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Quantum vertex $\mathbb{C}((t))$-algebras and quantum affine algebras

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Abstract

We give a summary of the theory of (weak) quantum vertex $\mathbb{C}((t))$-algebras and the association of quantum affine algebras with (weak) quantum vertex $\mathbb{C}((t))$-algebras.

1 Introduction

In the earliest days of vertex (operator) algebra theory, Lie algebras had played an important role, and in particular, an important family of vertex operator algebras (see [FLM], [FZ], [DL]) was associated with untwisted affine Lie algebras. A fundamental problem, posed in [FJ] (cf. [EFK]), has been to establish a suitable theory of quantum vertex algebras, so that quantum affine algebras can be canonically associated with quantum vertex algebras in the same (or a similar) way that affine Lie algebras are associated with vertex operator algebras. In literature, there have been a few of notions of quantum vertex (operator) algebra ([eFR], [EK], [B3], [Li3], [AB]), however this particular problem is still to be solved.

In a series of papers, starting with [Li3], we have been investigating vertex algebra-like structures arising from various algebras including quantum affine algebras and Yangians, with an ultimate goal to solve the aforementioned problem. Our key idea is to start with the algebraic structures that the generating functions of those quantum algebras on highest weight modules could possibly "generate." This is the fundamental guideline of this series of studies.

The first paper [Li3] was to provide a foundation for the whole series. Starting with an *arbitrary* vector space W, we studied general (formal) vertex operators (=quantum fields) on W, which are elements of $\mathrm{Hom}(W, W((x)))$. Let $\mathcal{E}(W)$ denote the space $\mathrm{Hom}(W, W((x)))$ alternatively. Then we studied various types of subsets of $\mathcal{E}(W)$ and the algebraic structures generated by such subsets, where the operations of vertex operators on vertex operators are given by what is often called "operator product expansion." The most general type consists of what were called "quasi compatible subsets," whereas compatible subsets are relatively restrictive, but still very general. It was proved therein that any (quasi) compatible subset of $\mathcal{E}(W)$ generates a nonlocal vertex algebra with W as a (quasi) module in a certain sense. (Nonlocal vertex algebras

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are analogs of noncommutative associative algebras, in contrast to that vertex algebras are analogs of commutative and associative algebras.) This generalizes the main result of [Li2], which states that every compatible subset generates a nonlocal vertex algebra with W as a module. It follows from this general result that a wide variety of algebras can be associated with nonlocal vertex algebras. In particular, if W is taken to be a highest weight module for a quantum affine algebra, the generating functions in the Drinfeld realization form a quasi compatible subset of $\mathcal{E}(W)$, and therefore they generate a nonlocal vertex algebra with W as a quasi module.

Furthermore, with the defining relations of quantum affine algebras in mind, we formulated and studied a notion of "pseudo local subset" of $\mathcal{E}(W)$ with W a general vector space, to single out a family of quasi compatible subsets. Now, given a pseudo local subset U, one has a nonlocal vertex algebra $\langle U\rangle$ with W as a quasi module. Under a certain assumption, we proved in [Li3] that there exists a unitary quantum Yang-Baxter operator on $\langle U\rangle$ with two spectral parameters, which describes the braided commutativity relation of the vertex operators from the set $\langle U\rangle$. Roughly speaking, one obtains a deformed chiral algebra structure in the sense of [eFR]. Note that this quantum Yang-Baxter operator is for vertex operators on *the quasi module* W, not for the adjoint vertex operators on *the algebra* $\langle U\rangle$.

Motivated by Etingof-Kazhdan's notion of quantum vertex operator algebra [EK], in particular by the $\mathcal{S}$-locality axiom, we studied in [Li3] a notion of "$\mathcal{S}$-local subset" of $\mathcal{E}(W)$ (with W a vector space), which singles out a family of compatible subsets. It was proved that if U is an $\mathcal{S}$-local subset of $\mathcal{E}(W)$, the adjoint vertex operators on the nonlocal vertex algebra $\langle U\rangle$ satisfies $\mathcal{S}$-locality (commutativity). This lead us to a theory of (weak) quantum vertex algebras and their modules. A conceptual result is that for any $\mathcal{S}$-local subset U of $\mathcal{E}(W)$, $\langle U\rangle$ is a weak quantum vertex algebra with W as a canonical module. As the set of the Drinfeld generating functions of quantum affine algebras is not $\mathcal{S}$-local, this theory of (weak) quantum vertex algebras leaves quantum affine algebras out. Nevertheless, it has been proved to be suitable for studying Yangians. More specifically, in [Li6] we have associated certain versions of double Yangians with quantum vertex algebras. Furthermore, in [Li7], we formulated a notion of $\hbar$-adic (weak) quantum vertex algebra and we associated centrally extended double Yangians with $\hbar$-adic quantum vertex algebras and their modules.

