数理解析研究所講究録
第 1656 巻 2009 年 118-126



# NEW RESULTS ON DADE'S CONJECTURE

SHIH-CHANG HUANG

## 1. INTRODUCTION

Let $G$ be a finite group and $p$ a prime dividing the order of $G$. There are several conjectures connecting the representation theory of $G$ with the representation theory of certain $p$-local subgroups (i.e. the $p$-subgroups and their normalizers) of $G$. For example, it seems to be true, that if $P$ is a Sylow $p$-subgroup of $G$, then the number of complex irreducible characters of $G$ of degree coprime with $p$ equals the same number for the normalizer $N_G(P)$.

This conjecture, called McKay conjecture [58], and its block-theoretic version due to Alperin [1] were generalized by various authors. In [52], Isaacs and Navarro proposed a refinement of the McKay conjecture that deals with congruences of character degrees mod $p$, and in [53], Isaacs, Malle and Navarro suggested a version of the McKay conjecture that includes characters of normal subgroups.

In a series of papers [32], [33], [34], Dade developed several conjectures expressing the number of complex irreducible characters with a fixed defect in a given $p$-block of $G$ in terms of an alternating sum of related values for $p$-blocks of certain $p$-local subgroups of $G$. The ordinary conjecture is the simplest one among others, and the most complicated one is called the inductive form, which implies all the other. If $G$ has a trivial Schur multiplier and a cyclic outer automorphism group, it follows that Dade's inductive conjecture is also true for $G$ in this case. Dade claimed that, if the inductive form is true for all finite simple groups, then it is true for all finite groups. In [33], Dade proved that his (projective) conjecture implies the McKay conjecture. Motivated by the Isaacs-Navarro conjecture [52], Uno [63] suggested a further refinement of Dade's conjecture including the $p'$-parts of character degrees.

This note is organised as follows: In Section 2, we fix notation and state Dade's and Uno's invariant conjectures in detail. In Section 3, we sketch the proof of Dade's and Uno's invariant conjecture for some exceptional groups in the defining characteristic. In Section 4, we sketch the proof of Dade's and Uno's invariant conjecture for some classical groups in the defining characteristic. In Section 5, we present some new results on Dade's conjecture.

## 2. CONJECTURES OF DADE AND UNO

Let $R$ be a $p$-subgroup of a finite group $G$. Then $R$ is *radical* if $O_p(N(R)) = R$, where $O_p(N(R))$ is the largest normal $p$-subgroup of the normalizer $N(R) := N_G(R)$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of $G$, and by $\mathrm{Blk}(G)$ the set of $p$-blocks. If $H \leq G$, $\widetilde{B} \in \mathrm{Blk}(G)$, and $d$ is an integer, we denote by $\mathrm{Irr}(H, \widetilde{B}, d)$ the set of characters $\chi \in \mathrm{Irr}(H)$ satisfying $\mathrm{d}(\chi) = d$ and $b(\chi)^G = \widetilde{B}$ (in the sense of Brauer), where $\mathrm{d}(\chi) = \log_p(|H|_p) - \log_p(\chi(1)_p)$ is the $p$-defect of $\chi$ and $b(\chi)$ is the block of $H$ containing $\chi$.

Given a $p$-subgroup chain

$$C : P_0 < P_1 < \cdots < P_n$$

of $G$, define the length $|C| := n$, $C_k : P_0 < P_1 < \cdots < P_k$ and

$$N(C) = N_G(C) := N_G(P_0) \cap N_G(P_1) \cap \cdots \cap N_G(P_n).$$

The chain $C$ is said to be *radical* if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$ and
(b) $P_k = O_p(N(C_k))$ for $1 \leq k \leq n$.

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical $p$-chains of $G$.

