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<tr>
<td>Author(s)</td>
<td>WYSOCZANSKI, JANUSZ</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2009), 1658: 124-135</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2009-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/140904">http://hdl.handle.net/2433/140904</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
On the Woronowicz's twisted product construction of quantum groups, with comments on related cubic Hecke algebra. *

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Abstract

We study the construction of compact quantum groups, based on the method invented by Woronowicz [SLW3], which uses a twisted determinant. As an example Woronowicz considered the function $S_N \ni \sigma \mapsto \text{inv} (\sigma)$, where $\text{inv} (\sigma)$ is the number of inversions in the permutation $\sigma$. Our twisted determinant is related to the function $S_N \ni \sigma \mapsto c (\sigma)$, where $c (\sigma)$ is the number of cycles in a permutation $\sigma$. For $N = 3$ it gave the quantum group $U_q(2)$. Here we show how the construction works if $N = 4$. We also describe the cubic Hecke algebra, associated with the quantum group $U_q(2)$.

1 Introduction

In [SLW3] Woronowicz provided a general method for constructing compact matrix quantum groups. The method depends on finding an $N^N$-element array $E = (E_{i_1 \ldots i_N})_{i_1, \ldots, i_N = 1}^N$ of complex numbers, called twisted determinant, which is (left and right) non-degenerate. Theorem 1.4 of [SLW3] says that if a $C^*$-algebra $A$, is generated by $N^2$ elements $u_{jk}$ which satisfy the unitarity condition:

$$\sum_{r=1}^N u_{jr}^* u_{rk} = \delta_{jk} I = \sum_{r=1}^N u_{jr} u_{rk}^*$$

and the following twisted determinant condition:

$$\sum_{k_1, \ldots, k_N = 1}^N u_{j_1, k_1} \ldots u_{j_N, k_N} E_{k_1, \ldots, k_N} = E_{j_1, \ldots, j_N} I$$

*Research partially supported by the European Commission Marie Curie Host Fellowship for the Transfer of Knowledge "Harmonic Analysis, Nonlinear Analysis and Probability" MTKD-CT-2004-013389, the Polish Ministry of Science’s research grants 1P03A01330 and N N201 270735

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and if the array $E$ is non-degenerate, then $(\mathcal{A}, u)$ is a compact matrix quantum group, where $u = (u_{jk})_{j,k=1}^{N}$. Woronowicz described the following example. For $\mu \in (0, 1]$, he defined

$$E_{i_{1}, \ldots, i_{N}} = (-\mu)^{\text{inv}(\sigma)} \text{ if } \sigma = \begin{pmatrix} 1 & 2 & \cdots & N \\ i_{1} & i_{2} & \cdots & i_{N} \end{pmatrix} \in S_{N}$$

is a permutation ($S_{N}$ denotes the set of permutations of $\{1, 2, \ldots, N\}$) and $E_{i_{1}, \ldots, i_{N}} = 0$ otherwise. Here, for a permutation $\sigma \in S_{N}$, $\text{inv}(\sigma)$ is the number of inversions of $\sigma$, which is the number of pairs $(j, k)$ such that $j < k$ and $i_{j} = \sigma(j) > \sigma(k) = i_{k}$. Then as $(\mathcal{A}, u)$ one gets the quantum group $S_{\mu}U(N)$, called the twisted $SU(N)$ group.

In [W3] we considered another array $E$ for $N = 3$, related to the number of cycles in a permutation. It was defined for a parameter $0 < q < 1$ as follows:

$$E(i, j, k) = \begin{cases} (-q)^{3-c(i, j, k)} & \text{if } \{i, j, k\} = \{1, 2, 3\} \\ 0 & \text{otherwise} \end{cases}$$

Here $c(i, j, k)$ is the number of cycles of the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$$

(which makes sense if and only if $\{i, j, k\} = \{1, 2, 3\}$). Then, following the Woronowicz's scheme, we obtained a quantum group, which turned out to be $U_{q}(2)$, the quantum deformation of the unitary $2 \times 2$ group. Moreover, the construction provided a description of it as a twisted product of its quantum subgroups

$$U_{q}(2) = SU_{q}(2) \ltimes U(1)$$

with the *-isomorphism $\sigma : \mathcal{A}_{1} \otimes \mathcal{A}_{2} \rightarrow \mathcal{A}_{2} \otimes \mathcal{A}_{1}$ given by

$$\sigma(1 \otimes v) = v \otimes 1, \quad \sigma(a \otimes v^{k}) = v^{k} \otimes a, \quad \sigma(c \otimes v^{k}) = v^{k-1} \otimes c.$$  

The natural continuation of the construction given in [W3], was investigating the cases $N \geq 4$. However, as shall see below, after some tiresome computations it turned out that for $N = 4$ (and thus also for all $N \geq 4$) the quantum group we obtain (via the Woronowicz’s theorem) is classical abelian.

