $e_{34}$   $L$ -admissible? This is false because  $\text{lcm}(m_3, m_4) = x_1 x_2 x_3 x_5 x_6$  is divisible by  $m_2$ . Now, is  $e_{124}$   $L$ -admissible? This is true. To see this, we have to check 3 conditions: about  $\text{lcm}(m_4)$ ;  $\text{lcm}(m_2, m_4)$ ;  $\text{lcm}(m_1, m_2, m_4)$ . First,  $\text{lcm}(m_4)$  is not divisible by  $m_1, m_2, m_3$  because these are a part of the minimal system of monomial generators of  $I$ . Second,  $\text{lcm}(m_2, m_4) = x_1 x_2 x_3 x_6$  and it is not divisible by  $m_1 = x_1 x_2 x_4$ . Lastly, we have to check the condition about  $\text{lcm}(m_1, m_2, m_4)$ , but there are nothing to do because there are no generators before  $m_1$ .

The observation in Example 2.1 yields the following lemma:

**Lemma 2.2.** Suppose  $[i_1, \dots, i_s] \in L_s$ .

- (1)  $[i_{j_1}, \dots, i_{j_t}] \in L_t$  for all  $t < s$  and for all  $1 \leq j_1 < \dots < j_t \leq s$ .
- (2) If  $i_1 > 1$ , then  $[1, i_1, \dots, i_s] \in L_{s+1}$ . In particular, if  $[i_1, \dots, i_\lambda] \in L_\lambda$ , then  $i_1 = 1$ .
- (3) Suppose  $\ell < i_1$ . If  $[\ell, i_1, \dots, i_s] \notin L_{s+1}$ , then there exists some integer  $q < \ell$  such that  $m_q$  divides  $m_\ell m_{i_1} \dots m_{i_s}$ .

*Proof.* (1) The conditions for  $e_{i_{j_1} \dots i_{j_t}}$  to be  $L$ -admissible is weaker than those for  $e_{i_1 \dots i_s}$  to be  $L$ -admissible.

(2) This assertion follows from the note at the end of Example 2.1.

(3) The assumptions  $[i_1, \dots, i_s] \in L_s$  and  $[\ell, i_1, \dots, i_s] \notin L_{s+1}$  imply that the condition about  $\text{lcm}(m_\ell, m_{i_1}, \dots, m_{i_s})$  is not satisfied.  $\square$

As this, Lemma 2.2 follows immediately by the definition of the  $L$ -admissibility, but it plays a key role in the proof of Theorem 1.1.

Next, we give an example to explain how to take  $\lambda$  elements.

**Example 2.3.** Let  $I$  be the same ideal as in Example 2.1 with the same order of monomial generators of  $I$ . For this ideal,  $\lambda = \text{pd}_S S/I = 3$ . Sets  $L_1, L_2, L_3$  are given by the following:

$$\begin{aligned} L_1 &= \{[1], [2], [3], [4], [5]\}, \\ L_2 &= \{[1, 2], [1, 3], [1, 4], [1, 5], [2, 3], [2, 4], [3, 5], [4, 5]\}, \\ L_3 &= \{[1, 2, 3], [1, 2, 4], [1, 3, 5], [1, 4, 5]\}. \end{aligned}$$

All elements in  $L_3$  contain 1, which is based on Lemma 2.2 (2). Thus we take  $m_1$ . Next, we focus on  $L_2$ , and ignore elements which contain 1. Then the rests are elements containing 2 and  $[3, 5], [4, 5]$ . Therefore we take  $m_2 +$

$m_3m_5 + m_4m_5$ . Finally in  $L_1$ , we ignore  $[1], [2]$ , and from the rests, we take  $m_3 + m_4 + m_5$ . Then we have

$$\sqrt{I} = \sqrt{(m_1, m_2 + m_3m_5 + m_4m_5, m_3 + m_4 + m_5)}.$$

The key is this arrangement of monomial generators  $m_1, m_2, \dots, m_5$ . It is also true for general monomial ideals. Then we can use the  $L$ -admissibility effective.

