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Kyoto University
On the Uniqueness of Pairs of a Hamiltonian and a Strong Time Operator in Quantum Mechanics

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Abstract

Let $H$ be a self-adjoint operator (a Hamiltonian) on a complex Hilbert space $\mathcal{H}$. A symmetric operator $T$ on $\mathcal{H}$ is called a strong time operator of $H$ if the pair $(T, H)$ obeys the operator equation $e^{itH}Te^{-itH} = T + t$ for all $t \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers and $i$ is the imaginary unit). In this note we review some results on the uniqueness (up to unitary equivalences) of the pairs $(T, H)$.

Keywords: canonical commutation relation, Hamiltonian, strong time operator, weak Weyl relation, weak Weyl representation, Weyl representation, spectrum.

Mathematics Subject Classification 2000: 81Q10, 47N50

1 Introduction

A pair $(T, H)$ of a symmetric operator $T$ and a self-adjoint operator $H$ on a complex Hilbert space $\mathcal{H}$ is called a weak Weyl representation of the canonical commutation relation (CCR) with one degree of freedom if it obeys the weak Weyl relation: For all $t \in \mathbb{R}$ (the set of real numbers), $e^{-itH}D(T) \subset D(T)$ ($i$ is the imaginary unit and $D(T)$ denotes the domain of $T$) and

$$Te^{-itH}\psi = e^{-itH}(T + t)\psi, \quad \forall t \in \mathbb{R}, \forall \psi \in D(T).$$  (1.1)

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It is easy to see that the weak Weyl relation is equivalent to the operator equation
\[ e^{itH}T e^{-itH} = T + t, \quad \forall t \in \mathbb{R}, \] (1.2)
implying that \( e^{-itH}D(T) = D(T), \forall t \in \mathbb{R}. \)

One can prove that, if \((T, H)\) is a weak Weyl representation of the CCR, then 
\((T, H)\) obeys the CCR
\[ [T, H] = i \] (1.3)
on \(D(TH) \cap D(HT)\), where \([X, Y] := XY - YX\). But the converse is not true.

In the context of quantum theory where \(H\) is the Hamiltonian of a quantum system, \(T\) is called a strong time operator of \(H\) [3, 5].

We remark that a standard time operator (simply a time operator) of \(H\) is defined to be a symmetric operator \(T\) on \(\mathcal{H}\) obeying CCR (1.3) on a subspace \(\mathcal{D} \neq \{0\}\) (not necessarily dense) of \(\mathcal{H}\) (i.e., \(\mathcal{D} \subset D(TH) \cap D(HT)\) and \([T, H] \psi = i \psi, \forall \psi \in \mathcal{D}\) \) (cf. [1]). Obviously this notion of time operator is weaker than that of strong time operator. General classes of time operators (not strong ones) of a Hamiltonian with discrete eigenvalues have been investigated by Galapon [12], Arai-Matsuzawa [9] and Arai [7].

Weak Weyl representations of the CCR were first discussed by Schm{"u}dgen [19, 20] from a purely operator theoretical point of view and then by Miyamoto [14] in application to a theory of time operator in quantum theory. A generalization of a weak Weyl relation was presented by the present author [2] to cover a wider range of applications to quantum physics including quantum field theory.

Arai-Matsuzawa [8] discovered a general structure for construction of a weak Weyl representation of the CCR from a given weak Weyl representation and established a theorem for the former representation to be a Weyl representation of the CCR. These results were extended by Hiroshima-Kuribayashi-Matsuzawa [13] to a wider class of Hamiltonians.

In the previous paper [6] the author considered the problem on uniqueness (up to unitary equivalences) of weak Weyl representations. In the context of theory of time operators, this is a problem on uniqueness (up to unitary equivalences) of pairs \((T, H)\) with \(H\) a Hamiltonian and \(T\) a strong time operator of \(H\). This problem has an independent interest in the theory of weak Weyl representations. This note is a review of some results obtained in [6].

