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Dihedral groups and subalgebras of Moonshine VOA

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1 Introduction

This article is based on a joint work with G. Höhn and H.Yamauchi and an ongoing research with H.Yamauchi.

The classifiation of finite simple groups asserts that there are exactly 26 simple groups (called sporadic groups) which does not belong to any infinite family. The largest group of such kind is called the Monster and denoted by $\mathbb{M}$. Among the 26 sporadic groups, 20 are involved in the Monster. They are often be realized by the centralizers of some elements.

Sporadic groups realized by the centralizers in $\mathbb{M}$

Classes	Centralizer in $\mathbb{M}$	Sporadic groups
$2A$	$2.\mathbb{B}$	Babymonster
$2B$	$2^{1+24}.Co_1$	Co_1
$3A$	$3.Fi_{24}'$	Fi_{24}'
$3B$	$3^{1+12}.Suz$	Suz
$3C$	$3 \times Th$	Th
$5A$	$5 \times Ha$	Ha
$5B$	$5^{1+6}.HJ$	HJ

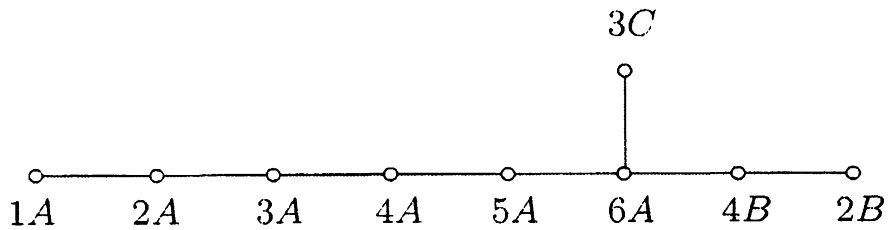
The main purpose of this article is to study some of these groups using certain subalgebras of the Moonshine VOA $V^{\natural}$. To explain our results more precisely, let us review the background of our method and the results established in [LYY1, LYY2, LM].

E_8-observation. John McKay has discovered many mysterious properties about finite groups, especially about the Monster simple group. Among them, there is a observation of McKay which relates the Monster group and some sporadic groups involved in the Monster to certain affine Coxeter-Dynkin diagrams [Mc]. It is known that the Monster group satisfies a 6-transposition property, i.e., given any two 2A-involutions a, b of the Monster simple group $\mathbb{M}$, the product of a and b has order less than or equal to 6. More precisely, the product ab will fall into one of nine conjugacy classes in the Monster [ATLAS, C] as follows:

$$1A,\ 2A,\ 3A,\ 4A,\ 5A,\ 6A,\ 4B,\ 2B,\ \text{or } 3C.$$

Here, the first number denotes the order of the elements in the conjugacy class and the second letter is arranged in descending order of the size of the centralizer of the elements.

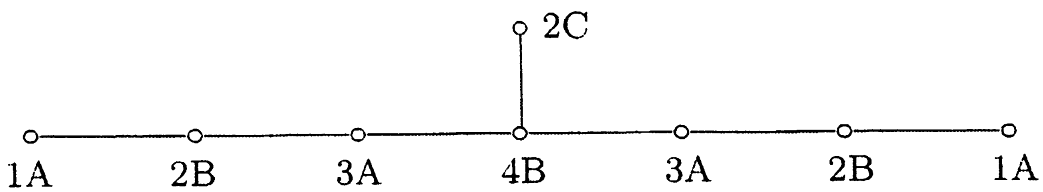
It was pointed out by McKay [Mc] that the orders of the elements in these conjugacy classes coincide with the numerical labels of the nodes in an affine E_8 Dynkin diagram and he observed there is an interesting correspondence between these nine conjugacy classes of $\mathbb{M}$ with the nine nodes of the affine diagram as follows:



(1.1)

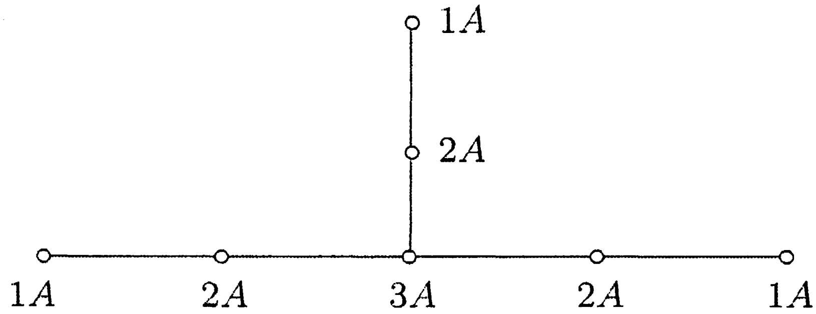
There are similar relations that associate the Babymonster to the E_7-diagram and Fischer's largest 3-transposition group Fi_{24} to the E_6-diagram as follows.