### 3. EXAMPLES

First, we give an example of squarefree monomial ideals which satisfy the assumption of Corollary 1.2.

**Example 3.1.** Let  $I \subset S$  be a squarefree monomial ideal. If  $\mu(I) - \text{pd}_S S/I \leq 1$ , then the  $L$ -length of  $I$  is equal to the projective dimension of  $S/I$ . Moreover, if  $\mu(I) - \text{height } I \leq 1$ , then the Lyubeznik resolution of  $I$  with respect to some order of monomial generators of  $I$  is minimal.

For example,

$$I_1 = (x_1x_2x_3, x_4x_5x_6, x_1x_4, x_2x_5, x_3x_6)$$

satisfies  $\mu(I_1) - \text{pd}_S S/I_1 = 1$  and the length of the Lyubeznik resolution of  $I_1$  with respect to this order of monomial generators is equal to  $\text{pd}_S S/I_1 = 4$ . Also,

$$I_2 = (x_1x_2x_3, x_1x_4, x_2x_5, x_3x_6)$$

satisfies  $\mu(I_2) - \text{height } I_2 = 1$  and the Lyubeznik resolution of  $I_2$  with respect to this order of monomial generators is minimal. For more details about these ideals, see [7, Section 2].

The next example implies the limit of Theorem 1.1.

**Example 3.2.** Let  $I = (m_1, m_2, m_3, m_4) \subset S$  be a squarefree monomial ideal. Then  $\mu(I) = 4$ . Suppose that  $\text{height } I = 2$  and  $S/I$  is Cohen–Macaulay. Then  $\text{pd}_S S/I = \text{height } I = 2$  and  $\mu(I) - \text{height } I = 2$ . Thus we have  $\text{ara } I = \text{pd}_S S/I = 2$  by [8, Theorem 4.1, Proposition 4.4].

If there exists a generator, say  $m_1$ , such that  $m_1$  divides  $m_2m_3, m_2m_4, m_3m_4$ , then the Lyubeznik resolution of  $I$  with respect to this order is minimal. Otherwise, the  $L$ -length of  $I$  is equal to 3 and it is bigger than  $\text{pd}_S S/I = \text{ara } I = 2$ .

For example,

$$I_1 = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_5)$$

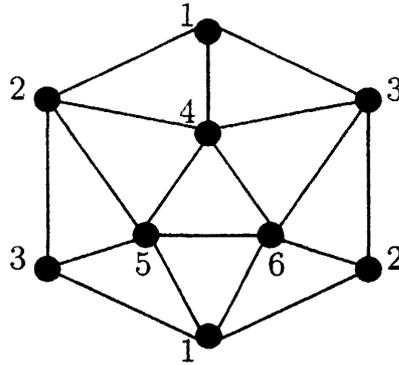
satisfies the former condition and

$$I_2 = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_2x_4x_5)$$

satisfies the latter condition.

Lastly, we give an example which shows the usability of Theorem 1.1.

**Example 3.3.** Let  $I \subset S$  be the Stanley–Reisner ideal of the following triangulation of the projective plane with 6 vertices:



That is,  $I$  is generated by the following 10 elements:

$$x_1x_2x_3, x_1x_2x_5, x_1x_3x_6, x_1x_4x_5, x_1x_4x_6, x_2x_3x_4, x_2x_4x_6, x_2x_5x_6, x_3x_4x_5, x_3x_5x_6.$$

Then  $\mu(I) = 10$ , height  $I = 3$ , and

$$\text{pd}_S S/I = \begin{cases} 3 & \text{when } \text{char } K \neq 2, \\ 4 & \text{when } \text{char } K = 2. \end{cases}$$

Yan [16] proved that  $\text{ara } I = 4$ .

The length of the Taylor resolution of  $I$  is equal to  $\mu(I) = 10$ , which is rather bigger than  $\text{ara } I = 4$ . On the other hand, the  $L$ -length of  $I$  is equal to 4, which is equal to  $\text{ara } I$ , although  $\text{pd}_S S/I = 3 < 4$  when  $\text{char } K \neq 2$ .

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