Regarding the quantum group $U_{q}(2)$, we shall present also a construction of a cubic Hecke algebra. In [SLW3] Woronowicz showed that there are Hecke algebras associated with the quantum groups $SU_{q}(N)$, for every $N \in \mathbb{N}$, $N \geq 2$. The Hecke algebra $H_{q,n}$ described the intertwining operators for the $n^{th}$ tensor power of the fundamental representation of the group. In this note we shall show similar construction for $U_{q}(2)$. The construction depends on defining an operator $\alpha : \mathbb{C}^{3} \otimes \mathbb{C}^{3} \rightarrow \mathbb{C}^{3} \otimes \mathbb{C}^{3}$, which satisfies the Yang-Baxter equation (3.1). The operator is not self-adjoint (contrary to the $SU_{q}(N)$ cases), although its square is so $(\alpha^{2} = (\alpha^{\ast})^{2})$. Nevertheless, it satisfies a generalization of the Hecke equation, namely $(\alpha^{2} - I)(\alpha + q^{2}I) = 0$ (see (4.1)). Therefore the operators $h_{j} := I_{j} \otimes \alpha \otimes I_{n-j-2}$, defined for $j = 1, \ldots, n-2$, generate a cubic Hecke algebra (Theorem 4.3).

The paper is organized as follows. In Section 2 we give the computation showing the generalization of our $U_{q}(2)$ construction, for $N = 4$. Then, in Section 3, we give the construction of the operator $\alpha$, and show that it satisfies the Yang-Baxter equation. The last Section 4, contains the construction of the cubic Hecke algebra, associated with $U_{q}(2)$. In particular, we show there that $\alpha$ satisfies the cubic equation.
2 The construction associated with $E$

Let $N_4 = \{(i,j,k,l) : \{i,j,k,l\} \subset \{1,2,3,4\}\}$, let $E : N_4 \hookrightarrow \mathbb{C}$ be zero outside $S_4 \subset N_4$, where the inclusion is given by $(i,j,k,l) \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & l \end{pmatrix}$ if $\{i,j,k,l\} = \{1,2,3,4\}$, and, for $0 < q < 1$, let the (non-zero) values of $E$ (with the notation $E((i,j,k,l)) = E_{ijkl}$) be given by the function

$$S_4 \ni \sigma \mapsto (-q)^{4-c(\sigma)}.$$

Explicitly, it can be written in the following way:

$$
\begin{align*}
E_{1234} &= 1 & E_{1243} &= -q & E_{1324} &= -q & E_{1342} &= q^2 & E_{1423} &= q^2 & E_{1432} &= -q \\
E_{2134} &= -q & E_{2143} &= q^2 & E_{2314} &= q^2 & E_{2341} &= -q^3 & E_{2413} &= -q^3 & E_{2431} &= q^2 \\
E_{3124} &= q^2 & E_{3142} &= -q^3 & E_{3214} &= -q & E_{3241} &= q^2 & E_{3412} &= q^2 & E_{3421} &= -q^3 \\
E_{4123} &= -q^3 & E_{4132} &= q^2 & E_{4213} &= q^2 & E_{4231} &= -q & E_{4312} &= -q^3 & E_{4321} &= q^2
\end{align*}

(2.1)

The function $S_4 \ni \sigma \mapsto 4 - c(\sigma) = t(\sigma)$ counts the number of transpositions in $\sigma$. It follows from [SLW3], Theorem 4.1, that this way we obtain a compact quantum group $(A, \mathfrak{u})$, where $A$ is the $C^*$-algebra generated by 16 matrix elements $\{u_{jk} : 1 \leq j, k \leq 4\}$ of $\mathfrak{u}$, which satisfy the unitarity condition:

$$\sum_{r=1}^{4} u_{jr}^* u_{rk} = \delta_{jk} I = \sum_{r=1}^{4} u_{jr} u_{rk}^* \quad (2.2)$$

and the twisted determinant condition:

$$\sum_{i,j,k,l=1}^{4} u_{\alpha i} u_{\beta j} u_{\gamma k} u_{\delta l} E_{ijkl} = E_{\alpha \beta \gamma \delta} I \quad (2.3)$$

