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<tbody>
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Entangled Quantum Markov Chain satisfying Entanglement Condition

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Abstract

The entropic criterion of entanglement is applied to prove that entangled Markov chain with unitarily implementable transition operator is indeed an entangled state on infinite multiple algebras.

1 Introduction and Preliminaries

Accardi and Fidaleo [2] proposed a construction to relate, based on classical Markov chain with discrete state space, to a quantum Markov chain (in the sense of [1]) on infinite tensor products of type I factors. They called entangled Markov chain (EMC) the special class of quantum Markov chains obtained in this way.

Using the PPT entanglement criterion [13, 8] (positivity of the partial transpose of the density matrix) Miyadera showed [9] that the finite volume restriction of a class of EMC on infinite tensor products of $2 \times 2$ matrix algebras is entangled.

In our previous paper [3], using the entropic type of entanglement criterion for pure states [11, 3], which is based on the notion of degree of entanglement, we proved that the vector states defining the EMC's on infinite tensor products of $d \times d$ matrix algebras ($d \in \mathbb{N}$) "generically" are entangled (see Definition (3) below).

Our result did not include Miyadear's one because, by restricting an EMC to some local algebra, one obtains a mixed state to which the above criterion for a pure state is not applicable. However our entanglement criterion gives the sufficient condition for entanglement in the case of mixtures (for pure states this condition is necessary and sufficient) [4]. Moreover our entanglement criterion, being based on a numerical inequality, is in many cases easier to verify than the positivity condition required by the PPT criterion.
In this note we will show some results obtained in [4] with proof for the reader’s convenience. Our entanglement condition is applied to the restriction of EMC’s, generated by a unitarily implementable $d \times d$ stochastic matrix, to algebras localized which is obtained as a mixed state. This allows to prove that the above EMC induce an entangled state on infinite tensor products of $d \times d$ matrix algebras for any $d \in \mathbb{N}$.

We consider a classical Markov chain $(S_n)$ with state space $S = \{1, 2, \cdots, d\}$, initial distribution $p = (p_j)$ and transition probability matrix $P = (p_{ij})$

$$p_{ij} \geq 0 \quad ; \quad \sum_j p_{ij} = 1$$

Let $\{e_i\}_{i \leq d}$ be an orthonormal basis (ONB) of $\mathbb{C}^{|S|}$. For a fixed vector $e_0$ in this basis, denote

$$\mathcal{H}_N := \bigotimes_N^{(e_0)} \mathbb{C}^{|S|}$$

the infinite tensor product of $N$-copies of the Hilbert space $\mathbb{C}^{|S|}$ with respect to the constant sequence $(e_0)$. An orthonormal basis of $\mathcal{H}_N$ is given by the vectors

$$|e_{j_0}, \cdots, e_{j_n}\rangle := \left(\bigotimes_{\alpha \in [0, n]} e_{j_\alpha}\right) \otimes \left(\bigotimes_{\alpha \in [0, n]} e_{0}\right).$$

For any Hilbert space $\mathcal{H}$ we denote $\mathcal{H}^*$ its dual and $\xi \in \mathcal{H} \mapsto \xi^* \in \mathcal{H}^*$ the canonical embedding. Thus, if $\xi \in \mathcal{H}$ is a unit vector, $\xi \xi^*$ denotes the projection onto the subspace generated by $\xi$.

Let $M_d$ denote the algebra of complex $d \times d$ matrices and let $\mathcal{A} := \otimes_N M_d = M_d \otimes M_d \otimes \cdots$ be the $\mathbb{C}$-infinite tensor product of $N$-copies of $M_d$.

An element $A_\Lambda \in \mathcal{A}$ (observable) will be said to be localized in a finite region $\Lambda \subset \mathbb{N}$ if there exists an operator $\overline{A}_\Lambda \in \otimes \Lambda M_d$ such that

$$A_\Lambda = \overline{A}_\Lambda \otimes 1_{\Lambda^c}$$

In the following we will identify $A_\Lambda = \overline{A}_\Lambda$ and we denote $A_\Lambda$ the local algebra at $\Lambda$. Let $\sqrt{p_i}$ (resp. $\sqrt{p_{ij}}$) be any complex square root of $p_i$ (resp. $p_{ij}$) (i.e. $|\sqrt{p_i}|^2 = p_i$ (resp. $|\sqrt{p_{ij}}|^2 = p_{ij}$)) and define the vector

$$\Psi_n = \sum_{j_0, \cdots, j_n} \sqrt{p_{j_0}} \prod_{0 \leq \alpha < n} \sqrt{p_{j_\alpha j_{\alpha+1}}} |e_{j_0}, \cdots, e_{j_n}\rangle$$

(2)

Although the limit $\lim_{n \to \infty} \Psi_n$ will not exist, the basic property of $\Psi_n$ is the following [2].