In [Li8], we came back to the problem with quantum affine algebras again. On the basis of [Li3], we developed a theory of (weak) quantum vertex $\mathbb{C}((t))$-algebras and we successfully associated quantum affine algebras with weak quantum vertex $\mathbb{C}((t))$-algebras. In this theory, a weak quantum vertex $\mathbb{C}((t))$-algebra is a $\mathbb{C}((t))$-module and a nonlocal vertex algebra over $\mathbb{C}$, that satisfies a Jacobi-like identity. As a nonlocal vertex algebra, a weak quantum vertex $\mathbb{C}((t))$-algebra satisfies the associativity for ordinary vertex algebras. Furthermore, a quantum vertex $\mathbb{C}((t))$-algebra is a weak quantum vertex $\mathbb{C}((t))$-algebra equipped with a unitary quantum Yang-Baxter operator on V with two formal parameters, which describes the braided commutativity relation of vertex operators $Y(v,x)$ for $v \in V$ and satisfies some other properties. Even though a weak quantum vertex $\mathbb{C}((t))$-algebra V is a $\mathbb{C}((t))$-module, the vertex operator map Y is not $\mathbb{C}((t))$-linear, as by definition

$$Y(f(t)u,x)g(t)v = f(t+x)g(t)Y(u,x)v \quad \text{for } f(t), g(t) \in \mathbb{C}((t)),\ u,v \in V$$

(where linearity is deformed). Thus, the formal variable t in this theory is not a deformation parameter, unlike the formal variable $\hbar$ in Etingof-Kazhdan's theory of quantum vertex operator algebras [EK].

The notion of weak quantum vertex $\mathbb{C}((t))$-algebra naturally arisen from our study on the nonlocal vertex algebras generated by pseudo local subsets of $\mathcal{E}(W)$ with W a vector space over $\mathbb{C}$. (Recall that any quasi compatible subset U of $\mathcal{E}(W)$ generates a nonlocal vertex algebra $\langle U\rangle$ over $\mathbb{C}$.) It has been realized in [Li3] that $\langle U\rangle$ is not large enough to describe the braided commutativity relation and one needs to consider the span $\mathbb{C}((x))\langle U\rangle$, noticing that $\mathcal{E}(W)$ $(= \mathrm{Hom}(W, W((x))))$ is naturally a $\mathbb{C}((x))$-module. It has also been proved therein that $\mathbb{C}((x))\langle U\rangle$ is a nonlocal vertex algebra over $\mathbb{C}$, but it is not a nonlocal vertex algebra over $\mathbb{C}((x))$, as the adjoint vertex operator map is not $\mathbb{C}((x))$-linear. This lead us to a notion of nonlocal vertex $\mathbb{C}((t))$-algebra, where a nonlocal vertex $\mathbb{C}((t))$-algebra V is simply a $\mathbb{C}((t))$-module and a nonlocal vertex algebra over $\mathbb{C}$ whose vertex operator map Y satisfies the deformed $\mathbb{C}((t))$-linear property mentioned previously. In terms of this notion, for any quasi compatible subset U of $\mathcal{E}(W)$, $\mathbb{C}((x))\langle U\rangle$ is a nonlocal vertex $\mathbb{C}((t))$-algebra with $f(t) \in \mathbb{C}((t))$ acting as $f(x)$. Furthermore, we considered what we called "quasi $\mathcal{S}(x_1,x_2)$-local subsets" and "$\mathcal{S}(x_1,x_2)$-local subsets" of $\mathcal{E}(W)$, where quasi $\mathcal{S}(x_1,x_2)$-local subsets are quasi compatible while $\mathcal{S}(x_1,x_2)$-local subsets are compatible. (The notion of quasi $\mathcal{S}(x_1,x_2)$-local subset is a slight reformulation of the notion of pseudo local subset.) Our conceptual result is that for any (quasi) $\mathcal{S}(x_1,x_2)$-local subset U of $\mathcal{E}(W)$, $\mathbb{C}((x))\langle U\rangle$ is a weak quantum vertex $\mathbb{C}((t))$-algebra with $f(t) \in \mathbb{C}((t))$ acting as $f(x)$, and the space W becomes a (quasi) module. Furthermore, to construct a quantum vertex $\mathbb{C}((t))$-algebra from a weak quantum vertex $\mathbb{C}((t))$-algebra, we extended Etingof-Kazhdan's notion of non-degeneracy for nonlocal vertex $\mathbb{C}((t))$-algebras and we proved that just as with quantum vertex algebras [Li5], every non-degenerate weak quantum $\mathbb{C}((t))$-algebra has a (unique) canonical quantum vertex $\mathbb{C}((t))$-algebra structure.

Now, taking W to be a highest weight module for a quantum affine algebra and taking U to be the set of the Drinfeld generating functions, one has a quasi $\mathcal{S}(x_1,x_2)$-local subset U of $\mathcal{E}(W)$, and then one has a weak quantum vertex $\mathbb{C}((t))$-algebra $\mathbb{C}((x))\langle U\rangle$ with W as a canonical quasi module. To a certain extent, this solves the problem mentioned at the very beginning, though we still have to show that this weak quantum vertex $\mathbb{C}((t))$-algebra is a quantum vertex $\mathbb{C}((t))$-algebra, or sufficiently to show that it is non-degenerate.