for each $\{\alpha, \beta, \gamma, \delta\} \subset \{1,2,3,4\}$. The matrix $\mathfrak{u} = (u_{jk})_{k=1}^{4}$ is the fundamental unitary co-representation of the quantum group. In our case the co-representation $\mathfrak{u} = (u_{kl})_{k,l=1}^{4}$ is reducible by the following reason. The operator $P = (E^* \otimes I)(I \otimes E)$, which acts on $\mathbb{C}^4$, intertwines the fundamental representation with itself: $(P \otimes I) \mathfrak{u} = \mathfrak{u} (P \otimes I)$. Moreover, $P$ has a diagonal matrix for the standard basis of $\mathbb{C}^4 : P = \text{diag}\{c_1, c_2, c_3, c_4\}$, with $c_j = \sum_{\alpha, \beta, \gamma \delta} E_{j\alpha \beta \gamma} E_{\alpha \beta \gamma j}$, and therefore $c_1 = c_4 = -(5q^3 + q^5)$, $c_2 = c_3 = -(2q^3 + 4q^5)$. Hence, for $q \neq 0, -1, 1$, which shall be the case in the sequel, $c_1 \neq c_2$, so $P$ is not a multiple of the identity operator $I$. The condition $(P \otimes I) \mathfrak{u} = \mathfrak{u} (P \otimes I)$ is equivalent to $c_j \cdot u_{jk} = c_k \cdot u_{jk}$ for all natural numbers $1 \leq j, k \leq 4$. This yields $u_{12} = u_{21} = 0, u_{13} = u_{31} = 0, u_{24} = u_{42} = 0, u_{34} = u_{43} = 0$, and therefore

$$
\mathfrak{u} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & u_{22} & u_{23} & 0 \\
0 & u_{32} & u_{33} & 0 \\
u_{41} & 0 & 0 & u_{44}
\end{pmatrix} = \begin{pmatrix}
a & 0 & 0 & b \\
0 & x & y & 0 \\
0 & z & w & 0 \\
c & 0 & 0 & d
\end{pmatrix} \quad (2.4)

This yields the decomposition of $\mathfrak{u}$ decomposes into two irreducible subrepresentations

$$\mathfrak{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus \begin{pmatrix} x & y \\ z & w \end{pmatrix} \quad (2.5)$$
Substitution in (2.3) of appropriate sequences \((\alpha, \beta, \gamma, \delta)\) gives the following relations between the generators of the \(C^*\)-algebra \(\mathcal{A}\) (the associated sequence is left of the relation):

\[
\begin{align*}
(1423) \quad & I = (ad - qbc)(xw - q^{-1}yz) \quad (1) \\
(1432) \quad & I = (ad - qbc)(wx - qzy) \quad (3) \\
(2314) \quad & I = (xw - q^{-1}yz)(ad - qbc) \quad (5) \\
(3214) \quad & I = (wx - q^{-1}yz)(ad - qbc) \quad (7)
\end{align*}
\]

Let \(W = ad - qbc\) and \(V = xw - q^{-1}yz\), then the above relation give \(VW = I = WV\) and also \(W = da - q^{-1}cb, V = wx - qzy\). Hence these relations are pairwise equivalent: (1) \(\Leftrightarrow\) (5), (2) \(\Leftrightarrow\) (6), (3) \(\Leftrightarrow\) (7) and (4) \(\Leftrightarrow\) (8). The operators \(V, W\), being the inverse of each other, are twisted determinants for the two matrix co-representations:

\[
W = \det_q \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad V = \det_{q^{-1}} \begin{pmatrix} x & y \\ z & w \end{pmatrix}.
\]

Let us observe here that a change of order in the basis for \(\begin{pmatrix} x & y \\ z & w \end{pmatrix}\) gives us the matrix \(\begin{pmatrix} w & z \\ y & x \end{pmatrix}\) which satisfies the same relations and for which the twisted determinant is

\[
\det_q \begin{pmatrix} w & z \\ y & x \end{pmatrix} = wx - qzy = V.
\]

Using the invertibility of \(W\) and \(V\) one can easily get the following relations:

\[
\begin{align*}
(1123) \quad & ab = qba \quad (9) \\
(4423) \quad & yx = qxy \quad (11) \\
(2214) \quad & cd = qdc \quad (10) \\
(3314) \quad & wz = qzw \quad (12)
\end{align*}
\]

In addition, the relations (2.2) can be written as:

\[
\begin{align*}
I &= aa^* + bb^* \quad (13) \\
I &= cc^* + dd^* \quad (14) \\
I &= a^*a + c^*c \quad (15) \\
I &= b^*b + d^*d \quad (16) \\
0 &= a^*b + c^*d \quad (17) \\
0 &= ca^* + db^* \quad (18)
\end{align*}
\]

and

\[
\begin{align*}
I &= xx^* + yy^* \quad (19) \\
I &= zz^* + ww^* \quad (20) \\
I &= x^*x + z^*z \quad (21) \\
I &= y^*y + w^*w \quad (22) \\
0 &= x^*y + z^*w \quad (23) \\
0 &= xx^* + wy^* \quad (24)
\end{align*}
\]

Multiplication of (16) from the left by \(a^*\) and using (9) and then (17) gives the equation \(d^*W = a\), or, equivalently, \(d = V^*a^*\). On the other hand, multiplication (15) from the right by \(d\) and using (10) and (17) gives \(d = a^*W\). These two combined ensure also that \(W^*a = aV\). Similarly, by multiplying (16) from the right by \(c\) and using (10) and then (17) one gets \(b^*W = -qc\), or equivalently, \(b^* = -qcV\). Then, multiplying (15) from the right by \(b\) and using (9) and (17) one obtains \(b = -qc^*W\). These two yield also \(cV = W^*c\). Therefore we have

\[
d = V^*a^* = a^*W, \quad b = -qV^*c^* = -qc^*W; \tag{2.8}
\]
\[ x = w^*V = W^*w^*, \quad z = -qy^*V = -qW^*y^*. \tag{2.9} \]