Suppose $1 \to G \to E \to \overline{E} \to 1$ is an exact sequence, so that $E$ is an extension of $G$ by $\overline{E}$. Then $E$ acts on $\mathcal{R}$ by conjugation. Given $C \in \mathcal{R}$ and $\psi \in \mathrm{Irr}(N_G(C))$, let $N_E(C, \psi)$ be the stabilizer of $(C, \psi)$ in $E$, and

$$N_{\overline{E}}(C, \psi) := N_E(C, \psi)/N_G(C).$$

For $\widetilde{B} \in \mathrm{Blk}(G)$, an integer $d \geq 0$ and $U \leq \overline{E}$, we define the set

$$\mathrm{Irr}(N_G(C), \widetilde{B}, d, U) := \{\psi \in \mathrm{Irr}(N_G(C), \widetilde{B}, d) \mid N_{\overline{E}}(C, \psi) = U\}.$$

Dade's invariant conjecture can be stated as follows:

**Dade's Invariant Conjecture** ([34]) *If $O_p(G) = 1$ and $\widetilde{B} \in \mathrm{Blk}(G)$ with defect group $D(\widetilde{B}) \neq 1$, then*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \, |\mathrm{Irr}(N_G(C), \widetilde{B}, d, U)| = 0,$$

*where $\mathcal{R}/G$ is a set of representatives for the $G$-orbits of $\mathcal{R}$.*

Let $H$ be a subgroup of $G$, $\varphi \in \mathrm{Irr}(H)$, and let $r(\varphi) = r_p(\varphi)$ be the integer $0 < r(\varphi) \leq (p-1)$ such that the $p'$-part $(|H|/\varphi(1))_{p'}$ of $|H|/\varphi(1)$ satisfies

$$\left(\frac{|H|}{\varphi(1)}\right)_{p'} \equiv r(\varphi) \bmod p.$$

Given $1 \leq r < (p+1)/2$, let $\mathrm{Irr}(H, [r])$ be the subset of $\mathrm{Irr}(H)$ consisting of those characters $\varphi$ with $r(\varphi) \equiv \pm r \bmod p$. For $\widetilde{B} \in \mathrm{Blk}(G)$, $C \in \mathcal{R}$, an integer $d \geq 0$ and $U \leq \overline{E}$, we define the set

$$\mathrm{Irr}(N_G(C), \widetilde{B}, d, U, [r]) := \mathrm{Irr}(N_G(C), \widetilde{B}, d, U) \cap \mathrm{Irr}(N_G(C), [r]).$$

The following refinement of Dade's conjecture is due to Uno.

**Uno's Invariant Conjecture** ([63], Conjecture 3.2) *If $O_p(G) = 1$ and $\widetilde{B} \in \mathrm{Blk}(G)$ with defect group $D(\widetilde{B}) \neq 1$, then for all integers $d \geq 0$ and $1 \leq r < (p+1)/2$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \, |\mathrm{Irr}(N_G(C), \widetilde{B}, d, U, [r])| = 0.$$

Note that if $p = 2$ or $3$, then Uno's conjecture is equivalent to Dade's conjecture.

## 3. Dade's/Uno's invariant conjecture for some exceptional groups

In this section, we sketch the proof of Dade's/Uno's invariant conjecture for some exceptional groups in the defining characteristic. Let $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ be the automorphism and outer automorphism groups of $G$, respectively. Let $n$ be a positive integer and

$$G \in \{G_2(p^n)\,(p \geq 5),\ {}^3D_4(p^n)\,(p = 2 \text{ or odd}),\ {}^2F_4(2^{2n+1})\}.$$

Then $\mathrm{Out}(G)$ is cyclic and the Schur multiplier of $G$ is trivial. So the invariant conjecture for $G$ is equivalent to the inductive conjecture.

Let $O = \mathrm{Out}(G) = \langle \alpha \rangle$, where $\alpha$ is a field automorphism of order

$$|\alpha| = \begin{cases} n & \text{if } G = G_2(p^n)\,(p \geq 5), \\ 3n & \text{if } G = {}^3D_4(p^n), \\ 2n+1 & \text{if } G = {}^2F_4(2^{2n+1}). \end{cases}$$

We fix a Borel subgroup $B$ and maximal parabolic subgroups $P$ and $Q$ of $G$ containing $B$ as in [17], [42], [41], [44] and [45]. In particular, we may assume that $\alpha$ stabilizes $B$, $P$ and $Q$. We note that the maximal parabolic subgroups $P$, $Q$ are the groups denoted by $P_a$, $P_b$ respectively in [45].

