On the Woronowicz's twisted product construction of quantum groups, with comments on related cubic Hecke algebra (Non-Commutative Analysis and Micro-Macro Duality)

Author(s): WYSOCZANSKI, JANUSZ

Citation: 数理解析研究所講究録 (2009), 1658: 124-135

Issue Date: 2009-07

URL: http://hdl.handle.net/2433/140904

Type: Departmental Bulletin Paper

Textversion: publisher

Kyoto University
On the Woronowicz's twisted product construction
of quantum groups,
with comments on related cubic Hecke algebra. *

JANUSZ WYSOCZAŃSKI †
Institute of Mathematics, Wroclaw University
pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland

Abstract

We study the construction of compact quantum groups, based on the method invented
by Woronowicz [SLW3], which uses a twisted determinant. As an example Woronowicz
considered the function $S_N \ni \sigma \mapsto \text{inv}(\sigma)$, where \text{inv}(\sigma) is the number of inversions in the
permutation $\sigma$. Our twisted determinant is related to the function $S_N \ni \sigma \mapsto c(\sigma)$, where
c(\sigma) is the number of cycles in a permutation $\sigma$. For $N = 3$ it gave the quantum group
$U_q(2)$. Here we show how the construction works if $N = 4$. We also describe the cubic
Hecke algebra, associated with the quantum group $U_q(2)$.

1 Introduction

In [SLW3] Woronowicz provided a general method for constructing compact matrix quantum
groups. The method depends on finding an $N^N$-element array $E = (E_{i_1 \ldots i_N})_{i_1 \ldots i_N=1}^N$ of complex
numbers, called twisted determinant, which is (left and right) non-degenerate. Theorem 1.4 of
[SLW3] says that if a $C^*$-algebra $\mathcal{A}$, is generated by $N^2$ elements $u_{jk}$ which satisfy the unitarity
condition:

$$ \sum_{r=1}^{N} u_{jr}^* u_{rk} = \delta_{jk} I = \sum_{r=1}^{N} u_{jr} u_{rk}^* $$

and the following twisted determinant condition:

$$ \sum_{k_1, \ldots, k_N=1}^{N} u_{j_1 k_1} \ldots u_{j_N k_N} E_{k_1, \ldots, k_N} = E_{j_1, \ldots, j_N} I $$

*Research partially supported by the European Commission Marie Curie Host Fellowship for the Transfer
of Knowledge "Harmonic Analysis, Nonlinear Analysis and Probability" MTKD-CT-2004-013389, the Polish
Ministry of Science’s research grants 1P03A01330 and N N201 270735
†e-mail: jwys@math.uni.wroc.pl
and if the array $E$ is non-degenerate, then $(\mathcal{A}, u)$ is a compact matrix quantum group, where $u = (u_{jk})_{j,k=1}^{N}$. Woronowicz described the following example. For $\mu \in (0, 1]$, he defined

$$E_{i_1, \ldots, i_N} = (-\mu)^{\text{inv}(\sigma)} \quad \text{if} \quad \sigma = \begin{pmatrix} 1 & 2 & \cdots & N \\ i_1 & i_2 & \cdots & i_N \end{pmatrix} \in S_N$$

is a permutation ($S_N$ denotes the set of permutations of $\{1, 2, \ldots, N\}$) and $E_{i_1, \ldots, i_N} = 0$ otherwise. Here, for a permutation $\sigma \in S_N$, $\text{inv}(\sigma)$ is the number of inversions of $\sigma$, which is the number of pairs $(j, k)$ such that $j < k$ and $i_j = \sigma(j) > \sigma(k) = i_k$. Then as $(\mathcal{A}, u)$ one gets the quantum group $S_\mu U(N)$, called the twisted $SU(N)$ group.

In [W3] we considered another array $E$ for $N = 3$, related to the number of cycles in a permutation. It was defined for a parameter $0 < q < 1$ as follows:

$$E(i, j, k) = \begin{cases} (-q)^{3-c(i,j,k)} & \text{if } \{i, j, k\} = \{1, 2, 3\} \\ 0 & \text{otherwise} \end{cases}$$

Here $c(i, j, k)$ is the number of cycles of the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$$

(which makes sense if and only if $\{i, j, k\} = \{1, 2, 3\}$). Then, following the Woronowicz's scheme, we obtained a quantum group, which turned out to be $U_q(2)$, the quantum deformation of the unitary $2 \times 2$ group. Moreover, the construction provided a description of it as a twisted product of its quantum subgroups

$$U_q(2) = SU_q(2) \ltimes U(1)$$

with the $*$-isomorphism $\sigma : \mathcal{A}_1 \otimes \mathcal{A}_2 \rightarrow \mathcal{A}_2 \otimes \mathcal{A}_1$ given by

$$\sigma(1 \otimes v) = v \otimes 1, \quad \sigma(a \otimes v^k) = v^k \otimes a, \quad \sigma(c \otimes v^k) = v^{k-1} \otimes c.$$ 

The natural continuation of the construction given in [W3], was investigating the cases $N \geq 4$. However, as shall see below, after some tiresome computations it turned out that for $N = 4$ (and thus also for all $N \geq 4$) the quantum group we obtain (via the Woronowicz's theorem) is classical abelian.