There are also other relations obtained from (2.3). They are listed in the following, with the associated sequences \((\alpha \beta \gamma \delta)\) on the left-hand side:

\[
\begin{align*}
(2143) & \quad I = x(ad - qbc)w - qy(ad - q^{-1}bc)z & \tag{25} \\
(2413) & \quad I = x(da - q^{-1}cb)w - q^{-1}y(da - qcb)z & \tag{26} \\
(3142) & \quad I = w(ad - q^{-1}bc)x - q^{-1}z(ad - qbc)y & \tag{27} \\
(3412) & \quad I = w(da - qcb)x - qz(da - q^{-1}cb)y & \tag{28}
\end{align*}
\]

and

\[
\begin{align*}
(1234) & \quad I = a(wx - qyz)d - qb(wx - qyz)c & \tag{29} \\
(4231) & \quad I = d(wx - q^{-1}zy)a - q^{-1}c(wx - q^{-1}zy)b & \tag{30} \\
(1324) & \quad I = a(wx - q^{-1}zy)d - q^{-1}c(wx - q^{-1}zy)b & \tag{31} \\
(4321) & \quad I = d(wx - q^{-1}zy)a - q^{-1}c(wx - q^{-1}zy)b & \tag{32}
\end{align*}
\]

From now on we shall assume the following additional relation:

\[ V = W^* \tag{2.10} \]

meaning that the twisted determinants are unitary operators. This yields that we are dealing with the quantum groups \(U_q(2)\) (for the generators \(a, b, c, d\) and another copy of \(U_q(2)\) (for the generators \(w, y, z, x\)). This assumption is also necessary to allow the technical procedure used in [W3].

Let us substitute (2.8) into the (1) - (32). In (1) - (8) we do the substitution in one of the bracket and put \(V\) or \(V^*\) for the other. Thus for each equation we get two:

\[
\begin{align*}
V a V^* a^* + q^2 c^* c &= 1 & \tag{1'a} \\
a^* a + V c V^* c^* &= 1 & \tag{2'a} \\
V a V^* a^* + q^2 c^* c &= 1 & \tag{3'a} \\
a^* a + V c V^* c^* &= 1 & \tag{4'a}
\end{align*}
\]

\[
\begin{align*}
w V w V^* + y y^* &= 1 & \tag{1'b} \\
w V w V^* + y y^* &= 1 & \tag{2'b} \\
V w w^* + q^2 y^* y V V^* &= 1 & \tag{3'b} \\
w w^* + q^2 y^* y V V^* &= 1 & \tag{4'b}
\end{align*}
\]

We see that \((1'a) \iff (3'a), (2'a) \iff (4'a), (1'b) \iff (2'b)\) and \((3'b) \iff (4'b)\). For (9) - (12) we obtain:

\[
\begin{align*}
c V a^* &= qa^* c V & \tag{9'} \\
w V c^* &= q c^* a V & \tag{10'} \\
w y^* V &= q w^* V y & \tag{11'} \\
w y^* V &= q y^* V w & \tag{12'}
\end{align*}
\]

The relation (13) - (18) give:

\[
\begin{align*}
a a^* + q^2 V^* c^* c V &= 1 & \tag{13'} \\
c c^* + V^* a^* a V &= 1 & \tag{14'} \\
a^* a + c^* c &= 1 & \tag{15'} \\
a a^* + q^2 c c^* &= 1 & \tag{16'} \\
V c V^* &= q c V a & \tag{17'} \\
V c a^* &= q a^* c V & \tag{18'}
\end{align*}
\]

and for (19) - (24) we get:

\[
\begin{align*}
w^* w + y y^* &= 1 & \tag{19'} \\
w w^* + q^2 y^* y &= 1 & \tag{20'} \\
w w^* + q^2 y^* y &= 1 & \tag{21'} \\
w y^* &= q y w & \tag{23'} \\
w y^* &= q y w & \tag{24'}
\end{align*}
\]
Let us first deal with the relations (2.14) involving \( w \) and \( y \). Comparing (19') with (21') one gets easily that \( y \) is normal: \( yy^* = y^*y \). Comparing (3'\( b \)) with (20') gives

\[
y^*Vy = y^*Vw
\]  
(2.15)

and (1'\( b \)) with (19') yield

\[
w^*Vw = w^*Vw, \tag{2.16}
\]

Putting (24') into (11') gives

\[
w^*Vy = w^*Vy
\]  
(2.17)