We mention that there is a very interesting and closely related work [AB], in which Anguelova and Bergvelt studied a class of vertex algebra-like structures, called H_D-quantum vertex algebras. The notion of H_D-quantum vertex algebra generalizes Etingof-Kazhdan's notion of braided vertex operator algebra [EK] in certain directions. In particular, the underlying space of an H_D-quantum vertex algebra is a topologically free $\mathbb{C}[[t]]$-module and the vertex operator map Y is $\mathbb{C}[[t]]$-linear, where the variable t plays the same role as $\hbar$ does in [EK]. One of the generalizations is that the braiding operator $\mathcal{S}$ (for the vertex operators on the algebra) is allowed to have two (independent) spectral parameters, instead of one. A fact is that general H_D-quantum vertex algebras fail to satisfy the associativity for ordinary vertex algebras, though they do satisfy a braided associativity, just as Etingof-Kazhdan's braided vertex operator algebras do. (On the other hand, Etingof-Kazhdan's quantum vertex operator algebras by definition satisfy the associativity.)

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2 Nonlocal vertex algebras and their modules

In this section, we review the basics on nonlocal vertex algebras and their modules and quasi modules, and we give a summary of the conceptual construction of nonlocal vertex algebras and their (quasi) modules.

For this paper, letters such as $t, x, y, z, x_0, x_1, x_2, \dots$ are mutually commuting independent formal variables. We shall use the formal variable notations and conventions as established in [FLM] and [FHL] (cf. [LL]). For this paper we shall be working on the field $\mathbb{C}$ of complex numbers. For any positive integer r, denote by $\mathbb{C}[[x_1, x_2, \dots, x_r]]$ the algebra of formal nonnegative power series and by $\mathbb{C}((x_1, \dots, x_r))$ the algebra of formal Laurent series which are globally truncated with respect to all the variables. Note that in the case $r = 1$, $\mathbb{C}((x))$ is in fact a field. By $\mathbb{C}_*(x_1, x_2, \dots, x_r)$ we denote the extension of $\mathbb{C}[[x_1, x_2, \dots, x_r]]$ by inverting all the nonzero polynomials.

For any permutation $(i_1, i_2, \dots, i_r)$ on $\{1, \dots, r\}$, $\mathbb{C}((x_{i_1}))\cdots((x_{i_r}))$ is a field, containing $\mathbb{C}[[x_1, \dots, x_r]]$ as a subalgebra, so (by a basic fact in classical ring theory), there exists a unique algebra embedding

$$\iota_{x_{i_1},\dots,x_{i_r}} : \mathbb{C}_*(x_1, x_2, \dots, x_r) \to \mathbb{C}((x_{i_1}))\cdots((x_{i_r})), \tag{2.1}$$

extending the identity endomorphism of $\mathbb{C}[[x_1, \dots, x_r]]$ (cf. [FHL]). Note that both $\mathbb{C}_*(x_1, \dots, x_r)$ and $\mathbb{C}((x_{i_1}))\cdots((x_{i_r}))$ contain $\mathbb{C}((x_1, \dots, x_r))$ as a subalgebra. We see that $\iota_{x_{i_1},\dots,x_{i_r}}$ preserves $\mathbb{C}((x_1, \dots, x_r))$ element-wise and that $\iota_{x_{i_1},\dots,x_{i_r}}$ is also $\mathbb{C}((x_1, \dots, x_r))$-linear. For any nonzero polynomial $p \in$

$\mathbb{C}[x_1, \ldots, x_r]$, (as $\iota_{x_{i_1},\ldots,x_{i_r}}$ is an algebra homomorphism) we have

$$\iota_{x_{i_1},\ldots,x_{i_r}}(1/p) = p^{-1}, \tag{2.2}$$

where p^{-1} denotes the inverse of p in $\mathbb{C}((x_{i_1}))\cdots((x_{i_r}))$.

In the general field of vertex algebras, a very basic notion is that of nonlocal vertex algebra, which generalizes the notion of vertex algebra in the way that the notion of associative algebra generalizes that of commutative associative algebra.

Definition 2.1. A *nonlocal vertex algebra* over $\mathbb{C}$ is a vector space V, equipped with a linear map

$$\begin{aligned} Y: \quad & V \to \mathrm{Hom}(V, V((x))) \subset (\mathrm{End}V)[[x, x^{-1}]],\\ & v \mapsto Y(v, x) \end{aligned}$$

and a vector $\mathbf{1} \in V$, satisfying the conditions that $Y(\mathbf{1}, x) = 1$,

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x\to 0} Y(v, x)\mathbf{1} = v \quad \text{for } v \in V,$$

and that for $u, v, w \in V$, there exists a nonnegative integer l such that

$$(x_0 + x_2)^l Y(u, x_0 + x_2)Y(v, x_2)w = (x_0 + x_2)^l Y(Y(u, x_0)v, x_2)w. \tag{2.3}$$

Remark 2.2. The notion of nonlocal vertex algebra, which was defined in [Li3], is exactly the notion of weak axiomatic G_1-vertex algebra in [Li2], and it is essentially the same as the notion of field algebra, studied in [BK] (cf. [K]).