By the remarks on p. 152 in [50], $G$ has only two $p$-blocks, the principal block $B_0$ and one defect-0-block (corresponding to the Steinberg character). Hence we have to verify Dade's/Uno's conjecture only for the principal block $B_0$.

By a corollary of the Borel-Tits theorem [28], the normalizers of radical $p$-subgroups are parabolic subgroups. The radical $p$-chains of $G$ (up to $G$-conjugacy) are given in Table 1.

**Table 1** *Radical p-chains of G.*

| $C$ | | $N_G(C)$ | $N_A(C)$ | Parity |
|---|---|---|---|---|
| $C_1$ | $\{1\}$ | $G$ | $A$ | $+$ |
| $C_2$ | $\{1\} < O_p(P)$ | $P$ | $P \rtimes \langle \alpha \rangle$ | $-$ |
| $C_3$ | $\{1\} < O_p(P) < O_p(B)$ | $B$ | $B \rtimes \langle \alpha \rangle$ | $+$ |
| $C_4$ | $\{1\} < O_p(Q)$ | $Q$ | $Q \rtimes \langle \alpha \rangle$ | $-$ |
| $C_5$ | $\{1\} < O_p(Q) < O_p(B)$ | $B$ | $B \rtimes \langle \alpha \rangle$ | $+$ |
| $C_6$ | $\{1\} < O_p(B)$ | $B$ | $B \rtimes \langle \alpha \rangle$ | $-$ |

Since $C_5$ and $C_6$ have the same normalizers $N_G(C_5) = N_G(C_6)$ and $N_A(C_5) = N_A(C_6)$, it follows that

$$|\mathrm{Irr}(N_G(C_5), B_0, d, u, [r])| = |\mathrm{Irr}(N_G(C_6), B_0, d, u, [r])|$$

for all $d \in \mathbb{N}$, $u \mid |\alpha|$ and $1 \leq r < (p+1)/2$. Thus the contribution of $C_5$ and $C_6$ in the alternating sum of Dade's/Uno's invariant conjecture is zero. So Dade's/Uno's invariant conjecture for $G$ is equivalent to

(1)

$$|\mathrm{Irr}(G, B_0, d, u, [r])| + |\mathrm{Irr}(B, B_0, d, u, [r])| = |\mathrm{Irr}(P, B_0, d, u, [r])| + |\mathrm{Irr}(Q, B_0, d, u, [r])|$$

for all $d \in \mathbb{N}$, $u \mid |\alpha|$ and $1 \leq r < (p+1)/2$.

In order to verify (1), we need to determine the character tables of parabolic subgroups of $G$. Up to conjugacy, $G$ has four parabolic subgroups: $G$, $B$, $P$ and $Q$. Here, we present the results on the character tables of parabolic subgroups of $G$:

| $G$ | $\mathrm{Irr}(G)$ | $\mathrm{Irr}(B)$, $\mathrm{Irr}(P)$, $\mathrm{Irr}(Q)$ |
|---|---|---|
| $G_2(p^n)$ $(p \geq 5)$ | Chang, Ree [30] | An, Huang [17] |
| ${}^3D_4(p^n)$ | Deriziotis, Michler [36] | Himstedt [41], [42] |
| ${}^2F_4(2^{2n+1})$ | Malle [57] | Himstedt, Huang [44], [45] |