Multiplying both sides of (2.16) this from the left by \( w \) provides \( ww^*yV = wVwV \). Similarly, multiplying (2.14) from the right by \( y \) gives \( yy^*Vw = y^*Vw \). Adding these two side by side yields

\[
Vy = yV. \tag{2.18}
\]

In a similar manner one gets

\[
Vw = wV. \tag{2.19}
\]

This requires putting (24') into (12') to get \( y^*wV = y^*Vw \) which is then multiplied from the left by \( q^2y \) and added side by side to \( ww^*Vw = wVwV \), which is obtained from (2.15). These can be collected together as the following relations:

\[
\begin{align*}
w^*w + y^*y &= 1 \\
w^*Vw &= w^*Vw \\
wy &= qyw \\
y^*Vw &= y^*Vw \\
yV &= yV
\end{align*}
\]
(2.20)

The fundamental co-representation is thus \( \begin{pmatrix} w^*V & y \\ -qy^*V & w \end{pmatrix} \) and the above relations define the \( C^* \)-algebra of \( U_q(2) \), and \( V \) is the \((-q)^{-1}\)-determinant.

Let us now work with the relations for \( a \) and \( c \). From (4') and (15') one deduces that \( cVc^* = c^*cV \). Then, multiplying (9') from the right by \( a \) one gets \( cVaa^* = qa^*cVc \). The left-hand side of this can be transformed as follows (using (15')):

\[
cVaa^* = cV(1 - c^*c) = cV - (cVc^*)c = cV - c^*cVc.
\]

For the right-hand side one can use (17') and then (15') to get:

\[
qa^*cVc = a^*aVc = (1 - cc^*)Vc = Vc - c^*cVc.
\]

It follows from these two that \( cV = Vc \), and also \( c^*V = Ve^* \), since \( V \) is unitary. Using this combined with (14') and (15') one obtains \( cc^* = c^*c \), so \( c \) is normal. Then from (10') follows \( ac^* = qc^*a \). Comparing (1'a) with (16') one concludes \( aVa^* = aa^*V \). Then, multiplication of (17') by \( c^* \) from the right gives \( VVc^* = qcVac^* \). The left-hand side of this is \( aV - aa^*Va \). The right-hand side of this can be transformed, with the help of the above relations, into:

\[
qcVac^* = q^2Vc^*a = q^2c^*Vc = Va - aa^*Va.
\]
Hence one concludes $aV = Va$, and also $a^*V = Va^*$. Therefore the above relations may be written as follows:

$$
\begin{align*}
  a^*a + c^*c &= 1 \\
aa^* + q^2cc^* &= 1 \\
aV &= Va \\
cV &= Vc \\
a^*V &= Va^*
\end{align*}
$$

For $N = 4$ we have more nontrivial relations between $a, c, w, y$ given by (2.3) then in the case $N = 3$, since, for example the sequence $(1, 1, 2, 2)$ gives a nontrivial relation here, and gave trivial relation there. Let us write them as follows, indicating the associated sequence $(\alpha, \beta, \gamma, \delta)$ on the left-hand side of it and successive numbering on the right-hand side of it. In the first set of equations we put elements from the same $C^*$-subalgebra outside, and the other inside.

\begin{align*}
  (1231) \quad a(xw - qyz)b &= qb(xw - qyz)a \\
  (1321) \quad a(wx - \frac{1}{q}zy)b &= qb(wx - \frac{1}{q}zy)a \\
  (4234) \quad c(xw - qyz)d &= qd(xw - qyz)c \\
  (4324) \quad c(wx - \frac{1}{q}zy)d &= qd(wx - \frac{1}{q}zy)c \\
  (2142) \quad x(ad - qbc)y &= qy(ad - \frac{1}{q}bc)x \\
  (2412) \quad y(da - qcb)x &= qx(da - \frac{1}{q}cb)y \\
  (3143) \quad z(ad - qbc)w &= qw(ad - \frac{1}{q}bc)z \\
  (3413) \quad w(da - qcb)z &= qz(da - \frac{1}{q}cb)w
\end{align*}

In the second set of equations we have alternating sequences of elements from different $C^*$-subalgebras.

\begin{align*}
  (1243) \quad axdw - qaydz - qbxcw + q^2bycz &= I \\
  (4213) \quad dxaw - qdyaz - \frac{1}{q}cxbw + cybz &= I \\
  (1342) \quad awdx - \frac{1}{q}azdy - qbwcx + bcz &= I \\
  (4312) \quad dwax - \frac{1}{q}dzay - \frac{1}{q}cwbx + \frac{1}{q^2}czby &= I \\
  (2134) \quad xawd - qxbw - qyazd + q^2ybzc &= I \\
  (3124) \quad wazd - qwbxc - \frac{1}{q}zayd + zbyc &= I \\
  (2431) \quad xdwa - \frac{1}{q}xcwb - qydza + yczb &= I \\
  (3421) \quad wdxa - \frac{1}{q}wcxb - \frac{1}{q}zdy + \frac{1}{q^2}zeyb &= I
\end{align*}