Definition 2.3. Let V be a nonlocal vertex algebra. A *V-module* is a vector space W, equipped with a linear map

$$\begin{aligned} Y_W: \quad & V \to \mathrm{Hom}(W, W((x))) \subset (\mathrm{End}W)[[x, x^{-1}]],\\ & v \mapsto Y_W(v, x), \end{aligned}$$

satisfying the conditions that

$$Y_W(\mathbf{1}, x) = 1_W \quad \text{(the identity operator on } W)$$

and that for $u, v \in V,\ w \in W$, there exists a nonnegative integer l such that

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2)Y_W(v, x_2)w = (x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2)w.$$

A *quasi V-module* is defined by simply replacing the last condition with that for $u, v \in V,\ w \in W$, there exists a nonzero polynomial $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ such that

$$p(x_0 + x_2, x_2)Y_W(u, x_0 + x_2)Y_W(v, x_2)w = p(x_0 + x_2, x_2)Y_W(Y(u, x_0)v, x_2)w.$$

Remark 2.4. The notion of module for a nonlocal vertex algebra was introduced and a conceptual construction of nonlocal vertex algebras and their modules was established in [Li2]. The notion of quasi module for a vertex algebra was first introduced and studied in [Li4], in order to associate vertex algebras to a certain type of Lie algebras. Later, quasi modules for nonlocal vertex algebras were studied in [Li3] and a general construction of nonlocal vertex algebras and their quasi modules was established therein.

Let W be a general vector space over $\mathbb{C}$. Set

$$\mathcal{E}(W) = \mathrm{Hom}(W, W((x))) \subset (\mathrm{End}W)[[x, x^{-1}]]. \tag{2.4}$$

The identity operator on W, denoted by 1_W, is a special element of $\mathcal{E}(W)$.

Definition 2.5. A finite sequence $a_1(x), \dots, a_r(x)$ in $\mathcal{E}(W)$ is said to be *quasi compatible* if there exists a nonzero polynomial $p(x, y) \in \mathbb{C}[x, y]$ such that

$$\left(\prod_{1\le i<j\le r} p(x_i, x_j)\right) a_1(x_1)\cdots a_r(x_r) \in \mathrm{Hom}(W, W((x_1, \dots, x_r))). \tag{2.5}$$

The sequence $a_1(x), \dots, a_r(x)$ is said to be *compatible* if there exists a nonnegative integer k such that

$$\left(\prod_{1\le i<j\le r} (x_i - x_j)^k\right) a_1(x_1)\cdots a_r(x_r) \in \mathrm{Hom}(W, W((x_1, \dots, x_r))). \tag{2.6}$$

Furthermore, a subset T of $\mathcal{E}(W)$ is said to be *quasi compatible* (*compatible*) if every finite sequence in T is quasi compatible (compatible).

Let $(a(x), b(x))$ be a quasi compatible ordered pair in $\mathcal{E}(W)$. That is, there is a nonzero polynomial $p(x, y) \in \mathbb{C}[x, y]$ such that

$$p(x_1, x_2)a(x_1)b(x_2) \in \mathrm{Hom}(W, W((x_1, x_2))). \tag{2.7}$$

We define $Y_{\mathcal{E}}(a(x), x_0)b(x) \in \mathcal{E}(W)((x_0))$ by

$$Y_{\mathcal{E}}(a(x), x_0)b(x) = \iota_{x,x_0}\left(\frac{1}{p(x + x_0, x)}\right)(p(x_1, x)a(x_1)b(x))\,|_{x_1=x+x_0} \tag{2.8}$$

and we then define $a(x)_n b(x) \in \mathcal{E}(W)$ for $n \in \mathbb{Z}$ by

$$Y_{\mathcal{E}}(a(x), x_0)b(x) = \sum_{n\in\mathbb{Z}} a(x)_n b(x) x_0^{-n-1}. \tag{2.9}$$

One can show that this is well defined; the expression on the right-hand side is independent of the choice of $p(x, y)$. In this way we have defined partial operations $(a(x), b(x)) \mapsto a(x)_n b(x)$ for $n \in \mathbb{Z}$ on $\mathcal{E}(W)$. We say that a quasi compatible $\mathbb{C}$-subspace U of $\mathcal{E}(W)$ is *$Y_{\mathcal{E}}$-closed* if

$$a(x)_n b(x) \in U \quad \text{for } a(x), b(x) \in U,\ n \in \mathbb{Z}. \tag{2.10}$$

The main results of [Li3] (cf. [Li2]) can be summarized as follows:

Theorem 2.6. *Let W be a general vector space over $\mathbb{C}$. a) For any $Y_{\mathcal{E}}$-closed (quasi) compatible subspace V of $\mathcal{E}(W)$, that contains 1_W, $(V, Y_{\mathcal{E}}, 1_W)$ carries the structure of a nonlocal vertex algebra with W as a (quasi) module where $Y_W(\alpha(x), x_0) = \alpha(x_0)$ for $\alpha(x) \in V$. b) For every (quasi) compatible subset U of $\mathcal{E}(W)$, there exists a unique smallest $Y_{\mathcal{E}}$-closed (quasi) compatible subspace $\langle U\rangle$ that contains U and 1_W, and $\langle U\rangle$ is a nonlocal vertex algebra with U as a generating subset and with W as a (quasi) module.*

3 Quantum vertex $\mathbb{C}((t))$-algebras and their modules

In this section, we review the basics on nonlocal vertex $\mathbb{C}((t))$-algebras, (weak) quantum vertex $\mathbb{C}((t))$-algebras and their modules, and we summarize the conceptual construction of weak quantum vertex $\mathbb{C}((t))$-algebras and their (quasi) modules.