**Proposition 1** There exists a unique quantum Markov chain $\psi$ on $\mathcal{A}$ such that, for every $k \in \mathbb{N}$ and for every $A \in \mathcal{A}_{[0, k]}$, one has

$$\langle \Psi_{k+1}, A \Psi_{k+1} \rangle = \lim_{n \to \infty} \langle \Psi_n, A \Psi_n \rangle =: \psi(A)$$

(3)

Moreover $\psi$ is stationary if and only if the associated classical Markov chain $\{p := (p_i) ; P = (p_{ij})\}$ is stationary, i.e.

$$\sum_i p_{i} p_{ij} = p_j \quad ; \quad \forall j$$

(4)
2 Notions of entanglement and degree of entanglement

Definition 2 Let \( A_j \ (j \in \{1, 2, \cdots, n\}) \) with \( n < \infty \) be \( C^* \)-algebras and let \( A = \bigotimes_{j=1}^{n} A_j \) be a tensor product of \( C^* \)-algebras. A state \( \omega \in S(\bigotimes_{j=1}^{n} A_j) \) is called separable if

\[
\omega \in \overline{\text{Conv}} \left\{ \bigotimes_{j=1}^{n} \omega_j ; \omega_j \in S(A_j), j \in \{1, 2, \cdots, n\} \right\}
\]

where \( \text{Conv} \) denotes norm closure of the convex hull.

A nonseparable state is called entangled.

Notice that the notion of separability may depend on the choice of the tensor product of \( C^* \)-algebras. Unless otherwise specified, one realizes the \( C^* \)-algebras on Hilbert spaces and one considers the induced tensor product. In any case a separable pure state must be a product of pure states.

Definition 3 \([3]\) In the notations of Definition (2) a state \( \omega \in S(A) \) is called 2-separable if

\[
\omega \in \overline{\text{Conv}} \left\{ \omega_k \otimes \omega_{(k} : \omega_k \in S(A_k), \omega_{(k} \in S(A_{(k}), \forall k \in \{1, 2, \cdots, n\} \right\}
\]

where \( A = A_k \otimes A_{(k} := A_{[1,k]} \otimes A_{(k,n]}. \)

A non-2-separable state is called 2-entangled.

Remark Notice that, for \( n = 2 \), 2-entanglement is equivalent to usual entanglement. For \( n > 2 \), 2-entanglement is a strictly stronger property than usual entanglement.

Definition 4 Let \( \mathcal{H}_1, \mathcal{H}_2 \) be separable Hilbert spaces and let \( \theta \) be density matrices in \( \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \) with its marginal densities denoted by \( \rho \) and \( \sigma \) in \( \mathcal{B}(\mathcal{H}_1), \mathcal{B}(\mathcal{H}_2) \) respectively.

The quasi mutual entropy of \( \rho \) and \( \sigma \) w.r.t \( \theta \) is defined by \([10]\)

\[
I_{\theta}(\rho, \sigma) \equiv \text{tr} \theta (\log \theta - \log \rho \otimes \sigma)
\]

The degree of entanglement of \( \theta \), denoted by \( D_{EN}(\theta) \), is defined by \([11]\)

\[
D_{EN}(\theta) \equiv \frac{1}{2} \{ S(\rho) + S(\sigma) \} - I_{\theta}(\rho, \sigma)
\]

where \( S(\cdot) \) is the von-Neumann entropy.