Regarding the quantum group $U_q(2)$, we shall present also a construction of a cubic Hecke algebra. In [SLW3] Woronowicz showed that there are Hecke algebras associated with the quantum groups $SU_q(N)$, for every $N \in \mathbb{N}$, $N \geq 2$. The Hecke algebra $H_{q,n}$ described the intertwining operators for the $n^{th}$ tensor power of the fundamental representation of the group. In this note we shall show similar construction for $U_q(2)$. The construction depends on defining an operator $\alpha : \mathbb{C}^3 \otimes \mathbb{C}^3 \rightarrow \mathbb{C}^3 \otimes \mathbb{C}^3$, which satisfies the Yang-Baxter equation (3.1). The operator is not self-adjoint (contrary to the $SU_q(N)$ cases), although its square is so ($\alpha^2 = (\alpha^*)^2$). Nevertheless, it satisfies a generalization of the Hecke equation, namely $(\alpha^2 - 1)(\alpha + q^2 I) = 0$ (see (4.1)). Therefore the operators $h_j := I_j \otimes \alpha \otimes I_{n-j-2}$, defined for $j = 1, \ldots, n-2$, generate a cubic Hecke algebra (Theorem 4.3).

The paper is organized as follows. In Section 2 we give the computation showing the generalization of our $U_q(2)$ construction, for $N = 4$. Then, in Section 3, we give the construction of the operator $\alpha$, and show that it satisfies the Yang-Baxter equation. The last Section 4, contains the construction of the cubic Hecke algebra, associated with $U_q(2)$. In particular, we show there that $\alpha$ satisfies the cubic equation.
2 The construction associated with $E$

Let $N_4 = \{(i, j, k, l) : \{i, j, k, l\} \subset \{1, 2, 3, 4\}\}$, let $E : N_4 \rightarrow \mathbb{C}$ be zero outside $S_4 \subset N_1$, where the inclusion is given by $(i, j, k, l) \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & l \end{pmatrix}$ if $\{i, j, k, l\} = \{1, 2, 3, 4\}$, and, for $0 < q < 1$, let the (non-zero) values of $E$ (with the notation $E((i, j, k, l)) = E_{ijkl}$) be given by the function

$$S_4 \ni \sigma \mapsto (-q)^{4 - c(\sigma)}.$$

Explicitely, it can be written in the following way:

$$
\begin{align*}
E_{1234} &= 1 & E_{1243} &= -q & E_{1324} &= -q & E_{1342} &= q^2 & E_{1423} &= q^2 & E_{1432} &= -q \\
E_{2134} &= -q & E_{2143} &= q^2 & E_{2314} &= q^2 & E_{2341} &= -q^3 & E_{2413} &= -q^3 & E_{2431} &= q^2 \\
E_{3124} &= q^2 & E_{3142} &= -q^3 & E_{3214} &= -q & E_{3241} &= q^2 & E_{3412} &= q^2 & E_{3421} &= -q^3 \\
E_{4123} &= -q^3 & E_{4132} &= q^2 & E_{4213} &= q^2 & E_{4231} &= -q & E_{4312} &= -q^3 & E_{4321} &= q^2
\end{align*}
$$

(2.1)

The function $S_4 \ni \sigma \mapsto 4 - c(\sigma) = t(\sigma)$ counts the number of transpositions in $\sigma$. It follows from [SLW3], Theorem 4.1, that this way we obtain a compact quantum group $(\mathcal{A}, \mathfrak{u})$, where $\mathcal{A}$ is the $C^*$-algebra generated by 16 matrix elements $\{u_{jk} : 1 \leq j, k \leq 4\}$ of $\mathfrak{u}$, which satisfy the unitarity condition:

$$\sum_{r=1}^{4} u_{jr}^{*} u_{rk} = \delta_{jk} I = \sum_{r=1}^{4} u_{jr} u_{rk}^{*}$$

(2.2)

and the twisted determinant condition:

$$\sum_{i,j,k,l=1}^{4} u_{\alpha i} u_{\beta j} u_{\gamma k} u_{\delta l} E_{ijkl} = E_{\alpha\beta\gamma\delta} I$$

