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On Representations of $\widetilde{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and Newforms of half-integral weight

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Introduction

The aim of this paper is to give a theory of newforms of weight $k+1/2$, level $8\times M$ and a quadratic character χ with an odd positive squarefree integer M.

The main ingredients of our proof are two things. One is trace identities between Hecke operators of integral weight and those of half-integral weight. Another one is representations of the metaplectic covering group $\widetilde{SL}_2$ over $\mathbb{Z}/N\mathbb{Z}$.

For the sake of simplicity, we state the results for only the case of trivial character. See the forthcoming paper for the details for general cases.

We compose this paper as follows: In the section 1, first we recall the previous works of newform theory of half-integral weight which were obtained by several authors. And then we state our main result.

In the section 2, we study a certain representation of $\widetilde{SL}_2(\mathbb{Z}/N\mathbb{Z})$ defined by modular forms of half-integral weight and level N.

In the section 3, we give the irreducible decomposition of the above representation and describe a connection between this representation and a non-vanishing of Fourier coefficients of modular forms. And then we give two applications. One is a characterization of plus spaces of level $4M$ and $8M$. And another one is a theory of newforms of half-integral weight and level $8M$.

1. Let k and N be positive integers with $4\mid N$. We decompose $N=2^{\mu}M$ with an odd positive integer M and an integer $\mu \geq 2$. Let χ be an even quadratic character modulo N.

We denote by $S(k+1/2, N, \chi)$ the space of cusp forms of weight $k+1/2$, level N and character χ. In particular, if χ is trivial, we shortly denote it $S(k+1/2, N)$. Moreover we define the plus space $S(k+1/2, N, \chi)_{pl}$ for the case of $\mu = 2, 3$ as follows:

$$S(k+1/2, N, \chi)_{pl} := \left\{ \begin{array}{l} f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S(k+1/2, N, \chi) \ ; \\ a(n) = 0 \text{ if } \chi_2(-1)(-1)^k n \equiv 2, 3 \pmod 4 \end{array} \right\},$$

where χ_2 is the 2-primary component of χ and $e(z) = \exp(2\pi\sqrt{-1}z)$. We write $S(k+1/2, N)_{pl}$ if χ is trivial.

Several authors have already given theories of newforms in various cases. We list them below.

- $S(k+1/2, 4M, \chi)_{pl}$, M is squarefree (Kohnen (1982) [K])
- $S(k+1/2, 4M, \chi)$, M is squarefree (Manickam, Ramakrishnan, and Vasudevan (1990) [MRV])
- $S(k+1/2, 4M, \chi)_{pl}$, M is general (Ueda (1998) [U2])
- $S(k+1/2, 8M, \chi)_{pl}$, M is squarefree (Ueda–Yamana (2009) [UY])
- $S(k+1/2, 8M, \chi)$, M is squarefree (**Today's talk**)

We need the results of [K] and [MRV] in order to state our result. Then we will recall them more precisely.

We prepare some notation.

Let $S^0(2k, M)$ be the space of newforms of weight $2k$ (cf. [M]). For any positive integer m, let $U(m)$ be a shift operator defined as follows:

$$\sum_{n\geq 1} a(n)e(nz) \mid U(m) := \sum_{n \geq 1} a(mn)e(nz) , \qquad z \in \mathbb{H} .$$

Here, $\mathbb{H}$ is the complex upper half plane.

Let $T(n)$ be the n-th Hecke operator of integral weight and $\tilde{T}(n^2)$ the n^2-th Hecke operator of half-integral weight. Please see [U2] for the details of the above definitions.

Then, we have the following.

Theorem 1 (Kohnen [K]). Assume that M is squarefree. Then we have the following decomposition of Hecke modules

$$S(k+1/2,4M)_{pl} = \bigoplus_{\substack{0<e,d \\ ed|M}} S^{\text{new}}(k+1/2,4e)_{pl} \mid U(d^2) \ ,$$