In the following we identify normal states on \( \mathcal{B}(\mathcal{H}) \) (\( \mathcal{H} \) some separable Hilbert space) with their density matrices and, if \( \theta \) is such a state, we will use indifferently the notations

\[
\theta(x) = \text{tr}(\theta x) \quad ; \quad x \in \mathcal{B}(\mathcal{H})
\]
Recalling that, for density operators $\theta$, $\delta$ in $\mathcal{B}(\mathcal{H})$, the relative entropy of $\delta$ with respect to $\theta$ is defined by:

$$R(\theta|\delta) := tr\theta(\log \theta - \log \delta)$$

(8)

(see [5, 12] for a more general discussion) we see that $I_\theta(\rho, \sigma)$ is the relative entropy of the tensor product of its marginal densities with respect to $\theta$ itself. Since it is known that the relative entropy is a kind of distance between states, it is clear why the degree of entanglement of $\theta$ by (6) is a measure of how far $\theta$ is from being a product state. Moreover we see also that $D_{EN}$ is a kind of symmetrized quantum conditional entropy. In the classical case the conditional entropy always takes non-negative value, however our new criterion can be negative according to the strength of quantum correlation between $\rho$ and $\sigma$ [4].

**Theorem 5** A necessary condition for a (normal) state $\theta$ on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ to be separable is that

$$D_{EN}(\theta) \geq 0$$

(9)

Equivalently: a sufficient condition for $\theta$ to be entangled is that

$$D_{EN}(\theta) < 0.$$  

(10)

**Proof.** Let $\theta$ be a state on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. If $\theta$ is separable, there exist density matrices $\rho_n$, $\sigma_n$ respectively in $\mathcal{B}(\mathcal{H}_1)$, $\mathcal{B}(\mathcal{H}_2)$ such that

$$\theta = \sum_n p_n \rho_n \otimes \sigma_n$$

with

$$p_n \geq 0, \quad \forall n \quad ; \quad \sum_n p_n = 1$$

Let $\{x_n\}$ be an ONB in $\mathcal{H}_1$ and define the completely positive unital (CP1) map $\Lambda_0 : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_1)$ by

$$\Lambda_0(A) = \sum_n tr(A \rho_n)x_n x_n^* \quad ; \quad A \in \mathcal{B}(\mathcal{H}_1)$$

(11)

Then its dual is

$$\Lambda_0^*(\delta) = \sum_n \langle x_n, \delta x_n \rangle \rho_n \quad ; \quad \delta \in \mathcal{B}(\mathcal{H}_1)_*$$

(12)

so that defining the CP1 map

$$\Lambda := \Lambda_0 \otimes id : \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

and the density matrix

$$\theta_d := \sum_n p_n x_n x_n^* \otimes \sigma_n$$
one easily verifies that
\[ \Lambda^*(\theta_d) = \theta \]

Moreover, denoting
\[ \rho = \sum_n p_n \rho_n \quad \text{and} \quad \sigma = \sum_n p_n \sigma_n \]
the marginal densities of \( \theta \) and \( \rho_d = \sum_n p_n |x_n\rangle \langle x_n| \) the first marginal density of \( \theta_d \), one has:
\[ \Lambda^* (\rho_d \otimes \sigma) = \rho \otimes \sigma \]

Recall now that the monotonicity property of the relative entropy (see [12] for proof and history) that for any pair of von Neumann algebras \( \mathcal{M}, \mathcal{M}^0 \), for any normal CP1 map \( \Lambda : \mathcal{M} \to \mathcal{M}^0 \) and for any pair of normal states \( \omega_0, \varphi_0 \) on \( \mathcal{M}^0 \) one has
\[ R(\Lambda^*(\omega_0)|\Lambda^*(\varphi_0)) \leq R(\omega_0|\varphi_0) \tag{13} \]

Using this property one finds:
\[ I_{\theta}(\rho, \sigma) = R(\theta|\rho \otimes \sigma) = R(\Lambda^*(\theta_d)|\Lambda^*(\rho_d \otimes \sigma)) \leq R(\theta_d|\rho_d \otimes \sigma) = I_{\theta_d}(\rho_d, \sigma) \]
so that
\[ S(\sigma) - I_{\theta}(\rho, \sigma) \geq S(\sigma) - I_{\theta_d}(\rho_d, \sigma) = - \sum_n p_n tr(\sigma_n \log \sigma_n) \geq 0 \tag{14} \]