E_7-observation. Let s, t be 2A-involutions of the Babymonster $\mathbb{B}$. It is known that the product st belongs to one of the Babymonster conjugacy classes 1A, 2B, 2C, 3A or 4B and McKay noticed [Mc] that the order of these elements coincide with the numerical labels of the affine E_7 Dynkin diagram and there is a correspondence as below.



In this case, the correspondence is no longer one-to-one but only up to the diagram automorphism.

E_6-**observation.** Similarly, for the Fischer group Fi_{24}, the products of any two 2C-involutions of Fi_{24} belongs to one of the conjugacy classes 1A, 2A or 3A of Fi_{24}. It was again noted by McKay [Mc] that the order of these elements coincide with the numerical labels of the affine E_6 Dynkin diagram and there is a correspondence as follows:



This correspondence is again not one-to-one but only up to diagram automorphisms.

Involutions associated to Virasoro vectors. The main tool for studying the above phenomenon is the one-to-one correspondence between 2A-involutions of the Monster and simple $c = 1/2$ Virasoro vectors in $V^\natural$. Miyamoto showed in [Mi1] that given a Virasoro vector e of a VOA V, an involutive automorphism τ_e of V can be defined based on fusion rules of $\mathrm{Vir}(e)$-modules, where $\mathrm{Vir}(e)$ denotes the Virasoro sub VOA generated by e. If e is a simple $c = 1/2$ Virasoro vector and $V = V^\natural$ is the Moonshine VOA, it is known [C, Mi1, Ma1, Hö2] that there is a one-to-one correspondence between 2A-elements of the Monster and simple $c = 1/2$ Virasoro vectors of the Moonshine VOA via Miyamoto involutions.

By the one-to-one correspondence, properties of 2A-involutions of the Monster can be deduced by analyzing corresponding simple $c = 1/2$ Virasoro vectors of the Moonshine VOA.

In Dong et. al. [DLMN], a special Virasoro vector of central charge 1/2 is defined in the lattice VOA $V_{\sqrt{E_8}}$. It is given by

$$e_{\sqrt{2}E_8} = \frac{1}{16}\omega_{\sqrt{2}E_8} + \frac{1}{32}\sum_{\alpha\in\Phi^+(E_8))}(e^{\sqrt{2}\alpha} + \theta(e^{\sqrt{2}\alpha})),$$

where $\omega_{\sqrt{2}E_8}$ is the Virasoro element of $V_{\sqrt{2}E_8}$ and $\Phi^+(E_8)$ is a set of positvie roots.

Next, we shall explain the main ideas in [LYY1, LYY2, LM].

Dihedral subalgebras associated to the affine E_8 diagram. We shall construct some automorphisms of $V_{\sqrt{2}E_8}$ from the affine E_8 diagram. Let nX be one of the Monster conjugacy classes in

$$\{1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B, 3C\}$$

and let L_{nX} be a sublattice of E_8 obtained by removing the node labeled nX in (1.1). Then the index $[E_8 : L_{nX}] = n$ and we have a coset decomposition

$$E_8 = \bigsqcup_{j=0}^{n-1} (L_{nX} + j\alpha).$$

Correspondingly, we have a decomposition

$$V_{\sqrt{2}E_8} = \bigoplus_{j=0}^{n-1} V_{\sqrt{2}(L_{nX}+j\alpha)}.$$

Define an automorphism ρ_{nX} acting on the component $V_{\sqrt{2}(L_{nX}+j\alpha)}$ by $e^{2\pi\sqrt{-1}j/n}$. Then ρ_{nX} is an automorphism of $V_{\sqrt{2}E_8}$ of order n.