For $L \in \{G, B, P, Q\}$, the action of $O = \mathrm{Out}(G)$ on the conjugacy classes of elements of $L$ induces an action of $O$ on the sets of $\mathrm{Irr}(L)$ and then an action on the parameter sets. Using the degrees and character values on the conjugacy classes we can describe the action of $O$ on the parameter sets. Suppose $u \mid |\alpha|$ and set $t := \frac{|\alpha|}{u}$ and $H := \langle \alpha^t \rangle$. Let $\mathrm{Irr}(L, B_0, d, [r]) = \mathrm{Irr}(L, B_0, d) \cap \mathrm{Irr}(L, [r])$. Our main task is to show that

$$\mathrm{Irr}(G, B_0, d, [r]) \cup \mathrm{Irr}(B, B_0, d, [r]) \quad \text{and} \quad \mathrm{Irr}(P, B_0, d, [r]) \cup \mathrm{Irr}(Q, B_0, d, [r])$$

are isomorphic $O$-sets. Our approach is similar to that in [43]: we want to use [51, Lemma (13.23)], so we have to count fixed points of subgroups $H \leq O$. Then (1) is equivalent to

$$|\mathrm{Irr}(G, B_0, d, [r])^{\alpha^t}| + |\mathrm{Irr}(B, B_0, d, [r])^{\alpha^t}| = |\mathrm{Irr}(P, B_0, d, [r])^{\alpha^t}| + |\mathrm{Irr}(Q, B_0, d, [r])^{\alpha^t}|.$$

Then we compute the number of fixed points of $\mathrm{Irr}(L, B_0, d, [r])$ under the action of $H$ and prove that above equation holds.

## 4. Dade's/Uno's invariant conjecture for some classical groups

In this section, we sketch the proof of Dade's/Uno's invariant conjecture for some classical groups in the defining characteristic. Let $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ be the automorphism and outer automorphism groups of $G$, respectively. Let $n$ be a positive integer and

$$G \in \{Sp_4(2^n),\ SU_4(2^{2n}),\ Sp_4(p^n)\ (p \text{ odd})\}.$$

First, we consider $G \in \{Sp_4(2^n), SU_4(2^{2n})\}$. Then $\mathrm{Out}(G)$ is cyclic. So, the proof of Dade's invariant conjecture for $G$ is similar to that for exceptional groups. Next, we consider Uno's invariant conjecture for $G = Sp_4(q)$ in the defining characteristic $p$, where $q = p^n$ with an odd prime $p$.

By [54, Proposition 2.4.4], we have $O = \mathrm{Out}(G) = \langle \phi \rangle \times \langle \alpha \rangle$ and $A = \mathrm{Aut}(G) = G \rtimes O$, where $\phi$ is a diagonal automorphism of order 2 and $\alpha$ is a field automorphism of $G$ with order $n$. We fix a Borel subgroup $B$ and maximal parabolic subgroups $P$ and $Q$ of $G$ containing $B$ as in [65]. In particular, we may assume that $\alpha$ stabilizes $B$, $P$ and $Q$. In the remaining part of this section, we consider the action of $O$ on the irreducible characters of $G$, $B$, $P$ and $Q$.

Let $L \in \{G, B, P, Q\}$.

**(A)** Suppose $U \leq \langle \alpha \rangle$. In this case, we use character tables of $L$ to determine the action of subgroups $U \leq \langle \alpha \rangle$ on the irreducible characters of $L$. Since the group of field automorphism is cyclic, so the proof is similar to that for exceptional groups.

**(B)** Suppose $U = \langle \phi \rangle$. Using the degrees and values of irreducible characters of $L$ on certain $L$-conjugacy classes, we can count fixed points of the action of $\phi$ on the irreducible characters of $L$.

**(C)** Suppose $U \leq O = \langle \alpha \rangle \times \langle \phi \rangle$. By proofs **(A)** and **(B)**, there exists a bijection $\psi$ from

$$\mathrm{Irr}(G, B_0, d, [r]) \bigcup \mathrm{Irr}(B, B_0, d, [r]) \text{ onto } \mathrm{Irr}(P, B_0, d, [r]) \bigcup \mathrm{Irr}(Q, B_0, d, [r])$$

such that

$$\psi(\chi)^x = \psi(\chi^x), \quad \chi \in \mathrm{Irr}(G, B_0, d, [r]) \bigcup \mathrm{Irr}(B, B_0, d, [r])$$

for any $x \in \langle \alpha \rangle$ or $x \in \langle \phi \rangle$.