(2.3)

for each $\{\alpha, \beta, \gamma, \delta\} \subset \{1, 2, 3, 4\}$. The matrix $\mathfrak{u} = (u_{jk})_{k=1}^{4}$ is the fundamental unitary co-representation of the quantum group. In our case the co-representation $\mathfrak{u} = (u_{kl})_{l=1}^{4}$ is reducible by the following reason. The operator $P = (E^* \otimes I)(I \otimes E)$, which acts on $\mathbb{C}^4$, intertwines the fundamental representation with itself: $(P \otimes I)\mathfrak{u} = \mathfrak{u}(P \otimes I)$. Moreover, $P$ has a diagonal matrix for the standard basis of $\mathbb{C}^4$:

$$P = diag\{c_1, c_2, c_3, c_4\},$$

with $c_j = \sum_{\alpha, \beta, \gamma} E_{\alpha j \beta k} E_{\alpha \beta \gamma j}$, and therefore $c_1 = c_4 = -(5q^3 + q^5)$, $c_2 = c_3 = -(2q^3 + 4q^5)$. Hence, for $q \neq 0, -1, 1$, which shall be the case in the sequel, $c_1 \neq c_2$, so $P$ is not a multiple of the identity operator $I$. The condition $(P \otimes I)\mathfrak{u} = \mathfrak{u}(P \otimes I)$ is equivalent to $c_j \cdot u_{jk} = c_k \cdot u_{jk}$ for all natural numbers $1 \leq j, k \leq 4$. This yields $u_{12} = u_{21} = 0, u_{13} = u_{31} = 0, u_{24} = u_{42} = 0, u_{34} = u_{43} = 0$, and therefore

$$\mathfrak{u} = \begin{pmatrix} u_{11} & 0 & 0 & u_{14} \\ 0 & u_{22} & u_{23} & 0 \\ 0 & u_{32} & u_{33} & 0 \\ u_{41} & 0 & 0 & u_{44} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & x & y & 0 \\ 0 & z & w & 0 \\ c & 0 & 0 & d \end{pmatrix}$$

(2.4)

This yields the decomposition of $\mathfrak{u}$ decomposes into two irreducible subrepresentations

$$\mathfrak{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

(2.5)
Substitution in (2.3) of appropriate sequences \((\alpha, \beta, \gamma, \delta)\) gives the following relations between the generators of the \(C^*\)-algebra \(\mathcal{A}\) (the associated sequence is left of the relation):

\[
\begin{align*}
(1423) \quad I &= (ad-qbc)(xw-q^{-1}yz) & (1) & \quad I &= (da-q^{-1}cb)(xw-q^{-1}yz) & (2) \\
(1432) \quad I &= (ad-qbc)(wx-qzy) & (3) & \quad I &= (da-q^{-1}cb)(wx-qzy) & (4) \\
(2314) \quad I &= (wx-q^{-1}yz)(ad-qbc) & (5) & \quad I &= (xw-q^{-1}yz)(da-q^{-1}cb) & (6) \\
(3214) \quad I &= (wx-qzy)(ad-qbc) & (7) & \quad I &= (xw-q^{-1}yz)(da-q^{-1}cb) & (8)
\end{align*}
\]

Let \(W = ad-qbc\) and \(V = xw-q^{-1}yz\), then the above relation give \(VW = I = WV\) and also \(W = da-q^{-1}cb\), \(V = wx-qzy\). Hence these relations are pairwise equivalent: \((1) \Leftrightarrow (5)\), \((2) \Leftrightarrow (6)\), \((3) \Leftrightarrow (7)\) and \((4) \Leftrightarrow (8)\). The operators \(V, W\), being the inverse of each other, are twisted determinants for the two matrix co-representations:

\[
W = \det_q \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad V = \det_{q^{-1}} \begin{pmatrix} x & y \\ z & w \end{pmatrix}. \quad (2.6)
\]

Let us observe here that a change of order in the basis for \(\begin{pmatrix} x & y \\ z & w \end{pmatrix}\) gives us the matrix \(\begin{pmatrix} w & z \\ y & x \end{pmatrix}\) which satisfies the same relations and for which the twisted determinant is

\[
\det_q \begin{pmatrix} w & z \\ y & x \end{pmatrix} = wx-qzy = V. \quad (2.7)
\]

Using the invertibility of \(W\) and \(V\) one can easily get the following relations:

\[
\begin{align*}
(1123) \quad ab &= qba & (9) & \quad cd &= qdc & (10) \\
(4423) \quad yx &= qxy & (11) & \quad wz &= qzw & (12)
\end{align*}
\]

In addition, the relations (2.2) can be written as:

\[
\begin{align*}
I &= aa^* + bb^* & (13) & I &= cc^* + dd^* & (14) \\
I &= a^*a + c^*c & (15) & I &= b^*b + d^*d & (16) \\
0 &= a^*b + c^*d & (17) & 0 &= ca^* + db^* & (18)
\end{align*}
\]

and

\[
\begin{align*}
I &= xx^* + yy^* & (19) & I &= zz^* + ww^* & (20) \\
I &= x^*x + z^*z & (21) & 0 &= y^*y + w^*w & (22) \\
0 &= x^*y + z^*w & (23) & 0 &= z^*x + wy^* & (24)
\end{align*}
\]