Computing

\begin{align*}
  xw - qyz &= V - (1 - q^2)yy^*V \\
  wx - \frac{1}{q}zy &= V + (1 - q^2)yy^*V \\
  ad - \frac{1}{q}bc &= V^* + (1 - q^2)cc^*V^* \\
  da - qcb &= V^* - (1 - q^2)cc^*V^*
\end{align*}
and substituting these into (2.22) one obtains

\[
\begin{align*}
axy c^* &= qcy a^* = q^2 cy c^* \quad (33'), (34') \\
awy &= qyyc w^* \quad (37'), (39') \\
ycc w^* &= 0 \quad (38') \\
wcc y^* &= 0 \quad (40') \\
aw y^* + q^2 c w y c &= 0 \quad (41'), (43') \\
a^* w y a + cw y c^* &= 0 \quad (42'), (44') \\
yac y^* &= 0 \quad (45') \\
wac w^* &= 0 \quad (46') \\
ya c y^* &= 0 \quad (47') \\
wac c w^* &= 0 \quad (48')
\end{align*}
\]

Unfortunately, (37') combined with (38') give

\[w^* y = 0\]

and it follows from (2.20) that \( y = 0 \). To see this let us observe that \( w w^* y y^* + q^2 y y^* y y^* = y y^* \) implies \( q^2 (yy^*)^2 = yy^* \), and hence, by induction, \( q^{2n} (yy^*)^{n+1} = yy^* \) for any positive integer \( n \in \mathbb{N} \). This yields that the spectral radius \( r(yy^*) = \lim_{n \to \infty} \left\| (yy^*)^n \right\|^\frac{1}{n} \) satisfies \( r(yy^*) = q^2 > 1 \).

However, it follows from the description of the irreducible representations of the relations (2.20) (see [W3]) that \( \|y\| \leq 1 \), so that \( r(yy^*) \leq 1 \). This is a contradiction, except \( y = 0 \).

Then \( x w = V = wx \) and \( x x^* = 1 = x^* x \), \( w w^* = 1 = w^* w \), so that \( x, w \) are unitary. Moreover \( x = w^* V \), so that for the fundamental co-representation eventually we get \( \begin{pmatrix} w^* V & 0 \\ 0 & w \end{pmatrix} \). In a similar manner one gets that

\[a^* c = 0\]

and hence \( c = 0 \). Substitution of these to (2.23) gives

\[awa^* w = 1 = a^* w^* a w.\]

If we set \( t := aw \) and \( s := wa \), then \( tt^* = 1 = t^* t, ss^* = 1 = s^* s \) and \( ts^* = 1 = s^* t \). Therefore \( t = s \), which gives \( aw = wa \).

These computations show that the \( C^* \)-algebra of the constructed quantum group is generated by three commuting unitaries \( a, w, V \), so it is isomorphic to \( C(T) \otimes C(T) \otimes C(T) \). Therefore, the quantum group we consider is in fact the classical group \( U(1) \times U(1) \times U(1) \).

### 3 The Yang-Baxter operator associated with \( U_q(2) \)

In the next two Sections we are going to show a construction of a cubic Hecke algebra associated with the quantum group \( U_q(2) \). In [W3] we gave a construction of the quantum group \( U_q(2) \), in which the crucial role is played by the function counting the number of cycles in permutations from the symmetric group \( S_3 \). Namely, by considering the function \( S_3 \ni \sigma \mapsto (-q)^{v(\sigma)} \), where \( c(\sigma) \) is the number of cycles and \( q > 0 \), we constructed the following array:

\[
\begin{align*}
E_{1,2,3} &= 1 \\
E_{2,3,1} &= E_{3,1,2} = q^2 \\
E_{1,3,2} &= E_{2,1,3} = E_{3,2,1} = -q \quad (37'), (39') \\
E_{i,j,k} &= 0 \text{ if } \{i, j, k\} \not\subseteq \{1, 2, 3\}
\end{align*}
\]
This array defines an operator \( \rho \) on \( \mathbb{C}^3 \otimes \mathbb{C}^3 \) by

\[
\rho : \mathbb{C}^3 \otimes \mathbb{C}^3 \ni (a, b) \mapsto \sum_{i,j,k=1}^{3} E_{i,j,k}E_{k,a,b}(i,j) \in \mathbb{C}^3 \otimes \mathbb{C}^3,
\]  

(3.26)

where \((a, b)\) denotes in short the standard basis element \( \epsilon_a \otimes \epsilon_b \). In particular \( \epsilon_1 = (1, 0, 0) \), \( \epsilon_2 = (0, 1, 0) \) and \( \epsilon_3 = (0, 0, 1) \).