2 Preliminaries

We denote by \(W(\mathcal{H})\) the set of all the weak Weyl representations on \(\mathcal{H}\):
\[ W(\mathcal{H}) := \{(T, H) | (T, H) \text{ is a weak Weyl representation on } \mathcal{H}\}. \] (2.1)
It is easy to see that, if \((T, H)\) is in \(W(\mathcal{H})\), then so are \((\overline{T}, H)\) and \((-T, -H)\), where \(\overline{T}\) denotes the closure of \(T\).

For a linear operator \(A\) on a Hilbert space, \(\sigma(A)\) (resp. \(\rho(A)\)) denotes the spectrum (resp. the resolvent set) of \(A\) (if \(A\) is closable, then \(\sigma(A) = \sigma(\overline{A})\)). Let \(\mathbb{C}\) be the set of complex numbers and

\[
\Pi_+ := \{ z \in \mathbb{C} | \text{Im} \ z > 0 \}, \quad \Pi_- := \{ z \in \mathbb{C} | \text{Im} \ z < 0 \}. \tag{2.2}
\]

In the previous paper [4], we proved the following facts:

**Theorem 2.1** [4] Let \((T, H) \in W(\mathcal{H})\). Then:

(i) If \(H\) is bounded below, then either \(\sigma(T) = \overline{\Pi}_+\) (the closure of \(\Pi_+\)) or \(\sigma(T) = \mathbb{C}\).

(ii) If \(H\) is bounded above, then either \(\sigma(T) = \overline{\Pi}_-\) or \(\sigma(T) = \mathbb{C}\).

(iii) If \(H\) is bounded, then \(\sigma(T) = \mathbb{C}\).

This theorem has to be taken into account in considering the uniqueness problem of weak Weyl representations.

A form of representations of the CCR stronger than weak Weyl representations is known as a *Weyl representation* of the CCR which is a pair \((T, H)\) of *self-adjoint* operators on \(\mathcal{H}\) obeying the *Weyl relation*

\[ e^{itT}e^{isH} = e^{-its}e^{isH}e^{itT}, \quad \forall t, \forall s \in \mathbb{R}. \tag{2.3} \]

It is well known (the von Neumann uniqueness theorem [15]) that, every Weyl representation on a *separable* Hilbert space is unitarily equivalent to a direct sum of the Schrödinger representation \((q, p)\) on \(L^2(\mathbb{R})\), where \(q\) is the multiplication operator by the variable \(x \in \mathbb{R}\) and \(p = -iD_x\) with \(D_x\) being the generalized differential operator in \(x\) (cf. [3, §3.5], [16, Theorem 4.3.1], [17, Theorem VIII.14]).

It is easy to see that a Weyl representation is a weak Weyl representation (but the converse is not true). Therefore, as far as the Hilbert space under consideration is separable, the non-trivial case for the uniqueness problem of weak Weyl representations is the one where they are *not* Weyl representations. A general class of such weak Weyl representations \((T, H)\) are given in the case where \(H\) is semi-bounded (bounded below or bounded above). In this case, \(T\) is not essentially self-adjoint [2, Theorem 2.8], implying Theorem 2.1.

Two simple examples in this class are constructed as follows:
Example 2.1 Let $a \in \mathbb{R}$ and consider the Hilbert space $L^2(\mathbb{R}_a^+)$ with $\mathbb{R}_a^+ := (a, \infty)$. Let $q_{a,+}$ be the multiplication operator on $L^2(\mathbb{R}_a^+)$ by the variable $\lambda \in \mathbb{R}_a^+$:

\[ D(q_{a,+}) := \left\{ f \in L^2(\mathbb{R}_a^+) \mid \int_a^\infty \lambda^2 |f(\lambda)|^2 d\lambda < \infty \right\}, \]  

(2.4) \[ q_{a,+}f := \lambda f, \quad f \in D(q_{a,+}) \]  

(2.5) and

\[ p_{a,+} := -i \frac{d}{d\lambda} \]  