Let $y \in U$, so that $y = y_1 y_2$ with $y_1 \in \langle \alpha \rangle$ and $y_2 \in \langle \phi \rangle$. Thus $\psi(\chi)^y = \psi(\chi)^{y_1 y_2} = \psi(\chi^{y_1})^{y_2} = \psi(\chi^y)$. This completes the proof of Uno's invariant conjecture for $Sp_4(q)$ in the defining characteristic $p$, where $q = p^n$ with an odd prime $p$.

## 5. Results on Dade's conjecture

So far, Dade's conjecture has been proved for the following cases:

(a) Sporadic simple groups:

| | | |
|---|---|---|
| $M_{11}$, $J_1$ | final | Dade [32] |
| $M_{12}$ | final | Dade |
| $M_{22}$ | final | Huang [47] |
| $M_{23}$, $M_{24}$ | final | Schwartz, An, Conder [13] |
| $J_2$ | final | Dade |
| $J_3$ | final | Kotlica [55] |
| $McL$ | final | Murray [59], Entz, Pahlings [38] |
| $Ru$ | final | Dade, An, O'Brien [18] |
| $He$ | final | An [4] |
| $HS$ | final | Hassan, Horváth [39] |
| $Co_1$ | final | An, O'Brien [23] |
| $Co_2$ | final | An, O'Brien [19] |
| $Co_3$ | final | An [6] |
| $Suz$ | final | Himstedt [40] |
| $O'N$ | final | An, O'Brien [18], Uno, Yoshiara [64] |
| $Th$ | final | Uno [63] |
| $Ly$ | final | Sawabe, Uno [61] |
| $HN$ | final | An, O'Brien [22] |
| $Fi_{23}$ | final | An, O'Brien [20] |
| $Fi_{22}$ | invar. | An, O'Brien [21] |
| $J_4$ | | An, O'Brien, Wilson [24] |
| $B$ | $p$ odd | An, Wilson [25] |
| $Fi'_{24}$ | | An, Cannon, O'Brien, Unger [12] |

(b) Classical groups:

| | | |
|---|---|---|
| $GL_n(q)$ | ord., $p \mid q$ | Olsson, Uno [60] |
| $GU_n(q)$ | ord., $p \mid q$ | Ku [56] |
| $GL_n(q)$, $GU_n(q)$ | invar., $p \nmid q$ | An [9] |
| $Sp_{2n}(q)$, $SO^{\pm}_m(q)$ | ord., $p \nmid q$, $p$, $q$ odd | An [11] |
| $L_2(q)$ | final | Dade [35] |
| $L_3(q)$ | final, $p \mid q$ | Dade |
| $L_n(q)$ | ord., $p \mid q$ | Sukizaki [62] |

(c) Exceptional groups:

| | | |
|---|---|---|
| ${}^2B_2(2^{2n+1})$ | final | Dade [35] |
| ${}^2G_2(3^{2n+1})$ | final | $p \neq 3$ An [2], $p = 3$ Eaton [37] |
| $G_2(q)$ | final, 2, 3 $\mid q$, $p \nmid q$ $q \neq 3, 4$ | An [8], [10] |
| ${}^3D_4(q)$ | final, $p \nmid q$ | An [7] |
| ${}^2F_4(2^{2n+1})$ | ord, $p \neq 2$ | An [5] |
| ${}^2F_4(2)'$ | final | An [3] |

Here, we present new results on Dade's conjecture for exceptional groups and classical groups:

| | | |
|---|---|---|
| $G_2(q)$ | final, $p \mid q$ $(p \geq 5)$, $q = 3, 4$ | Huang [48], [49] |
| ${}^3D_4(q)$ | final, $p \mid q$ ($p = 2$ or odd) | An, Himstedt, Huang [14], [43] |
| ${}^2F_4(2^{2n+1})$ | final, $p = 2$ | Himstedt, Huang [46] |
| $Sp_4(2^n)$, $SU_4(2^{2n})$ | final, $p = 2$ | An, Himstedt, Huang [15] |
| $Sp_4(q)$ | invar., $p \mid q$ ($p$ odd) | An, Himstedt, Huang, Yamada [16] |

## Acknowledgments

The author would like to thank RIMS and the organizer for the opportunity to be here and present this work. Part of this work was done while he visited Chiba University in Japan. He would like to thank Professor Shigeo Koshitani for his support and great hospitality. He also acknowledges the support of a JSPS postdoctoral fellowship from the Japan Society for the Promotion of Science.