Definition 3.1. A *nonlocal vertex $\mathbb{C}((t))$-algebra* is a nonlocal vertex algebra V over $\mathbb{C}$, equipped with an $\mathbb{C}((t))$-module structure, such that

$$Y(f(t)u, x)(g(t)v) = f(t+x)g(t)Y(u, x)v \tag{3.1}$$

for $f(t), g(t) \in \mathbb{C}((t))$, $u, v \in V$.

For any field K, we denote by $\mathbf{Vec}(K)$ the category of vector spaces over K. Note that for any vector space W (over $\mathbb{C}$), $\mathrm{Hom}(W, W((x)))$, alternatively denoted as $\mathcal{E}(W)$, is naturally a $\mathbb{C}((x))$-module.

Definition 3.2. Let V be a nonlocal vertex $\mathbb{C}((t))$-algebra. A *V-module in category* $\mathbf{Vec}(\mathbb{C})$ is a module (W, Y_W) for V viewed as a nonlocal vertex algebra over $\mathbb{C}$, satisfying the condition that

$$Y_W(f(t)v, x)w = f(x)Y_W(v, x)w \quad \text{for } f(t) \in \mathbb{C}((t)),\ v \in V,\ w \in W. \tag{3.2}$$

A notion of *quasi V-module in category* $\mathbf{Vec}(\mathbb{C})$ is defined in the obvious way—with the word "module" replaced by "quasi module" in the two places.

The following is a conceptual result which was obtained in [Li8]:

Theorem 3.3. *Let W be a vector space over $\mathbb{C}$ and let U be any (quasi) compatible subset of $\mathcal{E}(W)$. Then $\mathbb{C}((x))\langle U\rangle$ is the smallest $Y_{\mathcal{E}}$-closed (quasi) compatible $\mathbb{C}((x))$-submodule of $\mathcal{E}(W)$, that contains U and 1_W, where $\langle U\rangle$ denotes the smallest $Y_{\mathcal{E}}$-closed $\mathbb{C}$-subspace of $\mathcal{E}(W)$, that contains U and 1_W. Furthermore, $(\mathbb{C}((x))\langle U\rangle, Y_{\mathcal{E}}, 1_W)$ carries the structure of a nonlocal vertex $\mathbb{C}((t))$-algebra, where*

$$f(t)a(x) = f(x)a(x) \quad \textit{for } f(t) \in \mathbb{C}((t)),\ a(x) \in \mathbb{C}((x))\langle U\rangle,$$

and (W, Y_W) *carries the structure of a (quasi)* $\mathbb{C}((x))\langle U\rangle$*-module in category* $\mathbf{Vec}(\mathbb{C})$*, where* $Y_W(a(x), x_0) = a(x_0)$ *for* $a(x) \in \mathbb{C}((x))\langle U\rangle$*.*

The following notion of weak quantum vertex $\mathbb{C}((t))$-algebra singles out an important family of nonlocal vertex $\mathbb{C}((t))$-algebras:

Definition 3.4. A *weak quantum vertex* $\mathbb{C}((t))$*-algebra* is a nonlocal vertex $\mathbb{C}((t))$-algebra V satisfying the condition that for any $u, v \in V$, there exist

$$u^{(i)}, v^{(i)} \in V,\ f_i(x_1, x_2) \in \mathbb{C}_*(x_1, x_2) \quad (i = 1, \dots, r)$$

such that

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2) \\ & \quad -x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) \sum_{i=1}^{r} \iota_{t,x_2,x_1}(f_i(t + x_1, t + x_2))Y(v^{(i)}, x_2)Y(u^{(i)}, x_1) \\ = \ & x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2). \end{aligned} \tag{3.3}$$

A refinement of Theorem 3.3 is that if a quasi compatible subset U of $\mathcal{E}(W)$ is of a certain type, $\mathbb{C}((x))\langle U\rangle$ is a weak quantum vertex $\mathbb{C}((t))$-algebra.

Definition 3.5. Let W be a vector space over $\mathbb{C}$. A subset U of $\mathcal{E}(W)$ is said to be *quasi* $\mathcal{S}(x_1, x_2)$*-local* if for any $a(x), b(x) \in U$, there exist finitely many

$$u^{(i)}(x),\ v^{(i)}(x) \in U,\quad f_i(x_1, x_2) \in \mathbb{C}_*(x_1, x_2) \quad (i = 1, \dots, r)$$

such that

$$p(x_1, x_2)a(x_1)b(x_2) = \sum_{i=1}^{r} p(x_1, x_2)\iota_{x_2,x_1}(f_i(x_1, x_2))u^{(i)}(x_2)v^{(i)}(x_1) \tag{3.4}$$

for some nonzero polynomial $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$, depending on $a(x)$ and $b(x)$. We say that U is $\mathcal{S}(x_1, x_2)$*-local* if for any $a(x), b(x) \in U$, there exist

$$u^{(i)}(x), v^{(i)}(x) \in U,\ f_i(x_1, x_2) \in \mathbb{C}_*(x_1, x_2) \quad (i = 1, \dots, r)$$

such that

$$(x_1 - x_2)^k a(x_1)b(x_2) = (x_1 - x_2)^k \sum_{i=1}^{r} \iota_{x_2,x_1}(f_i(x_1, x_2))u^{(i)}(x_2)v^{(i)}(x_1) \tag{3.5}$$

for some nonnegative integer k.