Multiplication of (16) from the left by \(a^*\) and using (9) and then (17) gives the equation \(d^*W = a\), or, equivalently, \(d = V^*a^*\). On the other hand, multiplication (15) from the right by \(d\) and using (10) and (17) gives \(d = a^*W\). These two combined ensure also that \(W^*a = aV\). Similarly, by multiplying (16) from the right by \(c\) and using (10) and then (17) one gets \(b^*W = -qc\), or equivalently, \(b^* = -qcV\). Then, multiplying (15) from the right by \(b\) and using (9) and (17) one obtains \(b = -qc^*W\). These two yield also \(cV = W^*c\). Therefore we have

\[
d = V^*a^* = a^*W, \quad b = -qV^*c^* = -qc^*W; \quad (2.8)
\]
\[ x = w^*V = W^*w^*, \quad z = -qy^*V = -qW^*y^*. \]  
\[ 2.9 \]

There are also other relations obtained from (2.3). They are listed in the following, with the associated sequences \((\alpha\beta\gamma\delta)\) on the left-hand side:

\begin{align*}
(2143) & \quad I = x(ad - qbc)w - qy(ad - q^{-1}bc)z & (25) \\
(2413) & \quad I = x(da - q^{-1}cb)w - q^{-1}y(da - qcb)z & (26) \\
(3142) & \quad I = w(ad - q^{-1}bc)x - q^{-1}z(ad - qbc)y & (27) \\
(3412) & \quad I = w(da - qcb)x - qz(da - q^{-1}cb)y & (28)
\end{align*}

and

\begin{align*}
(1234) & \quad I = a(xw - qyz)d - qb(xw - qyz)c & (29) \\
(4231) & \quad I = d(xw - q^{-1}zy)a - q^{-1}c(xw - q^{-1}zy)b & (30) \\
(3142) & \quad I = w(ad - q^{-1}bc)x - q^{-1}z(ad - qbc)y & (31) \\
(4321) & \quad I = d(wx - q^{-1}zy)a - q^{-1}c(wx - q^{-1}zy)b & (32)
\end{align*}

From now on we shall assume the following additional relation:

\[ V = W^* \]  
\[ 2.10 \]

meaning that the twisted determinants are unitary operators. This yields that we are dealing with the quantum groups \(U_q(2)\) (for the generators \(a, b, c, d\)) and another copy of \(U_q(2)\) (for the generators \(w, y, z, x\)). This assumption is also necessary to allow the technical procedure used in [W3].

Let us substitute (2.8) into the (1) - (32). In (1) - (8) we do the substitution in one of the bracket and put \(V\) or \(V^*\) for the other. Thus for each equation we get two:

\begin{align*}
V a V^* a^* + q^2 c^* c & = 1 \quad (1'a) \quad w^* V w V^* + y y^* = 1 \quad (1'b) \\
a^* a + V c V^* c^* & = 1 \quad (2'a) \quad w^* V w V^* + y y^* = 1 \quad (2'b) \\
V a V^* a^* + q^2 c^* c & = 1 \quad (3'a) \quad w w^* + q^2 y^* V y V^* = 1 \quad (3'b) \\
a^* a + V c V^* c^* & = 1 \quad (4'a) \quad w w^* + q^2 y^* V y V^* = 1 \quad (4'b)
\end{align*}

We see that \((1'a) \Leftrightarrow (3'a),\) \((2'a) \Leftrightarrow (4'a),\) \((1'b) \Leftrightarrow (2'b)\) and \((3'b) \Leftrightarrow (4'b).\) For (9) - (12) we obtain:

\begin{align*}
V a V^* a^* & = qa^* c V \quad (9') \quad a V c^* = qc^* a V \quad (10') \\
Y w^* V & = q w^* V y \quad (11') \quad w y^* V = q y^* V w \quad (12')
\end{align*}

The relation (13) - (18) give:

\begin{align*}
aa^* + q^2 V^* c^* c V & = 1 \quad (13') \quad cc^* + V^* a^* a V = 1 \quad (14') \\
a^* a + c^* c & = 1 \quad (15') \quad aa^* + q^2 c c^* = 1 \quad (16') \\
a V c = q c V a \quad (17') \quad V c a^* = q a^* c V \quad (18')
\end{align*}

and for (19) - (24) we get:

\begin{align*}
w^* w + y y^* & = 1 \quad (19') \quad w w^* + q^2 y^* y = 1 \quad (20') \\
w w^* + q^2 y y^* & = 1 \quad (21') \quad w w^* + y^* y = 1 \quad (22') \\
w y & = q y w \quad (23') \quad w y^* = q y^* w \quad (24')
\end{align*}
Let us first deal with the relations (2.14) involving $w$ and $y$. Comparing (19') with (21') one gets easily that $y$ is normal: $yy^* = y^*y$. Comparing (3'b) with (20') gives

$$y^*Vy = y^*yV$$  \hfill (2.15)

and (1'b) with (19') yield

$$w^*Vw = w^*wV. \hfill (2.16)$$

Putting (24') into (11') gives

$$w^*yV = w^*Vy.$$  \hfill (2.17)

Multiplying both sides of (2.16) this from the left by $w$ provides $ww^*yV = ww^*yV$. Similarly, multiplying (2.14) from the right by $y$ gives $yy^*Vy = yy^*yV$. Adding these two side by side yields

$$Vy = yV.$$  \hfill (2.18)

In a similar manner one gets

$$Vw = wV.$$  \hfill (2.19)

This requires putting (24') into (12') to get $y^*wV = y^*Vw$ which is then multiplied from the left by $q^2y$ and added side by side to $ww^*Vw = ww^*wV$, which is obtained from (2.15). These can be collected together as the following relations:

$$w^*w + y^*y = 1 \quad ww^* + q^2yy^* = 1$$
$$wy = qyw \quad wy^* = qy^*w$$
$$wV = Vw \quad yV = Vy$$  \hfill (2.20)

The fundamental co-representation is thus $\begin{pmatrix} w^*V & y \\ -qy^*V & w \end{pmatrix}$ and the above relations define the $C^*$-algebra of $U_q(2)$, and $V$ is the $(-q)^{-1}$-determinant.