Let e be the special $c = 1/2$ Virasoro vector of $V_{\sqrt{2}E_8}$. Denote by U_{nX} the the subalgebra of $V_{\sqrt{2}E_8}$ generated by e and $f := \rho_{nX}e$. In [LYY1, LYY2, LM], the subalgebra U_{nX} is studied in detail. The main result is a one to one correspond between the subalgebras U_{nX} and dihedral subgroups generated by two 2A-involutions of the Monster via the one-to-one correspondence between the simple $c = 1/2$ Virasoro and 2A-involutions. In other words, there exists a natural embedding of $U_{nX} \hookrightarrow V^\natural$ such that the product $\tau_e\tau_f$ on $V^\natural$ is exactly in the Monster conjugacy class nX.

Sakuma's 6-transposition theorem. On the other hand, Sakuma [S] showed that the 6-transposition property of the Monster can be deduced from the theory of VOAs. He has shown: Let $V = \oplus_{n\geq 0} V_n$ be a VOA over $\mathbb{R}$ with $V_0 = \mathbb{R}\mathbf{1}$ and $V_1 = 0$, and assume that the invariant bilinear form on V is positive definite. Then for any pair e, f of simple $c = 1/2$ Virasoro vectors in V, we always have $|\tau_e\tau_f| \leq 6$ on V. He also determined the possible structures for the Griess subalgebra generated by e and f in the degree two subspace V_2. There are exactly nine possible cases and they agree with the dihedral subalgebras U_{nX} discussed in the previous paragraph. In other words, the dihedral subalgebras U_{nX} exhaust all the possibilities and they are all involved in the Moonshine VOA.

2 Commutant subalgebras and automorphism group

We are mainly interested in the commutant subalgebra of U_{nX} in $V^\natural$ and its automorphism groups.

U_{1A} and Babymonster VOA. Let t be a 2A-involution of the Monster $\mathbb{M}$. Then the centralizer $C_{\mathbb{M}}(t)$ is a double cover $2.\mathbb{B}$ of the Babymonster simple group $\mathbb{B}$. By the one-to-one correspondence, there exists a unique simple $c = 1/2$ Virasoro vector e of the Moonshine VOA $V^\natural$ such that $t = \tau_e$. Denote by $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))$ the commutant subalgebra of $\mathrm{Vir}(e)$ in $V^\natural$. Then the centralizer $C_{\mathbb{M}}(\tau_e)$ naturally acts on it. Since all simple $c = 1/2$ Virasoro vectors of $V^\natural$ are mutually conjugate under the Monster, the VOA structure on $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))$ is independent of $e \in V^\natural$ so that we denote it by $V\mathbb{B}^\natural$ and call it the *Babymonster VOA*. It is proved in [Hö2, Y] that the Babymonster is indeed the full automorphism group of the Babymonster VOA and therefore the Babymonster VOA $V\mathbb{B}^\natural$ is probably the most natural object to be considered in the study of the Babymonster simple group.

W_3-algebra $L(4/5,0) + L(4/5,3)$ and $VF^\natural$. Let g be a $3A$-element of $\mathbb{M}$. Then the normalizer $M_{\mathbb{M}}(g) \cong 3.\mathrm{Fi}_{24}$. On the other hand, Miyamoto showed that if $\mathcal{W} = L(4/5,0) \oplus L(4/5,3)$ is a subalgebra of $V^\natural$, one can recover a 3A-element of the Monster.

Now fixed a subalgebra $\mathcal{W} = cong = L(4/5,0) \oplus L(4/5,3)$ in $V^\natural$ and define

$$VF^\natural := \mathrm{Com}_{V^\natural}(\mathcal{W}) \tag{2.1}$$

Let ρ be the 3A-elemet defined by $\mathcal{W}$. Then we have

Lemma 2.1. $N_{\mathbb{M}}(\rho) \subset \mathrm{Stab}_{\mathbb{M}}(\mathcal{W})$.

In fact, $N_{\mathbb{M}}(\rho)$ acts on $VF^{\natural}$ with the kernel $< \rho >$. We believe that $\mathrm{Aut}(VF^{\natural}) \cong N_M(\rho)/ < \rho >$ but we can only prove the following.