It was proved in [Li8] that quasi $\mathcal{S}(x_1, x_2)$-local subsets of $\mathcal{E}(W)$ are quasi compatible while $\mathcal{S}(x_1, x_2)$-local subsets are compatible. We have the following fundamental result (see [Li8]):

Theorem 3.6. *Let W be a vector space over $\mathbb{C}$ and let U be any (quasi) $\mathcal{S}(x_1, x_2)$-local subset of $\mathcal{E}(W)$. Denote by $\langle U\rangle$ the nonlocal vertex algebra over $\mathbb{C}$ generated by U. Then $\mathbb{C}((x))\langle U\rangle$ is a weak quantum vertex $\mathbb{C}((t))$-algebra with*

$$f(t)a(x) = f(x)a(x) \quad \text{for } f(t) \in \mathbb{C}((t)),\ a(x) \in \mathbb{C}((x))\langle U\rangle,$$

and (W, Y_W) is a (quasi) module for $\mathbb{C}((x))\langle U\rangle$ in category $\mathbf{Vec}(\mathbb{C})$, *where*

$$Y_W(a(x), x_0) = a(x_0) \quad \text{for } a(x) \in \mathbb{C}((x))\langle U\rangle.$$

For nonlocal vertex $\mathbb{C}((t))$-algebras, there is another category of modules which are like the adjoint modules.

Definition 3.7. Let V be a nonlocal vertex $\mathbb{C}((t))$-algebra. A *(quasi) V-module in category* $\mathbf{Vec}(\mathbb{C}((t)))$ is a $\mathbb{C}((t))$-module W which is also a (quasi) module for V viewed as a nonlocal vertex algebra over $\mathbb{C}$ such that

$$Y_W(f(t)v, x)(g(t)w) = f(t+x)g(t)Y_W(v, x)w \tag{3.6}$$

for $f(t), g(t) \in \mathbb{C}((t))$, $v \in V$, $w \in W$.

Definition 3.8. Let W be a $\mathbb{C}((t))$-module and let t_1 be a formal variable. We define a $\mathbb{C}((t_1))$-module structure on $\mathcal{E}(W)$ by

$$f(t_1)a(x) = f(t+x)a(x) \quad \text{for } f(t_1) \in \mathbb{C}((t_1)),\ a(x) \in \mathcal{E}(W). \tag{3.7}$$

With these notions we have:

Proposition 3.9. *Let W be a $\mathbb{C}((t))$-module and let U be a compatible subset of $\mathcal{E}(W)$. Denote by $\langle U\rangle$ the nonlocal vertex algebra over $\mathbb{C}$ generated by U. Then $\mathbb{C}((t_1))\langle U\rangle$ is a nonlocal vertex $\mathbb{C}((t_1))$-algebra, and W, viewed as a $\mathbb{C}((t_1))$-module with $f(t_1) \in \mathbb{C}((t_1))$ acting as $f(t)$, is a module in category* $\mathbf{Vec}(\mathbb{C}((t_1)))$.

The notion of quantum vertex $\mathbb{C}((t))$-algebra involves quantum Yang-Baxter operators. Let H be a vector space over $\mathbb{C}$. A *quantum Yang-Baxter operator* with two spectral parameters on H is a linear map

$$\mathcal{S}(x_1, x_2) : H \otimes H \to H \otimes H \otimes \mathbb{C}_*(x_1, x_2)$$

satisfying the quantum Yang-Baxter equation

$$\mathcal{S}_{12}(x_1, x_2)\mathcal{S}_{13}(x_1, x_3)\mathcal{S}_{23}(x_2, x_3) = \mathcal{S}_{23}(x_2, x_3)\mathcal{S}_{13}(x_1, x_3)\mathcal{S}_{12}(x_1, x_2), \tag{3.8}$$

where $\mathcal{S}_{ij}(x_i, x_j)$ are linear maps from $H^{\otimes 3} \to H^{\otimes 3} \otimes \mathbb{C}_*(x_i, x_j)$, defined by $\mathcal{S}_{12}(x, z) = \mathcal{S}(x, z) \otimes 1$, $\mathcal{S}_{23}(x, z) = 1 \otimes \mathcal{S}(x, z)$, and

$$\mathcal{S}_{13}(x, z) = P_{23}(\mathcal{S}(x, z) \otimes 1)P_{23}.$$

Furthermore, $\mathcal{S}(x_1, x_2)$ is said to be *unitary* if

$$\mathcal{S}_{21}(x_2, x_1)\mathcal{S}(x_1, x_2) = 1, \tag{3.9}$$

where $\mathcal{S}_{21}(x_2, x_1) = P\mathcal{S}(x_2, x_1)P$ with P the flip operator on $H \otimes H$.