(2.6)

with $D(p_{a,+}) = C_0^\infty(\mathbb{R}_a^+)$, the set of infinitely differentiable functions on $\mathbb{R}_a^+$ with bounded support in $\mathbb{R}_a^+$. Then it is easy to see that $q_{a,+}$ is self-adjoint, bounded below with $\sigma(q_{a,+}) = [a, \infty)$ and $p_{a,+}$ is a symmetric operator. Moreover, $(-p_{a,+}, q_{a,+})$ is a weak Weyl representation of the CCR. Hence, as remarked above, $(-\bar{p}_{a,+}, q_{a,+})$ also is a weak Weyl representation.

Note that $p_{a,+}$ is not essentially self-adjoint and

\[ \sigma(-p_{a,+}) = \sigma(-\bar{p}_{a,+}) = \overline{\Pi}_+. \]  

(2.7)

In particular, $\pm \bar{p}_{a,+}$ are maximal symmetric, i.e., they have no non-trivial symmetric extensions (e.g., [18, §X.1, Corollary]).

Example 2.2 Let $b \in \mathbb{R}$ and consider the Hilbert space $L^2(\mathbb{R}_b^-)$ with $\mathbb{R}_b^- := (-\infty, b)$. Let $q_{b,-}$ be the multiplication operator on $L^2(\mathbb{R}_b^-)$ by the variable $\lambda \in \mathbb{R}_b^-$. and

\[ p_{b,-} := -i \frac{d}{d\lambda} \]  

(2.8)

with $D(p_{b,-}) = C_0^\infty(\mathbb{R}_b^-)$. Then $q_{b,-}$ is self-adjoint, bounded above with $\sigma(q_{b,-}) = (-\infty, b]$, $p_{b,-}$ is a symmetric operator, and $(-p_{b,-}, q_{b,-})$ is a weak Weyl representation of the CCR. As in the case of $p_{a,+}$, $p_{b,-}$ is not essentially self-adjoint and

\[ \sigma(-p_{b,-}) = \overline{\Pi}_-. \]  

(2.9)

A relation between $(-p_{a,+}, q_{a,+})$ and $(-p_{b,-}, q_{b,-})$ is given as follows. Let $U_{ab} : L^2(\mathbb{R}_a^+) \to L^2(\mathbb{R}_b^-)$ be a linear operator defined by

\[ (U_{ab}f)(\lambda) := f(a + b - \lambda), \quad f \in L^2(\mathbb{R}_a^+), \text{ a.e.} \lambda \in \mathbb{R}_b^- \]

Then $U_{ab}$ is unitary and

\[ U_{ab}q_{a,+}U_{ab}^{-1} = a + b - q_{b,-}, \quad U_{ab}p_{a,+}U_{ab}^{-1} = -p_{b,-}. \]  

(2.10)
In view of the von Neumann uniqueness theorem for Weyl representations, the pair \((-\overline{p}_{a,+}, q_{a,+})\) (resp. \((-\overline{p}_{b,-}, q_{b,-})\)) may be a reference pair in classifying weak Weyl representations \((T, H)\) with \(H\) being bounded below (resp. bounded above).

By Theorem 2.1, we can define two subsets of \(W(\mathcal{H})\):

\[
W_{+}(\mathcal{H}) := \{ (T, H) \in W(\mathcal{H}) | H \text{ is bounded below and } \sigma(T) = \overline{\Pi}_{+} \}, \tag{2.11}
\]

\[
W_{-}(\mathcal{H}) := \{ (T, H) \in W(\mathcal{H}) | H \text{ is bounded above and } \sigma(T) = \overline{\Pi}_{-} \}. \tag{2.12}
\]

Then, as shown above, \((-p_{a,+}, q_{a,+}) \in W_{+}(L^{2}(\mathbb{R}_{a}^{+}))\) and \((-p_{b,-}, q_{b,-}) \in W_{-}(L^{2}(\mathbb{R}_{b}^{-}))\).