Introducing the density operator
\[ \hat{\theta}_d = \sum_n p_n \rho_n \otimes y_n y_n^* \]
where \( \{y_n\} \) is an ONB in \( \mathcal{H}_2 \), and using a variant of the above argument (in which the density \( \theta_d \) is replaced by \( \hat{\theta}_d \)) one proves the analogue inequality
\[ S(\rho) - I_{\theta}(\rho, \sigma) \geq S(\rho) - I_{\theta_d}(\rho, \sigma_d) = - \sum_n p_n tr(\rho_n \log \rho_n) \geq 0 \tag{15} \]

Combining (14) and (15) one obtains:
\[ D_{EN}(\theta; \rho, \sigma) = \frac{1}{2} ((S(\sigma) - I_{\theta}(\rho, \sigma)) + (S(\rho) - I_{\theta}(\rho, \sigma))) \geq \tag{16} \]
\[ \geq \frac{1}{2} \left( - \sum_n p_n tr(\rho_n \log \rho_n) - \sum_n p_n tr(\sigma_n \log \sigma_n) \right) \geq 0 \]
which is (9).

\[ \blacksquare \]

**Remark** For pure (normal) states \( \theta \) on \( B(\mathcal{H}_1 \otimes \mathcal{H}_2) \) condition (10) is also necessary for entanglement (see [11, 3]).
3 The localized EMC and its marginal states

We discuss the entanglement of the finite volume restrictions of a class of EMC on infinite tensor products of $d \times d$ matrix algebras. By restricting an EMC to some local algebra one obtains a mixed state to which our entanglement criterion $D_{EN}$ is applicable because of theorem 5. In the following arguments we will denote $u_{ij} = \sqrt{p_{ij}}$ any (fixed) complex square root of $p_{ij}$ so that

$$|u_{ij}|^2 = p_{ij} \quad ; \quad \forall i, j$$

and we assume that $U = (u_{ij})$ is a unitary matrix.

Let denote the unitarily implementable EMC state restricted to a finite region $[0, \nu]$ by $\rho_{[0, \nu]}$, then for every local observable $A \in \mathcal{A}_{[0, \nu]}$ one has $\rho_{[0, \nu]}(A) = \langle \Psi_{\nu+1}, (A \otimes I) \Psi_{\nu+1} \rangle$. Hence the density operator $\rho_{[0, \nu]}$ is given by:

$$\rho_{[0, \nu]} = \text{tr}_{\mathcal{H}_{\nu+1}} |\Psi_{\nu+1}\rangle \langle \Psi_{\nu+1}|$$

$$= \sum_{i_0, \ldots, i_{\nu+1}, j_0, \ldots, j_{\nu+1}, l_{\nu+1}} \sqrt{p_{i_0}} \prod_{\alpha=0}^{\nu} u_{i_\alpha i_{\alpha+1}}^{*} u_{j_\alpha j_{\alpha+1}}$$

$$\langle e_{l_{\nu+1}}, e_{j_{\nu+1}} \rangle \langle e_{i_{\nu+1}}, e_{j_{\nu+1}} \rangle |e_{l_0}, \ldots, e_{j_0}, \ldots, e_{l_{\nu+1}}, e_{j_{\nu+1}}|$$

From the unitarity of $U = (u_{ij})$ one has $\sum_l u^*_{i_{l}l}u_{j_{l}j_{l}} = (UU^*)_{i_{l}i_{l}} = \delta_{i_{l}j_{l}}$. Using this unitarity one has

$$\rho_{[0, \nu]} = \sum_{j_0, j_1, \ldots, j_{\nu}} \sqrt{p_{j_0}} \prod_{\alpha=0}^{\nu-2} u_{i_\alpha i_{\alpha+1}}^{*} u_{j_\alpha j_{\alpha+1}}$$

$$u^*_{i_{\nu-1}k}u_{j_{\nu-1}k} |e_{j_0}, e_{j_1}, \ldots, e_{j_{\nu-1}}, e_{\nu}(k) \rangle \langle e_{i_0}, e_{i_1}, \ldots, e_{j_{\nu-1}}, e_{\nu}(k)|$$