Let us now work with the relations for $a$ and $c$. From (4') and (15') one deduces that $cVc^* = c^*cV$. Then, multiplying (9') from the right by $a$ one gets $cVaa^* = qa^*cVa$. The left-hand side of this can be transformed as follows (using (15')):

$$cVaa^* = cV(1 - c^*c) = cV - (cVc^*)c = cV - c^*cVc.$$ 

For the right-hand side one can use (17') and then (15') to get:

$$qa^*cVa = a^*aVc = (1 - cc^*)Vc = Vc - c^*cVc.$$ 

It follows from these two that $cV = Vc$, and also $c^*V = Ve^*$, since $V$ is unitary. Using this combined with (14') and (15') one obtains $cc^* = c^*c$, so $c$ is normal. Then from (10') follows $ac^* = qc^*a$. Comparing (1'a) with (16') one concludes $aVc^* = aa^*V$. Then, multiplication of (17') by $c^*$ from the right gives $aVc^*c = qcVac^*$. The left-hand side of this is $aV - aa^*Va$. The right-hand side of this can be transformed, with the help of the above relations, into:

$$qcVac^* = q^2cVc^*a = q^2c^*cV = Va - aa^*Va.$$
Hence one concludes $aV = Va$, and also $a^*V = Va^*$. Therefore the above relations may be written as follows:

$$
\begin{align*}
    a^*a + c^*c &= 1 \\
    ac &= qa \\
    ac^* &= qc^*a \\
    aV &= Va \\
    cV &= Vc
\end{align*}
$$

(2.21)

For $N = 4$ we have more nontrivial relations between $a, c, w, y$ given by (2.3) then in the case $N = 3$, since, for example the sequence $(1, 1, 2, 2)$ gives a nontrivial relation here, and gave trivial relation there. Let us write them as follows, indicating the associated sequence $(\alpha, \beta, \gamma, \delta)$ on the left-hand side of it and successive numbering on the right-hand side of it. In the first set of equations we put elements from the same $C^*$-subalgebra outside, and the other inside.

\begin{align*}
    (1231) \quad a(xw - qyz)b &= qb(xw - qyz)a & (33) \\
    (1321) \quad a(wx - \frac{1}{q}zy)b &= qb(wx - \frac{1}{q}zy)a & (34) \\
    (4234) \quad c(xw - qyz)d &= qd(xw - qyz)c & (35) \\
    (4324) \quad c(wx - \frac{1}{q}zy)d &= qd(wx - \frac{1}{q}zy)c & (36) \\
    (2142) \quad x(ad - qbc)y &= qy(ad - \frac{1}{q}bc)x & (37) \\
    (2412) \quad y(da - qc)b &= qx(da - \frac{1}{q}cb)y & (38) \\
    (3143) \quad z(ad - qbc)w &= qw(ad - \frac{1}{q}bc)z & (39) \\
    (3413) \quad w(da - qc)z &= qz(da - \frac{1}{q}cb)w & (40) \\
    (1224) \quad axyd &= qbyxc & (41) \\
    (4221) \quad cxyb &= qdxya & (42) \\
    (1334) \quad azwd &= qbwzd & (43) \\
    (4331) \quad czwb &= qdzwa & (44) \\
    (2113) \quad yabz &= 0 = yabz & (45) \\
    (3112) \quad wabx &= 0 = wabx & (46) \\
    (2443) \quad ydcz &= 0 = ydcz & (47) \\
    (3442) \quad wdcx &= 0 = wdcx & (48)
\end{align*}

In the second set of equations we have alternating sequences of elements from different $C^*$-subalgebras.

\begin{align*}
    (1243) \quad axdw - qaydz - qbxcw + q^2bycz &= 1 & (49) \\
    (4213) \quad dxaw - qdyaz - \frac{1}{q}cxbw + cybz &= 1 & (50) \\
    (1342) \quad awdx - \frac{1}{q}azdy - qbwtx + bzcy &= 1 & (51) \\
    (4312) \quad dwx - \frac{1}{q}dzay - \frac{1}{q}cwbx + \frac{1}{q^2}czby &= 1 & (52) \\
    (2134) \quad xawd - qxwbc - qyazd + q^2ybzc &= 1 & (53) \\
    (3124) \quad wazd - qwtxc - \frac{1}{q}zayd + zbyc &= 1 & (54) \\
    (2431) \quad xdwa - \frac{1}{q}xcwb - qydsa + yzc = 1 & (55) \\
    (3421) \quad wdxz - \frac{1}{q}xczw - \frac{1}{q^2}dzya + \frac{1}{q^2}zocyb &= 1 & (56)
\end{align*}