Here, $S^{\text{new}}(k+1/2,4e)_{pl}$ is the space of newforms (cf. [K]). And moreover we have an isomorphism as Hecke modules

$$S^{\text{new}}(k+1/2,4M)_{pl} \ \cong \ S^0(2k,M) \ .$$

□

Theorem 2 (Manickam, Ramakrishnan, Vasudevan [MRV]). Assume that M is squarefree. Then we have the following decomposition of Hecke modules

$$\begin{aligned} & S(k+1/2,4M) \\ & \quad = \bigoplus_{\substack{0<e,d \\ ed|M}} S^{\text{new}}(k+1/2,4e) \mid U(d^2) \\ & \qquad \oplus \bigoplus_{\substack{0<e,d \\ ed|M}} \left\{ S^{\text{new}}(k+1/2,4e)_{pl} \mid U(d^2) \oplus S^{\text{new}}(k+1/2,4e)_{pl} \mid U(4d^2) \right\} . \end{aligned}$$

Here, $S^{\text{new}}(k+1/2,4e)$ is the space of newforms (cf. [MRV]). And moreover we have an isomorphism as Hecke modules

$$S^{\text{new}}(k+1/2,4M) \ \cong \ S^0(2k,2M) \ .$$

□

The proofs of the above decompositions and isomorphisms are based on the following two facts.

$\boxed{\textbf{Fact 1.}}$ (Trace identity) For any positive integer n with $(n,4M)=1$,

$$\mathrm{tr}\Big(\tilde{T}(n^2); S(k+1/2,4M)_{pl}\Big) \ = \ \mathrm{tr}(T(n); S(2k,M)) \qquad \text{(by Kohnen)}$$

$$\mathrm{tr}\Big(\tilde{T}(n^2); S(k+1/2,4M)\Big) \ = \ \mathrm{tr}(T(n); S(2k,2M)) \qquad \text{(by Niwa)}$$

$\boxed{\textbf{Fact 2.}}$ Linear independence of the spaces of oldforms $S^{\text{new}}(k+1/2,4e) \mid U(d^2)$, $S^{\text{new}}(k+1/2,4e)_{pl} \mid U(d^2)$, and $S^{\text{new}}(k+1/2,4e)_{pl} \mid U(4d^2)$ ($0 < e,d$ with $ed \mid M$ and $ed < M$).

Here, concerning Fact 1, we note that we have also the same trace identity for the case of level $8M$ ([U1]) : For any positive integer n with $(n,8M)=1$

$$\mathrm{tr}\Big(\tilde{T}(n^2); S(k+1/2,8M)\Big) \;=\; \mathrm{tr}(T(n); S(2k,4M)) \;.$$

Hence we can expect a similar theory of newforms also in this case. In fact, we can give such a theory as follows:

First, we define the space of newforms $S^{\text{new}}(k+1/2,8M)$ to be the orthogonal complement of

$$\begin{aligned} S(k+1/2,4M) + S(k+1/2,4M) \mid Y_8 \\ + \sum_{p\mid M}\{S(k+1/2,8M/p) + S(k+1/2,8M/p) \mid U(p^2)\} \end{aligned}$$

with respect to the Petersson inner product. Here, p in the last sum runs over all prime divisors of M. Moreover, $Y_{2^n} = e\big(-(2k+1)/8\big)2^{n(-k/2+3/4)}\,U(2^n)\widetilde{W}(2^n)$ and $\widetilde{W}(2^n)$ is the Atkin-Lehner operator of half-integral weight.

Then we can prove the following Theorem.

Theorem 3. *Let M be a squarefree odd positive integer. Then we have the following decomposition of Hecke modules*

$$\begin{aligned} &S(k+1/2,8M) \\ &\quad = \bigoplus_{\substack{0<e,d\\ ed\mid M}} S^{\text{new}}(k+1/2,8e) \mid U(d^2) \\ &\quad \oplus \bigoplus_{\substack{0<e,d\\ ed\mid M}} \{S^{\text{new}}(k+1/2,4e) \mid U(d^2) \oplus S^{\text{new}}(k+1/2,4e) \mid Y_8\, U(d^2)\} \\ &\quad \oplus \bigoplus_{\substack{0<e,d\\ ed\mid M}} \{S^{\text{new}}(k+1/2,4e)_{pl} \mid U(d^2) \\ &\qquad\qquad \oplus S^{\text{new}}(k+1/2,4e)_{pl} \mid Y_8\, U(d^2) \oplus S^{\text{new}}(k+1/2,4e)_{pl} \mid U(4d^2)\} \;. \end{aligned}$$