$$= \sum_k p_k e_{[0, \nu]}(k) e_{[0, \nu]}^*(k), \quad (17)$$

where

$$e_{[0, \nu]}(k) := \frac{1}{\sqrt{p_k}} \sum_{j_0, \ldots, j_{\nu-1}} \sqrt{p_{j_0}} \prod_{\alpha=0}^{\nu-2} u_{j_\alpha j_{\alpha+1}} u_{j_{\nu-1}k} |e_{j_0}, \ldots, e_{j_{\nu-1}}, e_{\nu}(k)\rangle.$$
The vectors \( \{ e_{[0, \nu]}(k) \}_k \) are normalized and orthogonal each other. In fact

\[
\| e_{[0, \nu]}(k) \|^2 = \frac{1}{p_k} \sum_{j_{\mu+1}, j_1, \cdots, j_{\nu-1}} p_{j_{\mu+1}} \prod_{\alpha=\mu+1}^{\nu-2} p_{j_{\alpha} j_{\alpha+1}} p_{j_{\nu-1} k} = \frac{1}{p_k} \sum_{j_{\nu-1}} p_{j_{\nu-1}} p_{j_{\nu-1} k} = \frac{p_k}{p_k} = 1,
\]

and the orthogonality of \( \{ e_{[0, \nu]}(k) \}_k \) is clear because of the orthogonality of \( \{ e_\nu(k) \}_k \). We see that the decomposition (17) gives a Schatten decomposition.

Let us consider the marginal states of density \( \rho_{[0, \nu]} \) for each \( \mu \in [0, \nu-1] \) given by

\[
\rho_{[0, \nu]} = tr_{H_{(\mu, \nu]}} \rho_{[0, \nu]}, \quad \rho_{\mu} = tr_{H_{[0, \mu]}} \rho_{[0, \nu]},
\]

(18)

Since, by Proposition (1), the family \( \rho_{[0, \nu]} \) is projective, for each \( \mu \in [0, \nu-1] \) the restriction of \( \rho_{[0, \nu]} \) to the algebra localized on \([0, \mu]\) is \( \rho_{[0, \mu]} \). This implies

\[
\rho_{[0, \nu]} = tr_{H_{(\mu, \nu]}} \rho_{[0, \nu]} = \rho_{[0, \mu]},
\]

(19)

On the other hand the marginal state \( \rho_{\mu} \) is given by

\[
\rho_{\mu} = tr_{H_{[0, \mu]}} \rho_{[0, \nu]} = \sum p_{j_{0}, \cdots, j_{\mu+1}, \cdots, j_{\nu-1}, i_{\mu+1}, \cdots, i_{\nu-1}, k} u_{j_{0} i_{\mu+1}}^* u_{j_{0} j_{\mu+1}} \prod_{\alpha=0}^{\mu-1} u_{j_{\alpha} j_{\alpha+1}}^* u_{j_{\alpha} j_{\alpha+1}} u_{i_{\nu-1} k}^* u_{j_{\nu-1} k} |e_{j_{\mu+1}}, \cdots, e_{j_{\nu-1}}, e_{\nu}(k) \rangle \langle e_{i_{\mu+1}}, \cdots, e_{i_{\nu-1}}, e_{\nu} (k)|
\]

\[
= \sum_{n, j_{\mu+1}, \cdots, j_{\nu-1}, i_{\mu+1}, \cdots, i_{\nu-1}, k} p_n u_{j_{\mu+1} i_{\mu+1}}^* u_{j_{\mu+1} j_{\mu+1}} \prod_{\alpha=\mu+1}^{\nu-2} u_{j_{\alpha} j_{\alpha+1}}^* u_{j_{\alpha} j_{\alpha+1}} u_{i_{\nu-1} k}^* u_{j_{\nu-1} k} |e_{j_{\mu+1}}, \cdots, e_{j_{\nu-1}}, e_{\nu}(k) \rangle \langle e_{i_{\mu+1}}, \cdots, e_{i_{\nu-1}}, e_{\nu} (k)|
\]

\[
= \sum_{n, k} p_n e_{(\mu, \nu]}^n(k) e_{(\mu, \nu]}^n(k)^* \]

(20)

where

\[
e_{(\mu, \nu]}^n(k) = \sum_{j_{\mu+1}, \cdots, j_{\nu-1}} u_{j_{\mu+1} j_{\mu+1}} \prod_{\alpha=\mu+1}^{\nu-2} u_{j_{\alpha} j_{\alpha+1}} u_{j_{\nu-1} k} |e_{j_{\mu+1}}, \cdots, e_{j_{\nu-1}}, e_{\nu}(k) \rangle.
\]

Remark If we put

\[
\rho_{(\mu, \nu]}(n) := \sum_k e_{(\mu, \nu]}^n(k) e_{(\mu, \nu]}^n(k)^*, \quad (21)
\]
then it is shown that (21) can be recognized as an orthogonal decompositions of a density operator. In fact we can show the following properties of $\rho_{(\mu,\nu]}(n)$.