Proposition 2.2 (cf. [HLY]). *Let $\mathfrak{X}$ be the subalgebra of $VF^{\natural}$ generated by the weight 2 subspace. Then* $\mathrm{Aut}(\mathfrak{X}) \simeq N_{\mathbb{M}}(\rho_u)/\langle \rho_u \rangle \simeq \mathrm{Fi}_{24}$.

General cases. In general, there is no correspondence between $5A, 3C, 4B, \cdots$ elements with subVOA. Nevertheless, there is a correspondence bewteen the dihedral algebras U_{nX} and dihedral groups generated by two 2A-involutions.

Now let e, f be $c = 1/2$ Virasoro vectors such that the subVOA generated by e, f is isomorphic to U_{nX}. Denote by $D = \langle \tau_e, \tau_f \rangle$ the dihedral group generated by τ_e, τ_f. The centralizers and normalizers of the dihedral groups as follows.

Table 1: **Centralizers and Normalizers of dihedral subgroups in $\mathbb{M}$**

Conjugacy classes of $\tau_e\tau_f$	$M_{\mathbb{M}}(D)$	$C_{\mathbb{M}}(D)$	Simple groups involved	Outer automorphism
2A	$(2^2.{}^2E_6(2)) : S_3$	$2^2.{}^2E_6(2)$	${}^2E_6(2)$	S_3
3A	$S_3 \times Fi_{23}$	Fi_{23}	Fi_{23}	1
4A	$(2^{1+24}.McL).2$	$2^{1+22}.McL$	McL	2
5A	$(D_{10} \times Ha).2$	Ha	Ha	2
6A	$3.(2 \times 2.Fi_{22}).2$	$2.Fi_{22}$	Fi_{22}	2
4B	$2.(2^2.F_4(2)).2$	$2.F_4(2)$	$F_4(2)$	2
2B	$(2^{1+24}.Co_2)$	$2^{2+22}.Co_2$	Co_2	1
3C	$S_3 \times Th$	Th	Th	1

Let $VC^{\natural}(nX) = \mathrm{Com}_{V^{\natural}}(U(nX))$ be the commutant subalgebra of $U(nX)$

in $V^\natural$. Set $I_{nX} = \{e \in VC^\natural(nX) | \, e$ is a simple $C = 1/2$ Virsoro vector $\}$. Let $\mathfrak{X}_{nX}$ be the vertex operator subalgebra generated by I_{nX}.

Denote by $G(nX)$ the subgroup of $\mathrm{Aut}(VC^\natural(nX))$ generated by $\{\tau_e | \, e \in I_{nX}\}$ and $\tilde{G}(nX)$ the subgroup of $\mathrm{Aut}(V^\natural)$ generated by $\{\tau_e | \, e \in I_{nX}\}$. Then there is a natural group homomorphism

$$\begin{aligned} \phi : \tilde{G}(nX) &\longrightarrow G(nX) \\ g &\longmapsto g|_{\mathrm{Aut}(VC^\natural(nX))} \end{aligned}$$

by restriction. The followings can be proved by a case by case checking.

Theorem 2.3. *The homomorphism ϕ is injective and we have*

$$G(nX) \simeq C_{\mathbb{M}}(D)/Z(D).$$

Theorem 2.4. *Let $\mathfrak{X}_{nX}$ be the subalgebra generated by I. Then we also have*

$$\mathrm{Aut}(\mathfrak{X}_{nX}) \cong N_{\mathbb{M}}(D)/D.$$

Finally we shall end this article with the following few questions.

1. Are the Griess algebras of $VC^\natural(nX)$ and $\mathfrak{X}_{nX}$ equal?

If the Griess algebra of $VC^\natural(nX)$ can be decomposed as 1+ irred as a module of $N_{\mathbb{M}}(D)/D$, then it is trivial. However, it isn't always the case. For example, the Griess algebra of $VC^\natural(5A)$ is a sum of 3 irreducible modules of Ha?

2. Is $VC^\natural(nX)$ generated by weight 2 space?

Some cases (e.g. 1A, 2B,4A) may be easy to prove but probably difficult for 5A and 3A, 3C.

3. Is $\mathrm{Aut}(VC^\natural(nX)) \cong N_M(D)/D$?

If the answers for both 1 and 2 are yes, then the answer is also yes here.

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