## References

[1] J. L. Alperin, The main problem of block theory, in Proceedings of the Conference on Finite Groups, Univ. Utah, Park City, UT, Academic Press, New York, 1975, 341–356.

[2] J. An, Dade's conjecture for the simple Ree groups ${}^2G_2(q^2)$ in non-defining characteristics, *Indian J. Math.*, **36** (1994), 7–27.

[3] J. An, Dade conjecture for the Tits group, *New Zealand J. Math.*, **25** (1996), 107–131.

[4] J. An, The Alperin and Dade conjectures for the simple Held groups, *J. Algebra*, **189** (1997), 34–57.

[5] J. An, The Alperin and Dade conjectures for Ree groups ${}^2F_4(q^2)$ in non-defining characteristics, *J. Algebra*, **203** (1998), 30–49.

[6] J. An, The Alperin and Dade conjectures for the simple Conway's third group, *Israel J. Math.*, **112** (1999), 109–134.

[7] J. AN, Dade's conjecture for Steinberg triality groups ${}^3D_4(q)$ in non-defining characteristics, *Math. Z.*, **241** (2002), 445–469.
[8] J. AN, Dade's invariant conjecture for the Chevalley groups $G_2(q)$ in the defining characteristic, $q = 2^a, 3^a$, *Algebra Colloq.*, **10** (2003), 519–533.
[9] J. AN, Uno's invariant conjecture for the general linear and unitary groups in nondefining characteristics, *J. Algebra*, **284** (2005), 462–479.
[10] J. AN, Uno's invariant conjecture for Chevalley groups $G_2(q)$ in nondefining characteristics, *J. Algebra*, **313** (2007), 429–454.
[11] J. AN, Dade's ordinary conjectures for classical groups in non-defining characteristics, submitted.
[12] J. AN, J. CANNON, E. A. O'BRIEN AND W. R. UNGER, The Alperin weight conjecture and Dade's conjecture for the simple group $Fi_{24}'$, *LMS J. Comput. Math.*, **11** (2008), 100–145.
[13] J. AN AND M. CONDER, The Alperin and Dade conjectures for the simple Mathieu groups, *Comm. Algebra*, **34** (2006), 1763–1792.
[14] J. AN, F. HIMSTEDT AND S. HUANG, Uno's invariant conjecture for Steinberg's triality groups in defining characteristic, *J. Algebra*, 316 (2007), 79–108.
[15] J. AN, F. HIMSTEDT AND S. HUANG, Dade's invariant conjecture for the Symplectic group $Sp_4(2^n)$ and the special unitary group $SU_4(2^{2n})$ in defining characteristic, submitted.
[16] J. AN, F. HIMSTEDT, S. HUANG AND H. YAMADA, Uno's invariant conjecture for the finite Symplectic group $Sp_4(q)$ in the defining characteristic, preprint.
[17] J. AN AND S. C. HUANG, Character tables of parabolic subgroups of the Chevalley groups of type $G_2$, *Comm. Algebra*, **23** (1995), 2797–2823.
[18] J. AN AND E. A. O'BRIEN, The Alperin and Dade conjectures for the O'Nan and Rudivalis simple groups, *Comm. Algebra*, **30** (2002), 1305–1348.
[19] J. AN AND E. A. O'BRIEN, A local strategy to decide the Alperin and Dade conjectures, *J. Algebra*, **206** (1998), 183–207.
[20] J. AN AND E. A. O'BRIEN, The Alperin and Dade conjectures for the simple Fischer group $Fi_{23}$, *Internat. J. Algebra Comput.*, **9** (1999), 621–670.
[21] J. AN AND E. A. O'BRIEN, The Alperin and Uno's conjectures for the Fischer simple group $Fi_{22}$, *Comm. Algebra*, **33** (2005), 1529–1557.