(i) Orthogonality:

$$\langle e^n_{(\mu,\nu]}(k), e^n_{(\mu,\nu]}(l) \rangle = \delta_{k,l},$$

(ii) Density:

$$\|e^n_{(\mu,\nu]}(k)\|^2 = \sum_{j_{\mu+1},\cdots,j_{\nu-1}} p_{nj_{\mu+1}} \left( \prod_{\alpha=\mu+2}^{\nu-2} p_{j_{\alpha}j_{\alpha+1}} \right) p_{j_{\nu-1}k}$$

$$\equiv \left( P^{\nu-(\mu+1)} \right)_{nk}. $$

This matrix $P^{\nu-(\mu+1)}$ can be recognized as a transition probability matrix generated by $P = (p_{ij})$ (i.e. a classical ergodic Markov chain). This implies

$$tr\rho_{(\mu,\nu]}(n) = \sum_k \left( P^{\nu-(\mu+1)} \right)_{nk} = 1.$$

Let denote $\hat{e}^n_{(\mu,\nu]}(k)$ the normalized vector i.e.

$$\hat{e}^n_{(\mu,\nu]}(k) = \frac{1}{\sqrt{\left( P^{\nu-(\mu+1)} \right)_{nk}}} e^n_{(\mu,\nu]}(k).$$

Then $\rho_{(\mu,\nu]}(n)$ is represented by

$$\rho_{(\mu,\nu]}(n) = \sum_k \left( P^{\nu-(\mu+1)} \right)_{nk} \hat{e}^n_{(\mu,\nu]}(k) \hat{e}^n_{(\mu,\nu]}(k)^*$$

(22)

which is a Schatten decomposition.

### 4 The DEN of EMC generated by unitarily implemtable channel

We can define the entanglement criterion of EMC $\varphi$ via the DEN of a localized EMC $\rho_{[0,\nu]}$. According to the definition of DEN one can compute the DEN of $\rho_{[0,\nu]}$ as follows:

$$DEN\left(\rho_{[0,\nu]};\rho_{\mu}\right) = \frac{1}{2} \left\{ S\left(\rho_{\mu}\right) + S\left(\rho_{(\mu)}\right) \right\} - I_{\rho_{[0,\nu]}} \left(\rho_{\mu}\right).$$

$$= \frac{1}{2} \left\{ S\left(\rho_{\mu}\right) + S\left(\rho_{(\mu)}\right) \right\} - \left\{ S\left(\rho_{\mu}\right) + S\left(\rho_{(\mu)}\right) - S\left(\rho_{[0,\nu]}\right) \right\}$$

$$= S\left(\rho_{[0,\nu]}\right) - \frac{1}{2} \left\{ S\left(\rho_{\mu}\right) + S\left(\rho_{(\mu)}\right) \right\}.$$
Definition 6 For a fixed $\mu \in \mathbb{N}$ we define the 2-entangled DEN of EMC $\varphi$ by
\[
D_{EN} \left( \varphi; \rho_{\mu}, \rho_{(\mu)} \right) \equiv \lim_{\nu \to \infty} D_{EN} \left( \rho_{[0,\nu]}, \rho_{\mu}, \rho_{(\mu)} \right),
\]
where $\nu > \mu$. The $D_{EN}$ of EMC $\varphi$ is defined by the infimum of the 2-entangled DEN.
\[
D_{EN} (\varphi) \equiv \inf_{\mu \in \mathbb{N}} D_{EN} \left( \varphi; \rho_{\mu}, \rho_{(\mu)} \right). \tag{24}
\]

Then we have the following result [4].

Theorem 7
\[
D_{EN} (\varphi) = -\frac{1}{2} H(P) < 0 \tag{25}
\]
where $H(P)$ is a Shannon entropy of a initial distribution of $P$.