The definition of \( E \) implies that (3.26) simplifies to

\[
\rho(a, b) = E_{a,b,k}E_{k,a,b}(a, b) + E_{ba,k}E_{k,a,b}(b, a), \quad \text{where} \quad \{a, b, k\} = \{1, 2, 3\}
\]  

(3.27)

for \( a \neq b \) and \( a, b = 1, 2, 3 \). If \( a = b \) then we get \( \rho(a, a) = 0 \). The formulas can be written explicitly as follows.

\[
\begin{align*}
\rho(1, 2) &= E_{1,2,3}E_{3,1,2}(1, 2) + E_{2,1,3}E_{3,1,2}(2, 1) = q^2(1, 2) + q^3(2, 1) \\
\rho(2, 1) &= E_{2,1,3}E_{3,2,1}(2, 1) + E_{1,2,3}E_{3,2,1}(1, 2) = q^2(2, 1) + q(1, 2) \\
\rho(1, 3) &= E_{1,3,2}E_{2,1,3}(1, 3) + E_{3,1,2}E_{2,1,3}(3, 1) = q^2(1, 3) + q^3(3, 1) \\
\rho(3, 1) &= E_{3,1,2}E_{2,3,1}(3, 1) + E_{1,3,2}E_{2,3,1}(1, 3) = q^4(1, 3) + q^3(1, 3) \\
\rho(2, 3) &= E_{2,3,1}E_{1,2,3}(2, 3) + E_{3,2,1}E_{1,2,3}(3, 2) = q^2(2, 3) + q(3, 2) \\
\rho(3, 2) &= E_{3,2,1}E_{1,3,2}(3, 2) + E_{2,3,1}E_{1,3,2}(2, 3) = q^2(3, 2) + q^3(2, 3)
\end{align*}
\]

Therefore, the operator \( \alpha := I_2 - \frac{1}{q^2} \rho \) acts as: \( \alpha(a, a) = (a, a) \) for \( a = 1, 2, 3 \) and

\[
\begin{align*}
\alpha(1, 2) &= -q(2, 1) \\
\alpha(1, 3) &= -q(3, 1) \\
\alpha(3, 2) &= -q(2, 3) \\
\alpha(2, 1) &= -q^{-1}(1, 2) \\
\alpha(2, 3) &= -q^{-1}(3, 2) \\
\alpha(3, 1) &= (1 - q^2)(3, 1) - q(1, 3)
\end{align*}
\]  

(3.28)

This operator is not self-adjoint, but \( \alpha^2 = (\alpha^2)^* \) is so, since

\[
\begin{align*}
\alpha^2(1, 2) &= (2, 1) \\
\alpha^2(2, 1) &= (2, 1) \\
\alpha^2(2, 3) &= (3, 2) \\
\alpha^2(3, 2) &= (2, 3) \\
\alpha^2(1, 3) &= q^2(1, 3) - q(1 - q^2)(3, 1) \\
\alpha^2(3, 1) &= (1 - q^2 + q^4)(3, 1) - q(1 - q^2)(1, 3)
\end{align*}
\]  

(3.29)

The first important property of \( \alpha \) is that it is a Yang-Baxter operator.

**Proposition 3.1** The operator \( \alpha \) satisfies the Yang-Baxter equation

\[
(\alpha \otimes I)(I \otimes \alpha)(\alpha \otimes I) = (I \otimes \alpha)(\alpha \otimes I)(I \otimes \alpha).
\]  

(3.30)
Proof: Let $L = (\alpha \otimes I)(I \otimes \alpha)(\alpha \otimes I)$ be the left-hand side and $P = (I \otimes \alpha)(\alpha \otimes I)(I \otimes \alpha)$ be the right-hand side of (3.30). We have to show that $L(a, b, c) = P(a, b, c)$ for every $a, b, c \in \{1, 2, 3\}$ (with the notation: $(a, b, c) = \epsilon_a \otimes \epsilon_b \otimes \epsilon_c$). This requires checking 27 cases. It is clear that $L(a, a, a) = (a, a, a) = P(a, a, a)$ for any $a = 1, 2, 3$.

The direct calculation provides the following formulas for the other cases.

\[L(3, 2, 3) = (3, 2, 3) = P(3, 2, 3)\]
\[L(2, 3, 2) = (2, 3, 2) = P(2, 3, 2)\]
\[L(1, 2, 1) = (1, 2, 1) = P(1, 2, 1)\]
\[L(2, 1, 2) = (2, 1, 2) = P(2, 1, 2)\]
\[L(1, 2, 3) = -q(3, 2, 1) = P(1, 2, 3)\]
\[L(1, 3, 2) = -q^2(2, 3, 1) = P(1, 3, 2)\]
\[L(2, 1, 3) = -q^{-1}(3, 1, 2) = P(2, 1, 3)\]
\[L(2, 2, 3) = q^2(2, 3, 2) = P(2, 2, 3)\]
\[L(3, 3, 2) = q^2(2, 3, 3) = P(3, 3, 2)\]
\[L(2, 2, 3) = q^2(3, 2, 2) = P(2, 3, 2)\]
\[L(1, 1, 3) = q^2(3, 1, 1) = P(1, 1, 3)\]
\[L(1, 3, 3) = q^2(3, 3, 1) = P(1, 1, 3)\]
\[L(1, 1, 2) = q^2(2, 1, 1) = P(1, 1, 2)\]
\[L(1, 2, 2) = q^2(2, 2, 1) = P(1, 2, 2)\]
\[L(2, 3, 3) = q^{-2}(3, 3, 2) = P(2, 3, 3)\]
\[L(2, 1, 1) = q^{-2}(1, 1, 2) = P(2, 1, 1)\]
\[L(2, 2, 1) = q^{-2}(1, 2, 2) = P(2, 2, 1)\]