### 3 Irreducibility

For a set \(\mathcal{A}\) of linear operators on a Hilbert space \(\mathcal{H}\), we set

\[
\mathcal{A}' := \{ B \in B(\mathcal{H}) | BA \subset AB, \forall A \in \mathcal{A} \},
\]

called the strong commutant of \(\mathcal{A}\) in \(\mathcal{H}\), where \(B(\mathcal{H})\) is the set of all bounded linear operators on \(\mathcal{H}\) with \(D(B) = \mathcal{H}\).

We say that \(\mathcal{A}\) is irreducible if \(\mathcal{A}' = \{ cI | c \in \mathbb{C} \}\), where \(I\) is the identity on \(\mathcal{H}\).

**Proposition 3.1** For all \(a \in \mathbb{R}\), the set \(\{\overline{p}_{a,+}, p_{a,+}^{*}, q_{a,+}\}\) (Example 2.1) is irreducible.

To prove this proposition, we need a lemma.

Let \(a \in \mathbb{R}\) be fixed. For each \(t \geq 0\), we define a linear operator \(U_{a}(t)\) on \(L^{2}(\mathbb{R}_{a}^{+})\) as follows: For each \(f \in L^{2}(\mathbb{R}_{a}^{+})\),

\[
(U_{a}(t)f)(\lambda) := \begin{cases} f(\lambda - t) & \lambda > t + a \\
0 & a < \lambda \leq t + a \end{cases}
\] \(\tag{3.1}\)

Then it is easy to see that \(\{U_{a}(t)\}_{t \geq 0}\) is a strongly continuous one-parameter semigroup of isometries on \(L^{2}(\mathbb{R}_{a}^{+})\).

**Lemma 3.2** The generator of \(\{U_{a}(t)\}_{t \geq 0}\) is \(-i\overline{p}_{a,+}^{*}:

\[
\frac{dU_{a}(t)f}{dt} = -i\overline{p}_{a,+}U_{a}(t)f, \quad \forall f \in D(\overline{p}_{a,+}), t \in \mathbb{R}, \tag{3.2}\]

where the derivative in \(t\) is taken in the strong sense.

**Proof.** Let \(iA\) be the generator of \(\{U_{a}(t)\}_{t \geq 0}:

\[
\frac{dU_{a}(t)f}{dt} = iAU_{a}(t)f, \quad \forall f \in D(A), t \in \mathbb{R}.
\]

Then it follows from the isometry of \(U_{a}(t)\) that \(A\) is a closed symmetric operator. It is easy to see that \(-p_{a,+} \subset A\) and hence \(-\overline{p}_{a,+} \subset A\). As already remarked in Example 2.1, \(-\overline{p}_{a,+}\) is maximal symmetric. Hence \(A = -\overline{p}_{a,+}\).
Proof of Proposition 3.1

Let $B \in \{\overline{p}_{a,+}, p_{a,+}^{*}, q_{a,+}\}'$. Then

\begin{align*}
B\overline{p}_{a,+} &\subset \overline{p}_{a,+}B, \\
Bp_{a,+}^{*} &\subset p_{a,+}^{*}B, \\
Bq_{a,+} &\subset q_{a,+}B.
\end{align*}

(3.3) \hspace{1cm} (3.4) \hspace{1cm} (3.5)

As in the case of bounded linear operators on $L^2(\mathbb{R})$ strongly commuting with $q$ (the multiplication operator by the variable $x \in \mathbb{R}$) [3, Lemma 3.13], (3.5) implies that there exists an essentially bounded function $F$ on $\mathbb{R}_a^+$ such that $B = M_F$, the multiplication operator by $F$.