Proof. The localized state $\rho_{[0,\nu]}$ is decomposed to (17) and its marginal state $\rho_{\mu}$ has a similar decomposition because of (19) which implies
\[
S (\rho_{[0,\nu]}) = S (\rho_{\mu}) = -\sum_{n=1}^{d} p_{n} \log p_{n} = H(P). \tag{26}
\]

On the other hand the another marginal state $\rho_{(\mu)}$ is decomposed to (20) which can not be recognized as a orthogonal decomposition in general. However we can estimate the orthogonality of the vectors $e_{[\mu,\nu]}^{n}(k)$ and $e_{[\mu,\nu]}^{m}(k)$ ($n \neq m$) asymptotically as follows:

\[
\langle e_{[\mu,\nu]}^{n}(k), e_{[\mu,\nu]}^{m}(k) \rangle = \sum_{j_{\mu+1}, \cdots, j_{\nu-1}} u_{nj_{\mu+1}}^{*} u_{mj_{\mu+1}} \left( \prod_{\alpha=\mu+1}^{\nu-2} p_{j_{\alpha}j_{\alpha+1}} \right) p_{j_{\nu-1}k} = \sum_{j_{\mu+1}} u_{nj_{\mu+1}}^{*} u_{mj_{\mu+1}} \left( P^{\nu-\mu-2} \right)_{j_{\mu+1}k} = \lim_{\nu \to \infty} \left( P^{\nu-\mu-2} \right)_{j_{\mu+1}k} = p_{k}
\]

From the ergodic property of $(P^{\nu-\mu-2})$ we have
\[
\lim_{\nu \to \infty} \left( P^{\nu-\mu-2} \right)_{j_{\mu+1}k} = p_{k}
\]

Therefore
\[
\lim_{\nu \to \infty} \langle e_{[\mu,\nu]}^{n}(k), e_{[\mu,\nu]}^{m}(k) \rangle = p_{k} \sum_{j_{\mu+1}} u_{nj_{\mu+1}}^{*} u_{mj_{\mu+1}} = p_{k}\delta_{n,m}. \tag{27}
\]
In large $\nu \gg 0$ we can estimate the orthogonality of $\{\rho_{(\mu,\nu]}(n)\}_n$ approximately
\[
\rho_{(\mu,\nu]}(n) \rho_{(\mu,\nu]}(m) \simeq 0 \quad (n \neq m).
\] (28)

It is known (see [12]) that, if a density operator $\rho$ is a convex combination of densities $\rho_n$,
\[
\rho = \sum_n \lambda_n \rho_n, \quad \lambda_n \geq 0, \quad \sum_n \lambda_n = 1
\]
then the following inequality holds:
\[
S(\rho) \leq \sum_n \lambda_n S(\rho_n) - \sum_n \lambda_n \log \lambda_n
\] (29)
and the equality holds if $\rho_n \perp \rho_m$ for $n \neq m$. Thanks to (28) we can apply the equality of (29) to $\rho_{(\mu} = \sum_n p_n \rho_{(\mu,\nu]}(n)$.

\[
\lim_{\nu \to \infty} S(\rho_{(\mu}) = \lim_{\nu \to \infty} S\left(\sum_n p_n \rho_{(\mu,\nu]}(n)\right)
= \sum_{n=1}^d p_n \sum_{k=1}^d p_k \log p_k - \sum_{n=1}^d p_n \log p_n
= 2H(P).
\] (30)

From the above arguments we have
\[
\lim_{\nu \to \infty} D_{EN}(\rho_{[0,\nu]; \rho_{\mu}], \rho_{(\mu}) = H(P) - \frac{1}{2} \{H(P) + 2H(P)\}
= -\frac{1}{2} H(P).
\] (31)

It is clear that the equation (31) holds for any $\mu \in \mathbb{N}$. This fact shows that the equation (25) holds.

This theorem says that the unitary implementable EMC is entangled state in the sense of definition 6. On the base of theorem 7 we can compute another entropic criteria, introduced in [6, 7], for EMC. As a result of such computations we can conclude that EMC gives an example of maximal entangled state on infinite multiple algebras. The detailed discussion will appear in a forthcoming paper [4].

References