\[L(3, 2, 1) = (1 - q^2)(3, 2, 1) - q(1, 2, 3) = P(3, 2, 1)\]
\[L(3, 1, 2) = q^2(1 - q^2)(2, 3, 1) - q^3(2, 1, 3) = P(3, 1, 2)\]
\[L(2, 3, 1) = q^{-2}(1 - q^2)(3, 1, 2) - q^{-1}(1, 3, 2) = P(2, 3, 1)\]
\[L(1, 3, 1) = -q(1 - q^2)(3, 1, 1) + q^2(1, 3, 1) = P(1, 3, 1)\]
\[L(3, 1, 3) = -q(1 - q^2)(3, 3, 1) + q^2(3, 1, 3) = P(3, 1, 3)\]

From these formulas the Proposition follows. \(\square\)

4 The cubic Hecke algebra associated with \(U_q(2)\)

The second important property of the operator \(\alpha\) is that, even though it is not a Hecke operator, it does satisfy a cubic equation, and thus it generates a cubic Hecke algebra. This notion has been introduced by Funar in [F], where the cubic equation \(\alpha^3 - I = 0\) was considered.

Proposition 4.1 The operator \(\alpha\) satisfies the cubic equation:

\[(\alpha^2 - I)(\alpha + q^2 I) = 0.\] (4.34)
Proof: From the formulas (3.28), defining $\alpha$ it follows that it acts on the following subspaces by simple matricial formulas.

1. On the span of $(1, 2), (2, 1)$ as $\beta := \begin{pmatrix} 0 & -1 \\ -q & q \end{pmatrix}$

2. On the span of $(2, 3), (3, 2)$ as $\beta^* := \begin{pmatrix} 0 & -1 \\ -q & q \end{pmatrix}$

3. On the span of $(1, 3), (3, 1)$ as $\gamma := \begin{pmatrix} 0 & -q \\ -q & 1-q^2 \end{pmatrix}$

4. As identity on every $(a, a)$ with $a = 1, 2, 3$.

It is strightforward to see that $\beta^2 - I = 0 = (\beta^*)^2 - I$. On the other hand, since

$$\gamma^2 = \begin{pmatrix} q^2 & -q(1-q^2) \\ -q(1-q^2) & 1-q^2+q^4 \end{pmatrix},$$

we obtain

$$(\gamma^2 - I)(\gamma + q^2 I) = (q^2 - 1) \begin{pmatrix} 1 & q \\ q & q^2 \end{pmatrix} \begin{pmatrix} q^2 & -q \\ -q & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore both $\beta$ and $\gamma$ satisfy the equation (4.34), so the $\alpha$ docs.

Let us define the elements

$$h_j := I_j \otimes \alpha \otimes I_{n-j-2} \quad \text{for} \quad j = 1, \ldots, n-2,$$

where $I_k$ denotes the identity map on $(\mathbb{C}^N)^{\otimes k}$. Then by Propositions 3.1 and 4.1 the elements $h_1, \ldots, h_n$ generate a cubic Hecke algebra, associated with the quantum group $U_q(2)$.

Definition 4.2 The algebra $\mathcal{H}_{q,n}(2)$ generated by the elements $h_j, j = 1, \ldots, n$ defined by (4.35) will be called the cubic Hecke algebra associated with the quantum group $U_q(2)$.

The basic properties of this algebra are summarized in the following.

Theorem 4.3 The generators $\{h_j : 1 \leq j \leq n\}$ of $H_{q,n}(2)$ satisfy:

$$h_j h_{j+1} h_j = h_{j+1} h_j h_{j+1} \quad \text{for} \quad j = 1, \ldots, n-1,$$

$$h_j h_k = h_k h_j \quad \text{for} \quad |j - k| \geq 2,$$

$$(h_j^2 - 1)(h_j + q^2) = 0 \quad \text{for} \quad j = 1, \ldots, n.$$

The role of the Hecke algebra in the study of $SU_q(N)$ was that it was the intertwining algebra of the tensor powers of the fundamental co-representation. In [W3] the irreducible co-representations of $U_q(2)$ have been described, but it is not clear if the description is complete. So, it is still to be checked whether $\mathcal{H}_{q,n}(2)$ plays the same role as in $SU_q(N)$. 

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References