Let $f \in D(\overline{p}_{a,+})$ and $g(t) := BU_a(t)f$. Then, by Lemma 3.2, $g$ is strongly differentiable in $t \geq 0$ and

\[
\frac{dg(t)}{dt} = B(-i\overline{p}_{a,+})U_a(t)f = -i\overline{p}_{a,+}g(t),
\]

where we have used (3.3). Note that $g(0) = Bf$. Hence, by the uniqueness of solutions of the initial value problem on differential equation (3.2), we have $g(t) = U_a(t)Bf$. Therefore it follows that $BU_a(t) = U_a(t)B, \forall t \geq 0$. Hence $FU_a(t)f = U_a(t)Ff, \forall f \in L^2(\mathbb{R}_a^+)$, which implies that

\[
F(\lambda)f(\lambda - t) = F(\lambda - t)f(\lambda - t), \quad \lambda > t + a.
\]

Hence $F(\lambda) = F(\lambda + t)$, a.e. $\lambda > 0, \forall t > 0$. This means that $F$ is equivalent to a constant function. Hence $B = M_F = cI$ with some $c \in \mathbb{C}$. \hfill \Box

Proposition 3.3 For all $b \in \mathbb{R}$, the set $\{\overline{p}_{b,-}, p_{b,-}^{*}, q_{b,-}\}$ (Example 2.2) is irreducible.

Proof. Let $B \in \{\overline{p}_{b,-}, p_{b,-}^{*}, q_{b,-}\}'$. Then, by (2.10), the operator $C := U_{ab}^{-1}BU_{ab}$ is in $\{\overline{p}_{a,+}, p_{a,+}^{*}, q_{a,+}\}'$. Hence, by Proposition 3.1, $C = cI$ with some constant $c \in \mathbb{C}$. Thus $B = cI$. \hfill \Box

4 Uniqueness Theorem

One can prove the following theorem:

Theorem 4.1 Let $\mathcal{H}$ be separable and $(T, H) \in W_+(\mathcal{H})$ with $\varepsilon_0 := \inf \sigma(H)$. Suppose that $\{T, T^*, H\}$ is irreducible. Then there exists a unitary operator $U : \mathcal{H} \rightarrow L^2(\mathbb{R}_{\varepsilon_0}^+)$ such that

\[
U\overline{T}U^{-1} = -\overline{p}_{\varepsilon_0,+}, \quad UHU^{-1} = q_{\varepsilon_0,+}.
\]

(4.1)
In particular

\[ \sigma(H) = [\varepsilon_0, \infty). \]  

(4.2)

**Remark 4.1** It is known that, for every weak Weyl representation \((T, H) \in \mathcal{W}(\mathcal{H})\) (\(\mathcal{H}\) is not necessarily separable), \(H\) is purely absolutely continuous [14, 19].

We prove Theorem 4.1 in the next section. For the moment, we note a result which immediately follows from Theorem 4.1:

**Theorem 4.2** Let \(\mathcal{H}\) be separable and \((T, H) \in \mathcal{W}_-(\mathcal{H})\) with \(b := \sup \sigma(H)\). Suppose that \(\{\overline{T}, T^*, H\}\) is irreducible. Then there exists a unitary operator \(V : \mathcal{H} \to L^2(\mathbb{R}_b^-)\) such that

\[ V\overline{T}V^{-1} = -\overline{p}_{b,-}, \quad VH V^{-1} = q_{b,-}. \]  

(4.3)

In particular

\[ \sigma(H) = (-\infty, b]. \]  

(4.4)

**Proof.** As remarked in Section 2, \((-T, -H) \in \mathcal{W}_+(\mathcal{H})\) with \(a := \inf \sigma(-H) = -b\) and \(\sigma(-T) = \overline{\Pi}_+\). Hence, we can apply Theorem 4.1 to conclude that there exists a unitary operator \(U : \mathcal{H} \to L^2(\mathbb{R}_a^+)\) such that

\[ U\overline{T}U^{-1} = \overline{p}_{a,+}, \quad UHU^{-1} = -q_{a,+}. \]

By Example 2.2, we have

\[ U_{ab}\overline{p}_{a,+}U_{ab}^{-1} = -\overline{p}_{b,-}, \quad U_{ab}q_{a,+}U_{ab}^{-1} = -q_{b,-}, \]

where we have used that \(a + b = 0\). Hence, putting \(V := U_{ab}U\), we obtain the desired result.