Definition 3.10. A *quantum vertex* $\mathbb{C}((t))$*-algebra* is a weak quantum vertex $\mathbb{C}((t))$-algebra V equipped with a $\mathbb{C}$-linear unitary quantum Yang-Baxter operator $\mathcal{S}(x_1, x_2)$ on V, satisfying the conditions that

$$\mathcal{S}(x_1,x_2)(f(t)u \otimes g(t)v) = f(x_1)g(x_2)\mathcal{S}(x_1,x_2)(u \otimes v) \tag{3.10}$$

for $f(t), g(t) \in \mathbb{C}((t))$, $u, v \in V$, and that for any $u, v \in V$, (3.3) holds with

$$\mathcal{S}(x_1,x_2)(u \otimes v) = \sum_{i=1}^{r} u^{(i)} \otimes v^{(i)} \otimes f_i(x_1,x_2),$$

and that

$$[\mathcal{D} \otimes 1, \mathcal{S}(x_1,x_2)] = -\frac{\partial}{\partial x_1}\mathcal{S}(x_1,x_2), \tag{3.11}$$

$$\mathcal{S}(x_1,x_2)(Y(x) \otimes 1) = (Y(x) \otimes 1)\mathcal{S}_{23}(x_1,x_2)\mathcal{S}_{13}(x_1 + x - t, x_2), \tag{3.12}$$

where $\mathcal{D}$ is the $\mathbb{C}$-linear operator on V, defined by $\mathcal{D}(v) = v_{-2}\mathbf{1}$ for $v \in V$.

In the study of quantum vertex operator algebras, Etingof-Kazhdan [EK] introduced a notion of non-degeneracy, which has played a very important role. This notion is also a very important tool in the study of quantum vertex algebras in [Li3]. The following is a version of non-degeneracy for nonlocal vertex $\mathbb{C}((t))$-algebras (cf. [EK]):

Definition 3.11. Let V be a nonlocal vertex $\mathbb{C}((t))$-algebra. Denote by $V^{\otimes n}$ the tensor product in the category of $\mathbb{C}$-vector spaces and define $V^{\otimes n} \boxtimes \mathbb{C}_*(x_1, \dots, x_n)$ to be the quotient space of $V^{\otimes n} \otimes \mathbb{C}_*(x_1, \dots, x_n)$ by the relations

$$f_1(t)v^{(1)} \otimes \cdots \otimes f_n(t)v^{(n)} \otimes f = v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes f_1(x_1) \cdots f_n(x_n)f$$

for $f \in \mathbb{C}_*(x_1, \dots, x_n)$, $f_i(t) \in \mathbb{C}((t))$, $v^{(i)} \in V$ $(i = 1, \dots, n)$. For each positive integer n, define a $\mathbb{C}$-linear map

$$Z_n : V^{\otimes n} \boxtimes \mathbb{C}_*(x_1, \dots, x_n) \to V((x_1)) \cdots ((x_n))$$

by

$$Z_n(v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes f) = \iota_{t,x_1,\dots,x_n} f(t + x_1, \dots, t + x_n)Y(v^{(1)}, x_1) \cdots Y(v^{(n)}, x_n)\mathbf{1}.$$

We say that V is *non-degenerate* if for every positive integer n, Z_n is injective.

With this notion we have (see [Li8], cf. [EK], Proposition 1.11):

Theorem 3.12. *Let V be a weak quantum vertex $\mathbb{C}((t))$-algebra. Assume that V is non-degenerate. Then there exists a $\mathbb{C}$-linear map*

$$\mathcal{S}(x_1,x_2) : V \otimes V \to V \otimes V \otimes \mathbb{C}_*(x_1,x_2),$$

which is uniquely determined by the condition that for $u, v, w \in V$,

$$\begin{aligned}&(x_1 - x_2)^k Y(v, x_2)Y(u, x_1)w \\ = \;& (x_1 - x_2)^k Y(x_1)(1 \otimes Y(x_2))(\mathcal{S}(x_1 + t, x_2 + t)(u \otimes v) \otimes w) \quad (3.13)\end{aligned}$$

for some nonnegative integer k, *depending only on* u *and* v. *Furthermore,* $\mathcal{S}(x_1, x_2)$ *is a unitary quantum Yang-Baxter operator on* V, *and* V *equipped with* $\mathcal{S}(x_1, x_2)$ *is a quantum vertex* $\mathbb{C}((t))$*-algebra.*

The following is a general result on non-degeneracy (see [Li8], cf. [Li5]):

Proposition 3.13. *Let* V *be a nonlocal vertex* $\mathbb{C}((t))$*-algebra such that* V *as a* V*-module is irreducible with* $\mathrm{End}_V(V^{\mathrm{mod}}) = \mathbb{C}((t))$. *Then* V *is non-degenerate.*

4 Quantum affine algebras and weak quantum vertex $\mathbb{C}((t))$-algebras

In this section we give a summary of the association of quantum affine algebras with weak quantum vertex $\mathbb{C}((t))$-algebras and their quasi modules.