(2.23)

Computing

\begin{align*}
    xw - qyz &= V - (1 - q^2)yy^*V \\
    wx - \frac{1}{q}zy &= V + (1 - q^2)yy^*V \\
    ad - \frac{1}{q}bc &= V^* + (1 - q^2)cc^*V^* \\
    da - qcb &= V^* - (1 - q^2)cc^*V^*
\end{align*}

(2.24)
and substituting these into (2.22) one obtains

\[
\begin{align*}
a y y^* c^* &= q c^* y y^* a \quad (33'), (34') \\
c z y y^* a^* &= q a^* y y^* c \quad (35'), (36') \\
w^* y &= q y c^* y^* w^* \quad (37'), (39') \\
y c c^* w^* &= 0 \quad (38') \\
w c c^* y^* &= 0 \quad (40') \\
a w^* y a^* + q^2 c^* w^* y c &= 0 \quad (41'), (43') \\
a^* w^* y a + c w^* y c^* &= 0 \quad (42'), (44') \\
y a c^* y^* &= 0 \quad (45') \\
w a c^* w^* &= 0 \quad (46') \\
y a^* c y^* &= 0 \quad (47') \\
w a^* c w^* &= 0 \quad (48')
\end{align*}
\]

Unfortunately, (37') combined with (38') give

\[w^* y = 0\]

and it follows from (2.20) that \( y = 0 \). To see this let us observe that \( w w^* y y^* + q^2 y y^* y y^* = y y^* \) implies \( q^2 (y y^*)^2 = y y^* \), and hence, by induction, \( q^{2n} (y y^*)^{n+1} = y y^* \) for any positive integer \( n \in \mathbb{N} \). This yields that the spectral radius \( r(y y^*) = \lim_{n \to \infty} \| (y y^*)^n \|^{1/n} \) satisfies \( r(y y^*) = q^2 > 1 \).

However, it follows from the description of the irreducible representations of the relations (2.20) (see [W3]) that \( \| y \| \leq 1 \), so that \( r(y y^*) \leq 1 \). This is a contradiction, except \( y = 0 \).

Then \( x w = V = w x \) and \( x x^* = 1 = x^* x, w w^* = w^* w \), so that \( x, w \) are unitary. Moreover \( x = w^* V \), so that for the fundamental co-representation eventually we get \( \begin{pmatrix} w^* V & 0 \\
0 & w \end{pmatrix} \). In a similar manner one gets that

\[a^* c = 0\]

and hence \( c = 0 \). Substitution of these to (2.23) gives

\[a w a^* w^* = 1 = a^* w^* a w.\]

If we set \( t := a w \) and \( s := w a \), then \( t t^* = 1 = t^* t, s s^* = 1 = s^* s \) and \( t s^* = 1 = s^* t \). Therefore \( t = s \), which gives \( a w = w a \).

These computations show that the \( C^* \)-algebra of the constructed quantum group is generated by three commuting unitaries \( a, w, V \), so it is isomorphic to \( C(T) \otimes C(T) \otimes C(T) \). Therefore, the quantum group we consider is in fact the classical group \( U(1) \times U(1) \times U(1) \).

### 3 The Yang-Baxter operator associated with \( U_q(2) \)

In the next two Sections we are going to show a construction of a cubic Hecke algebra associated with the quantum group \( U_q(2) \). In [W3] we gave a construction of the quantum group \( U_q(2) \), in which the crucial role is played by the function counting the number of cycles in permutations from the symmetric group \( S_3 \). Namely, by considering the function \( S_3 \ni \sigma \mapsto (-q)^3 c(\sigma) \), where \( c(\sigma) \) is the number of cycles and \( q > 0 \), we constructed the following array:

\[
\begin{align*}
E_{1,2,3} &= 1 \\
E_{1,3,2} &= E_{2,1,3} = E_{3,2,1} = -q \\
E_{2,3,1} &= E_{3,1,2} = q^2 \\
E_{i,j,k} &= 0 \text{ if } \{i, j, k\} \not\subset \{1, 2, 3\}
\end{align*}
\]
This array defines an operator $\rho$ on $\mathbb{C}^3 \otimes \mathbb{C}^3$ by

$$\rho : \mathbb{C}^3 \otimes \mathbb{C}^3 \ni (a, b) \mapsto \sum_{i,j,k=1}^{3} E_{i,j,k} E_{k,a,b}(i,j) \in \mathbb{C}^3 \otimes \mathbb{C}^3,$$  \hfill (3.26)

where $(a, b)$ denotes in short the standard basis element $\epsilon_a \otimes \epsilon_b$. In particular $\epsilon_1 = (1,0,0)$, $\epsilon_2 = (0,1,0)$ and $\epsilon_3 = (0,0,1)$.