**Remark 4.2** In view of Theorems 4.1 and 4.2, it would be interesting to know when \(\sigma(T) = \overline{\Pi}_+\) (resp. \(\overline{\Pi}_-\)) for \((T, H) \in \mathcal{W}(\mathcal{H})\) with \(H\) bounded below (resp. above). Concerning this problem, we have the following results [5]:

(i) Let \((T, H) \in \mathcal{W}(\mathcal{H})\) and \(H\) be bounded below. Suppose that, for some \(\beta_0 > 0\), \(\text{Ran}(e^{-\beta_0 H}T)\) (the range of \(e^{-\beta_0 H}T\)) is dense in \(\mathcal{H}\). Then \(\sigma(T) = \overline{\Pi}_+\).

(ii) Let \((T, H) \in \mathcal{W}(\mathcal{H})\) and \(H\) be bounded above. Suppose that, for some \(\beta_0 > 0\), \(\text{Ran}(e^{\beta_0 H}T)\) is dense in \(\mathcal{H}\). Then \(\sigma(T) = \overline{\Pi}_-\).
5 Proof of Theorem 4.1

Lemma 5.1 Let $S$ be a closed symmetric operator on $\mathcal{H}$ such that $\sigma(S) = \Pi_+$. Then there exists a unique strongly continuous one-parameter semi-group $\{Z(t)\}_{t \geq 0}$ whose generator is $iS$. Moreover, each $Z(t)$ is an isometry:

$$Z(t)^*Z(t) = I, \quad \forall t \geq 0. \quad (5.1)$$

Proof. This fact is probably well known. But, for completeness, we give a proof. By the assumption $\sigma(S) = \Pi_+$, we have $\sigma(iS) = \{z \in \mathbb{C} | \Re z \leq 0\}$. Therefore the positive real axis $(0, \infty)$ is included in the resolvent set $\rho(iS)$ of $iS$. Since $S$ is symmetric, it follows that

$$\| (iS - \lambda)^{-1} \| \leq \frac{1}{\lambda}, \quad \lambda > 0.$$

Hence, by the Hille-Yosida theorem, $iS$ generates a strongly continuous one-parameter semi-group $\{Z(t)\}_{t \geq 0}$ of contractions. For all $\psi \in D(iS) = D(S)$, $Z(t)\psi$ is in $D(S)$ and strongly differentiable in $t \geq 0$ with

$$\frac{d}{dt} Z(t)\psi = iSZ(t)\psi = Z(t)iS\psi.$$

This equation and the symmetricity of $S$ imply that $\|Z(t)\psi\|^2 = \|\psi\|^2, \forall t \geq 0$. Hence (5.1) follows.

Lemma 5.2 Let $(T, H) \in W_+(\mathcal{H})$. Then there exists a unique strongly continuous one-parameter semi-group $\{U_T(t)\}_{t \geq 0}$ whose generator is $i\overline{T}$. Moreover, each $U_T(t)$ is an isometry and

$$U_T(t)e^{-isH} = e^{its}e^{-isH}U_T(t), \quad t \geq 0, s \in \mathbb{R}. \quad (5.2)$$

Proof. We can apply Lemma 5.1 to $S = \overline{T}$ to conclude that $i\overline{T}$ generates a strongly continuous one-parameter semi-group $\{U_T(t)\}_{t \geq 0}$ of isometries on $\mathcal{H}$. For all $\psi \in D(\overline{T})$ and all $t \geq 0$, $U_T(t)\psi$ is in $D(\overline{T})$ and strongly differentiable in $t \geq 0$ with

$$\frac{d}{dt} U_T(t)\psi = i\overline{T}U_T(t)\psi = U_T(t)i\overline{T}\psi.$$

Let $s \in \mathbb{R}$ be fixed and $V(t) := e^{its}e^{-isH}U_T(t)e^{isH}$. Then $\{V(t)\}_{t \geq 0}$ is a strongly continuous one-parameter semi-group of isometries. Let $\psi \in D(\overline{T})$. Then $e^{-isH}\psi \in D(\overline{T})$ and