First, we follow [FJ] (cf. [Dr]) to present the quantum affine algebras. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank l of type A, D, or E and let $A = (a_{ij})$ be the Cartan matrix. Let q be a nonzero complex number. For $1 \leq i, j \leq l$, set

$$f_{ij}(x) = (q^{a_{ij}}x - 1)/(x - q^{a_{ij}}) \in \mathbb{C}(x). \tag{4.1}$$

Then we set

$$g_{ij}(x)^{\pm 1} = \iota_{x,0} f_{ij}(x)^{\pm 1} \in \mathbb{C}[[x]], \tag{4.2}$$

where $\iota_{x,0} f_{ij}(x)^{\pm 1}$ are the formal Taylor series expansions of $f_{ij}(x)^{\pm 1}$ at 0. Let $\mathbb{Z}_+$ denote the set of positive integers. The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ is (isomorphic to) the associative algebra with identity 1 with generators

$$X_{ik}^{\pm}, \quad \phi_{im}, \quad \psi_{in}, \quad \gamma^{1/2}, \quad \gamma^{-1/2} \tag{4.3}$$

for $1 \leq i \leq l$, $k \in \mathbb{Z}$, $m \in -\mathbb{Z}_+$, $n \in \mathbb{Z}_+$, where $\gamma^{\pm 1/2}$ are central, satisfying the relations below, written in terms of the following generating functions in a formal variable z:

$$X_i^{\pm}(z) = \sum_{k \in \mathbb{Z}} X_{ik}^{\pm} z^{-k}, \quad \phi_i(z) = \sum_{m \in -\mathbb{Z}_+} \phi_{im} z^{-m}, \quad \psi_i(z) = \sum_{n \in \mathbb{Z}_+} \psi_{in} z^{-n}. \tag{4.4}$$

The relations are

$$\begin{aligned}
&\gamma^{1/2}\gamma^{-1/2} = \gamma^{-1/2}\gamma^{1/2} = 1,\\
&\phi_{i0}\psi_{i0} = \psi_{i0}\phi_{i0} = 1,\\
&[\phi_i(z), \phi_j(w)] = 0, \quad [\psi_i(z), \psi_j(w)] = 0,\\
&\phi_i(z)\psi_j(w)\phi_i(z)^{-1}\psi_j(w)^{-1} = g_{ij}(z/w\gamma)/g_{ij}(z\gamma/w),\\
&\phi_i(z)X_j^{\pm}(w)\phi_i(z)^{-1} = g_{ij}(z/w\gamma^{\pm 1/2})^{\pm 1}X_j^{\pm}(w),\\
&\psi_i(z)X_j^{\pm}(w)\psi_i(z)^{-1} = g_{ij}(w/z\gamma^{\pm 1/2})^{\mp 1}X_j^{\pm}(w),\\
&(z - q^{\pm 4a_{ij}}w)X_i^{\pm}(z)X_j^{\pm}(w) = (q^{\pm 4a_{ij}}z - w)X_j^{\pm}(w)X_i^{\pm}(z),\\
&[X_i^+(z), X_j^-(w)] = \frac{\delta_{ij}}{q-q^{-1}}\left(\delta\left(\frac{z}{w\gamma}\right)\psi_i(w\gamma^{1/2}) - \delta\left(\frac{z\gamma}{w}\right)\phi_i(z\gamma^{1/2})\right),
\end{aligned}$$

and there is one more set of relations of Serre type.

A $U_q(\hat{\mathfrak{g}})$-module W is said to be *restricted* if for any $w \in W$, $X_{ik}^{\pm}w = 0$ and $\psi_{ik}w = 0$ for $1 \le i \le l$ and for k sufficiently large. We say W is of *level* $\ell \in \mathbb{C}$ if $\gamma^{\pm 1/2}$ act on W as scalars $q^{\pm \ell/4}$. (Rigorously speaking, one needs to choose a branch of $\log q$.) We have (cf. [Li3], Proposition 4.9):

Proposition 4.1. *Let q and ℓ be complex numbers with $q \neq 0$ and let W be a restricted $U_q(\hat{\mathfrak{g}})$-module of level ℓ. Set*

$$U_W = \{\phi_i(x), \psi_i(x), X_i^{\pm}(x) \mid 1 \le i \le l\}.$$

Then U_W is a quasi $S(x_1, x_2)$-local subset of $\mathcal{E}(W)$ and $\mathbb{C}((x))\langle U_W\rangle$ is a weak quantum vertex $\mathbb{C}((t))$-algebra with W as a quasi module in category **Vec**$(\mathbb{C})$, *where $\langle U_W\rangle$ denotes the nonlocal vertex algebra over $\mathbb{C}$ generated by U_W.*

With Proposition 4.1 on hand, the remaining problem is to determine the weak quantum vertex $\mathbb{C}((t))$-algebras V_W explicitly and to show that they are quantum vertex $\mathbb{C}((t))$-algebras, sufficiently by establishing the non-degeneracy. We expect that just as with vertex algebras associated with affine Lie algebras (cf. [LL]), these weak quantum vertex $\mathbb{C}((t))$-algebras are vacuum modules for certain associative algebras derived from quantum affine algebras.

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