The definition of $E$ implies that (3.26) simplifies to

$$\rho(a,b) = E_{a,b,k}E_{k,a,b}(a,b), \quad \text{where} \quad \{a,b,k\} = \{1,2,3\} \quad \hfill (3.27)$$

for $a \neq b$ and $a,b = 1,2,3$. If $a = b$ then we get $\rho(a,a) = 0$. The formulas can be written explicitly as follows.

$$\begin{align*}
\rho(1,2) &= E_{1,2,3}E_{3,1,2}(1,2) + E_{2,1,3}E_{3,1,2}(2,1) = q^2(1,2) + q^3(2,1) \\
\rho(2,1) &= E_{2,1,3}E_{3,1,2}(2,1) + E_{1,2,3}E_{3,2,1}(1,2) = q^2(2,1) + q(1,2) \\
\rho(1,3) &= E_{1,3,2}E_{2,1,3}(1,3) + E_{3,1,2}E_{2,1,3}(3,1) = q^2(1,3) + q^3(3,1) \\
\rho(3,1) &= E_{3,1,2}E_{2,3,1}(3,1) + E_{1,3,2}E_{2,3,1}(1,3) = q^4(3,1) + q^2(1,3) \\
\rho(2,3) &= E_{2,3,1}E_{1,2,3}(2,3) + E_{3,2,1}E_{1,2,3}(3,2) = q^2(2,3) + q(3,2) \\
\rho(3,2) &= E_{3,2,1}E_{2,1,3}(3,2) + E_{2,3,1}E_{1,3,2}(2,3) = q^2(3,2) + q^3(2,3)
\end{align*}$$

Therefore, the operator $\alpha := I_2 - \frac{1}{q^2} \rho$ acts as: $\alpha(a,a) = (a,a)$ for $a = 1,2,3$ and

$$\begin{align*}
\alpha(1,2) &= -q(2,1) \\
\alpha(1,3) &= -q(3,1) \\
\alpha(3,2) &= -q(2,3) \\
\alpha(2,1) &= -q^{-1}(1,2) \\
\alpha(2,3) &= -q^{-1}(3,2) \\
\alpha(3,1) &= (1 - q^2)(3,1) - q(1,3)
\end{align*} \quad \hfill (3.28)$$

This operator is not self-adjoint, but $\alpha^2 = (\alpha^2)^*$ is so, since

$$\begin{align*}
\alpha^2(1,2) &= (2,1) \\
\alpha^2(2,1) &= (2,1) \\
\alpha^2(2,3) &= (3,2) \\
\alpha^2(3,2) &= (2,3) \\
\alpha^2(1,3) &= q^2(1,3) - q(1 - q^2)(3,1) \\
\alpha^2(3,1) &= (1 - q^2 + q^4)(3,1) - q(1 - q^2)(1,3)
\end{align*} \quad \hfill (3.29)$$

The first important property of $\alpha$ is that it is a Yang-Baxter operator.

**Proposition 3.1** The operator $\alpha$ satisfies the Yang-Baxter equation

$$\begin{align*}
(\alpha \otimes I)(I \otimes \alpha)(\alpha \otimes I) &= (I \otimes \alpha)(\alpha \otimes I)(I \otimes \alpha).
\end{align*} \quad \hfill (3.30)$$
Proof: Let \( L = (\alpha \otimes I)(I \otimes \alpha)(\alpha \otimes I) \) be the left-hand side and \( P = (I \otimes \alpha)(\alpha \otimes I)(I \otimes \alpha) \) be the right-hand side of (3.30). We have to show that \( L(a, b, c) = P(a, b, c) \) for every \( a, b, c \in \{1, 2, 3\} \) (with the notation: \( (a, b, c) = \epsilon_a \otimes \epsilon_b \otimes \epsilon_c \)). This requires checking 27 cases. It is clear that \( L(a, a, a) = (a, a, a) = P(a, a, a) \) for any \( a = 1, 2, 3 \).

The direct calculation provides the following formulas for the other cases.