$$\overline{T}e^{-isH}\psi = e^{-isH}\overline{T}\psi + se^{-isH}\psi.$$
Hence $V(t)\psi$ is in $D(\overline{T})$ and strongly differentiable in $t$ with
\[
\frac{d}{dt} V(t)\psi = i\overline{T}V(t)\psi.
\]
This implies that $V(t)\psi = U_T(t)\psi, \forall t \in \mathbb{R}$. Since $D(\overline{T})$ is dense, it follows that $V(t) = U_T(t), \forall t \in \mathbb{R}$, implying (5.2).

1

We recall a result of Bracci and Picasso [10]. Let $\{U(\alpha)\}_{\alpha \geq 0}$ and $\{V(\beta)\}_{\beta \in \mathbb{R}}$ be a strongly continuous one-parameter semi-group and a strongly continuous one-parameter unitary group on $\mathcal{H}$ respectively, satisfying
\[
U(\alpha)^*U(\alpha) = I, \quad \alpha \geq 0, \tag{5.3}
\]
\[
U(\alpha)V(\beta) = e^{i\alpha\beta}V(\beta)U(\alpha), \quad \alpha \geq 0, \beta \in \mathbb{R}. \tag{5.4}
\]
Then, by the Stone theorem, there exists a unique self-adjoint operator $P$ on $\mathcal{H}$ such that
\[
V(\beta) = e^{-i\beta P}, \quad \beta \in \mathbb{R}. \tag{5.5}
\]

Lemma 5.3 [10] Let $\mathcal{H}$ be separable and $P$ is bounded below with $\nu := \inf \sigma(P)$. Suppose that $\{U(\alpha), U(\alpha)^*, V(\beta)|\alpha \geq 0, \beta \in \mathbb{R}\}$ is irreducible. Then, there exists a unitary operator $Y : \mathcal{H} \rightarrow L^2(\mathbb{R}_+^\nu)$ such that
\[
YV(\beta)Y^{-1} = e^{-i\beta q_{\nu,+}}, \beta \in \mathbb{R}, \tag{5.6}
\]
\[
YU(\alpha)Y^{-1} = U_\nu(\alpha), \quad \alpha \geq 0. \tag{5.7}
\]

We denote the generator of $\{U(\alpha)\}_{\alpha \geq 0}$ by $iQ$. It follows that $Q$ is closed and symmetric.

Lemma 5.4 Under the assumption of Lemma 5.3,
\[
YPY^{-1} = q_{\nu,+}, \tag{5.8}
\]
\[
YQY^{-1} = -\overline{p}_{\nu,+}. \tag{5.9}
\]

In particular
\[
\sigma(P) = [\nu, \infty). \tag{5.10}
\]

Proof. Lemma 5.3 and (5.6) imply (5.8). Similarly (5.9) follows from Lemma 5.3, (5.7) and Lemma 3.2.

Lemma 5.5 Let $(T, H) \in W(\mathcal{H})$ with $\sigma(T) = \overline{\Pi}_+$. Suppose that $\{\overline{T}, T^*, H\}$ is irreducible. Then $\{U_T(t), U_T(t)^*, e^{-isH}|t \geq 0, s \in \mathbb{R}\}$ is irreducible.
Proof. Let \( B \in \mathcal{B} \mathcal{H} \) be such that
\[
BU_T(t) = U_T(t)B, \quad (5.11)
\]
\[
BU_T(t)^* = U_T(t)^*B, \quad (5.12)
\]
\[
Be^{-isH} = e^{-isH}B, \forall t \geq 0, \forall s \in \mathbb{R}. \quad (5.13)
\]
Let \( \psi \in D(\overline{T}) \). Then, by (5.11), we have \( BU_T(t)\psi = U_T(t)B\psi, \forall t \geq 0 \). By Lemma 5.2, the left hand side is strongly differentiable in \( t \) with \( d(BU_T(t)\psi)/dt = iB\overline{T}U_T(t)\psi \). Hence so does the right hand side and we obtain that \( B\psi \in D(\overline{T}) \) and \( B\overline{T}\psi = \overline{T}B\psi \). Therefore \( B\overline{T} \subset \overline{T}B \). Note that (5.12) implies that \( U_T(t)B^* = B^*U_T(t) \). Hence it follows that \( B^*\overline{T} \subset \overline{T}B^* \), which implies that \( BT^* \subset T^*B \), where we have used the following general facts: for every densely defined closable linear operator \( A \) on \( \mathcal{H} \) and all \( C \in \mathcal{B} \mathcal{H} \), \( (CA)^* = A^*C^* \), \( (AC)^* \supset C^*A^* \), \( (\overline{A})^* = A^* \).