[22] J. AN AND E. A. O'BRIEN, Conjectures on the character degrees of the Harada-Norton simple group HN, *Israel J. Math.*, **137** (2003), 157–181.
[23] J. AN AND E. A. O'BRIEN, The Alperin and Dade conjectures for the Conway simple group $Co_1$, *Algebr. Represent. J. Theory*, **7** (2004), 139–158.
[24] J. AN, E. A. O'BRIEN AND R. A. WILSON, The Alperin weight conjecture and Dade's conjecture for the simple group $J_4$, *LMS J. Comput. Math.*, **6** (2003), 119–140.
[25] J. AN AND R. WILSON, The Alperin weight conjecture and Uno's conjecture for the Baby Monster $B$, $p$ odd, *LMS J. Comput. Math.*, **7** (2004), 120–166.
[26] H. I. BLAU AND G. O. MICHLER, Modular representation theory of finite groups with T.I. Sylow $p$-subgroups, *Trans. Amer. Math. Soc.*, **319** (1990), 417–468.
[27] A. BOREL, ET. AL., Seminar on algebraic groups and related finite groups, Lecture Notes in Math., vol. **131**, Springer, Heidelberg, 1970.
[28] N. BURGOYNE AND C. WILLIAMSON, On a theorem of Borel and Tits for finite Chevalley groups, *Arch. Math. (Basel)*, **27** (1976), 489–491.
[29] R. W. CARTER, Finite groups of Lie Type - conjugacy classes and complex characters, A Wiley-Interscience publication', Chichester, 1985.
[30] B. CHANG AND R. REE, The characters of $G_2(q)$, Symposia Mathematica XIII, Instituto Nazionale di Alta Mathematica, 1974, 395–413.
[31] B. CHAR, K. GEDDES, G. GONNET, B. LEONG, M. MONAGAN AND S. WATT, Maple V, Language Reference Manual, Springer, 1991.
[32] E. C. DADE, Counting characters in blocks I, *Invent. Math.*, **109** (1992), 187–210.
[33] E. C. DADE, Counting characters in blocks II, *J. reine angew. Math.*, **448** (1994), 97–190.
[34] E. C. DADE, Counting characters in blocks 2.9, in R. SOLOMON, ed., Representation Theory of Finite Groups, 1997 pp. 45–59.
[35] E. C. DADE, Counting characters of (ZT)-groups, *J. Group Theory*, **2** (1999), 113–146.
[36] D. I. DERIZIOTIS AND G. O. MICHLER, Character tables and blocks of finite simple triality groups ${}^3D_4(q)$, *Trans. Amer. Math. Soc.*, **303** (1987), 39–70.

[37] C. W. EATON, Dade's inductive conjecture for the Ree groups of type $G_2$ in defining characteristic, *J. Algebra*, **226** (2000), 614–620.

[38] G. ENTZ AND H. PAHLINGS, The Dade conjecture for the McLaughlin group, Groups St. Andrews 1997 in Bath, LMS Lecture Notes Seri. 260, Cambridge Univ. Press, Cambridge, 1999.

[39] N. M. HASSAN AND E. HORVÁTH, Dade's conjecture for the simple Higman-Sims group, Groups St. Andrews 1997 in Bath, I, 329-345, LMS Lecture Notes Seri. 260, Cambridge Univ. Press, Cambridge, 1999.

[40] F. HIMSTEDT, Die Dade-Vermutungen für die sporadische Suzuki-Gruppe, Diploma thesis, RWTH Aachen (1999).

[41] F. HIMSTEDT, Character tables of parabolic subgroups of Steinberg's triality groups, *J. Algebra*, **281** (2004), 774–822.

[42] F. HIMSTEDT, Character tables of parabolic subgroups of Steinberg's triality groups ${}^3D_4(2^n)$, *J. Algebra*, **316** (2007), 254–283.