\[
\begin{align*}
L(3, 2, 3) &= (3, 2, 3) = P(3, 2, 3) \\
L(2, 3, 2) &= (2, 3, 2) = P(2, 3, 2) \\
L(1, 2, 1) &= (1, 2, 1) = P(1, 2, 1) \\
L(2, 1, 2) &= (2, 1, 2) = P(2, 1, 2) \\
L(1, 2, 3) &= -q(3, 2, 1) = P(1, 2, 3) \\
L(1, 3, 2) &= -q^3(2, 3, 1) = P(1, 3, 2) \\
L(2, 1, 3) &= -q^{-1}(3, 1, 2) = P(3, 1, 2) \\
L(3, 3, 2) &= q^2(2, 3, 3) = P(3, 3, 2) \\
L(2, 2, 3) &= q^2(3, 2, 2) = P(2, 2, 3) \\
L(3, 2, 2) &= q^2(2, 2, 3) = P(3, 2, 2) \\
L(1, 1, 3) &= q^2(3, 1, 1) = P(1, 1, 3) \\
L(1, 3, 3) &= q^2(3, 3, 1) = P(1, 3, 3) \\
L(1, 1, 2) &= q^2(2, 1, 1) = P(1, 1, 2) \\
L(1, 2, 2) &= q^2(2, 2, 1) = P(1, 2, 2) \\
L(2, 3, 3) &= q^{-2}(3, 3, 2) = P(2, 3, 3) \\
L(2, 1, 1) &= q^{-2}(1, 1, 2) = P(2, 1, 1) \\
L(2, 2, 1) &= q^{-2}(1, 2, 2) = P(2, 2, 1) \\
L(3, 2, 1) &= (1 - q^2)(3, 2, 1) - q(1, 2, 3) = P(3, 2, 1) \\
L(3, 1, 2) &= q^2(1 - q^2)(2, 3, 1) - q^3(2, 1, 3) = P(3, 1, 2) \\
L(2, 3, 1) &= q^{-2}(1 - q^2)(3, 1, 2) - q^{-1}(1, 3, 2) = P(2, 3, 1) \\
L(1, 3, 1) &= -q(1 - q^2)(3, 1, 1) + q^2(1, 3, 1) = P(1, 3, 1) \\
L(3, 1, 3) &= -q(1 - q^2)(3, 3, 1) + q^2(3, 1, 3) = P(3, 1, 3)
\end{align*}
\]

From these formulas the Proposition follows. \(\square\)

4 The cubic Hecke algebra associated with \( U_q(2) \)

The second important property of the operator \( \alpha \) is that, even though it is not a Hecke operator, it does satisfy a cubic equation, and thus it generates a cubic Hecke algebra. This notion has been introduced by Funar in [F], where the cubic equation \( \alpha^3 - I = 0 \) was considered.

Proposition 4.1 The operator \( \alpha \) satisfies the cubic equation:

\[
(\alpha^2 - I)(\alpha + q^2 I) = 0.
\]
Proof: From the formulas (3.28), defining $\alpha$ it follows that it acts on the following subspaces by simple matricial formulas.

1. On the span of $(1,2), (2,1)$ as $\beta := \begin{pmatrix} 0 & \frac{-1}{q} \\ -q & 0 \end{pmatrix}$

2. On the span of $(2,3), (3,2)$ as $\beta^* := \begin{pmatrix} 0 & -q \\ \frac{-1}{q} & 0 \end{pmatrix}$

3. On the span of $(1,3), (3,1)$ as $\gamma := \begin{pmatrix} 0 & -q \\ -q & 1-q^2 \end{pmatrix}$

4. As identity on every $(a,a)$ with $a=1,2,3$.

It is straightforward to see that $\beta^2 - I = 0 = (\beta^*)^2 - I$. On the other hand, since

$$\gamma^2 = \begin{pmatrix} q^2 & -q(1-q^2) \\ -q(1-q^2) & 1-q^2+q^4 \end{pmatrix},$$

we obtain

$$(\gamma^2 - I)(\gamma + q^2I) = (q^2 - 1) \begin{pmatrix} 1 & q \\ q & q^2 \end{pmatrix} \begin{pmatrix} q^2 & -q \\ -q & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Therefore both $\beta$ and $\gamma$ satisfy the equation (4.34), so the $\alpha$ docs. \hfill \square

Let us define the elements

$$h_j := I_j \otimes \alpha \otimes I_{n-j-2} \quad \text{for} \quad j = 1, \ldots, n-2,$$  \hspace{1cm} (4.35)

where $I_k$ denotes the identity map on $(\mathbb{C}^N)^{\otimes k}$. Then by Propositions 3.1 and 4.1 the elements $h_1, \ldots, h_n$ generate a cubic Hecke algebra, associated with the quantum group $U_q(2)$.

**Definition 4.2** The algebra $\mathcal{H}_{q,n}(2)$ generated by the elements $h_j, j = 1, 2, \ldots, n$ defined by (4.35) will be called the **cubic Hecke algebra** associated with the quantum group $U_q(2)$.

The basic properties of this algebra are summarized in the following.

**Theorem 4.3** The generators $\{h_j : 1 \leq j \leq n\}$ of $\mathcal{H}_{q,n}(2)$ satisfy:

$$h_j h_{j+1} h_j = h_{j+1} h_j h_{j+1} \quad \text{for} \quad j = 1, \ldots, n-1,$$

$$h_j h_k = h_k h_j \quad \text{for} \quad |j-k| \geq 2,$$

$$((h_j)^2 - 1)(h_j + q^2) = 0 \quad \text{for} \quad j = 1, \ldots, n.$$  \hspace{1cm} (4.36)

The role of the Hecke algebra in the study of $SU_q(N)$ was that it was the intertwining algebra of the tensor powers of the fundamental co-representation. In [W3] the irreducible co-representations of $U_q(2)$ have been described, but it is not clear if the description is complete. So, it is still to be checked whether $\mathcal{H}_{q,n}(2)$ plays the same role as in $SU_q(N)$.
References