Similarly (5.13) implies that \( BH \subset HB \). Hence \( B \in \{\overline{T}, T^*, H\}' \). Therefore \( B = cI \) for some \( c \in \mathbb{C} \).

Proof of Theorem 4.1

By Lemmas 5.2 and 5.5, we can apply Lemma 5.3 to the case where \( V(\beta) = e^{-i\beta H}, \beta \in \mathbb{R} \) and \( U(\alpha) = U_T(\alpha), \alpha \geq 0 \). Then the desired results follow from Lemmas 5.3 and 5.4.

Remark 5.1 Recently Bracci and Picasso [11] have obtained an interesting result on the reducibility of the von Neumann algebra generated by \( \{U(\alpha), U(\alpha)^*, V(\beta)\alpha \geq 0, \beta \in \mathbb{R}\} \) obeying (5.3) and (5.4). By employing the result, one can generalize Theorem 4.1 to the case where \( \{\overline{T}, T^*, H\} \) is not necessarily irreducible.

6 Application to Construction of a Weyl representation

In the previous paper [8], a general structure was found to construct a Weyl representation from a weak Weyl representation. Here we recall it.

Theorem 6.1 [8, Corollary 2.6] Let \( (T, H) \) be a weak Weyl representation on a Hilbert space \( \mathcal{H} \) with \( T \) closed. Then the operator
\[
L := \log |H| \quad (6.1)
\]
is well-defined, self-adjoint and the operator
\[
D := \frac{1}{2}(TH + \overline{HT}) \quad (6.2)
\]
is a symmetric operator. Moreover, if $D$ is essentially self-adjoint, then $(\overline{D}, L)$ is a Weyl representation of the CCR and $\sigma(|H|) = [0, \infty)$.

To apply this theorem, we need a lemma.

**Lemma 6.2** [6] Let $a \in \mathbb{R}$ and

$$d_a := -\frac{1}{2}(p_{a,+}q_{a,+} + \overline{q_{a,+}}p_{a,+})$$

acting in $L^2(\mathbb{R}^+_a)$. Then $d_a$ is essentially self-adjoint if and only if $a = 0$.

**Theorem 6.3** Let $\mathcal{H}$ be separable and $(T, H) \in W_+(\mathcal{H})$ with $\inf \sigma(H) = 0$ and $T$ closed. Suppose that $\{T, T^*, H\}$ is irreducible. Let $L$ and $D$ be as in (6.1) and (6.2) respectively. Then $D$ is essentially self-adjoint and $(\overline{D}, L)$ is a Weyl representation of the CCR.

**Proof.** Let $\hat{d}_0$ be the operator $d_0$ with $p_{0,+}$ replaced by $\overline{p}_{0,+}$. Then, by Theorem 4.1, $D$ is unitarily equivalent to $\hat{d}_0$. We have $d_0 \subset \hat{d}_0$. By Lemma 6.2, $d_0$ is essentially self-adjoint. Hence $\hat{d}_0$ is essentially self-adjoint. Therefore it follows that $D$ is essentially self-adjoint. The second half of the theorem follows from Theorem 6.1. \[\qed\]

**References**