[43] F. HIMSTEDT AND S. HUANG, Dade's invariant conjecture for Steinberg's triality groups ${}^3D_4(2^n)$ in defining characteristic. *J. Algebra* 316 (2007), 802–827.

[44] F. HIMSTEDT AND S. HUANG, Character table of a Borel subgroup of the Ree groups ${}^2F_4(q^2)$, to appear in LMS J. Comput. Math.

[45] F. HIMSTEDT AND S. HUANG, Character tables of the maximal parabolic subgroups of the Ree groups ${}^2F_4(q^2)$, submitted.

[46] F. HIMSTEDT AND S. HUANG, Dade's invariant conjecture for the Ree groups ${}^2F_4(q^2)$ in defining characteristic, preprint.

[47] J. HUANG, Counting characters in blocks of $M_{22}$, *J. Algebra*, **191** (1997), 1–75.

[48] S. HUANG, Dade's invariant conjecture for the Chevalley groups of type $G_2$ in the defining characteristic, *J. Algebra*, **292** (2005), 110–121.

[49] S. HUANG, Uno's conjecture for the Chevalley simple groups $G_2(3)$ and $G_2(4)$, *New Zealand J. Math.*, **35** (2006), 155–182.

[50] J. HUMPHREYS, Defect groups for finite groups of Lie type, *Math. Z.*, **119** (1971), 149–152.

[51] M. ISAACS, Character Theory of Finite Groups Dover, New York, 1976.

[52] I. M. ISAACS AND G. NAVARRO, New refinements of the McKay conjecture for arbitrary finite groups, *Ann. of Math.*, **156** (2002), 333–344.

[53] I. M. ISAACS, G. MALLE AND G. NAVARRO, A reduction theorem for the McKay conjecture, *Invent. Math.*, **170** (2007), 33–101.

[54] P. KLEIDMAN AND M. LIEBECK, The Subgroup Structure of the Finite Classical Groups, London Mathematical Society Lecture Note Series, 129. Cambridge University Press, Cambridge, 1990.

[55] S. KOTLICA, Verification of Dade's conjecture for Janko group $J_3$, *J. Algebra*, **187** (1997), 579–619.

[56] C. KU, Dade's conjecture for the finite unitary groups in the defining characteristic, PhD thesis, California Institute of Technology, June 1999.

[57] G. MALLE, Die unipotenten Charaktere von ${}^2F_4(q^2)$, *Comm. Algebra*, 18 (1990) 2361–2381.

[58] J. MCKAY, A new invariant for simple groups, *Notices Amer. Math. Soc*, **18** (1971), 397.

[59] J. MURRAY, Dade's conjecture for the McLaughlin simple groups, PhD thesis, University of Illinois at Urbana-Champaign, January 1998.

[60] J. B. OLSSON AND K. UNO, Dade's conjecture for general linear groups in the defining characteristic, *Proc. London Math. Soc.*, **72** (1996), 359–384.

[61] M. SAWABE AND K. UNO, Conjectures on character degrees for the simple Lyons group, *Q. J. Math.*, **54** (2003), 103–121.

[62] H. SUKIZAKI, Dade's conjecture for special linear groups in the defining characteristic. *J. Algebra*, **220** (1999), 261–283.

[63] K. UNO, Conjectures on character degrees for the simple Thompson group, *Osaka J. Math.*, **41** (2004), 11–36.

[64] K. UNO AND S. YOSHIARA, Dade's conjecture for the simple O'Nan group, *J. Algebra*, **249** (2002), 149–185.

[65] H. YAMADA, The characters of $Sp_4(q)$, $q$ odd, *preprint*.

DEPARTMENT OF MATHEMATICS AND INFORMATICS, GRADUATE SCHOOL OF SCIENCE, CHIBA UNIVERSITY, 1-33 YAYOI-CHO, INAGE-KU, CHIBA, 263-8522, JAPAN
*E-mail address*: huang@math.s.chiba-u.ac.jp