And moreover we have an isomorphism as Hecke modules

$$S^{\mathrm{new}}(k+1/2, 8M) \cong S^0(2k, 4M)\,.$$

Hence, $S^{\mathrm{new}}(k+1/2, 8M)$ *has an orthogonal basis* $\{f_i\}$ *consisting of common eigenforms of* $\tilde{T}(p^2)$ *if* $(p, 8M) = 1$ *and* $U(p^2)$ *if* $p \mid 8M$. *Moreover there exists bijection between* $\{f_i\}$ *and* $\{$*primitive forms* $F_i \in S^0(2k, 4M)$ $\}$ *such that*

$$\begin{cases} f_i \mid \tilde{T}(p^2) = \lambda_{i,p} f_i & \text{if } (p, 2M) = 1 \\ f_i \mid U(p^2) = \lambda_{i,p} f_i & \text{if } p \mid 2M \end{cases} \quad\leftrightsquigarrow\quad \begin{cases} F_i \mid T(p) = \lambda_{i,p} F_i & \text{if } (p, 2M) = 1 \\ F_i \mid U(p) = \lambda_{i,p} F_i & \text{if } p \mid 2M \end{cases}$$

$\square$

Remark 1. We can also establish this theorem for any quadratic character χ.

Remark 2. It seems that the operator Y_8 is slightly strange. However, we can see Y_8 is essentially equal to a certain modification of the shift operator $U(4)$ (cf. [UY]). And also Y_8 has an important role in a characterization of the plus spaces. See the section 3 below.

This theorem can be proved in a similar manner as the previous results. We already mentioned the trace identity in this case. Hence, in the following, we will discuss linear independence of the spaces of oldforms. For that purpose, we introduce a certain representation of metaplectic cover $\widetilde{SL}_2$ over a ring of residue classes modulo N.

2. Let $\widetilde{SL}_2(\mathbb{R}) := \{[\alpha, \zeta] \mid \alpha \in SL_2(\mathbb{R}), \zeta = \pm 1\}$ be a metaplectic covering of $SL_2(\mathbb{R})$. And we denote its projection $\boldsymbol{p} : \widetilde{SL}_2(\mathbb{R}) \ni [\alpha, \zeta] \mapsto \alpha \in SL_2(\mathbb{R})$. Then $\boldsymbol{p}$ splits on the congruent subgroup $\Gamma_1(4)$ and the section is given by

$$\Gamma_1(4) \ni \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \underline{\underline{\gamma}} := \left[\gamma, \left(\frac{c}{d}\right)\right] \in \widetilde{SL}_2(\mathbb{R})\,,$$

where $\left(\frac{*}{*}\right)$ is the Kronecker symbol. (cf. [Ge])

For any subgroup H of $SL_2(\mathbb{R})$, put $\widetilde{H} := \boldsymbol{p}^{-1}(H)$. Moreover if $H \subseteq \Gamma_1(4)$, put $\underline{\underline{H}} := \left\{\underline{\underline{\gamma}} \,\middle|\, \gamma \in H\right\}$.

Let $j : \widetilde{SL}_2(\mathbb{R}) \times \mathbb{H} \to \mathbb{C}$ be the usual automorphic factor of weight $1/2$. Then for any function $f : \mathbb{H} \to \mathbb{C}$ and $\xi = [\alpha, \zeta] \in \widetilde{SL}_2(\mathbb{R})$, put

$$(f \|_{k+1/2}\, \xi)(z) := j(\xi, z)^{-(2k+1)} f(\alpha z)\,, \qquad z \in \mathbb{H}\,.$$

Now, we introduce a representation on the space of cusp forms.

Let $S(k+1/2, \Delta(N))$ be the space of cusp forms of weight $k+1/2$ with respect to the principal congruence subgroup $\Gamma(N)$. (See [U2] for the definition of $\Delta(N)$.)

Since $\underline{\underline{\Gamma}}(N) \lhd \widetilde{SL}_2(\mathbb{Z})$, we can consider a quotient group $\widetilde{G} := \widetilde{G}(N) = \widetilde{SL}_2(\mathbb{Z})/\underline{\underline{\Gamma}}(N)$, which we denoted $\widetilde{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and called a metaplectic group over $\mathbb{Z}/N\mathbb{Z}$ in the above.

Then we can define a representation ϖ of $\widetilde{G}$ on $S(k+1/2, \Delta(N))$ by

$$\varpi(\xi_*)(f) := f\|_{k+1/2}\,\xi^{-1}\ , \qquad f \in S(k+1/2, \Delta(N))\ ,$$

where $\xi_* = \xi \bmod \underline{\underline{\Gamma}}(N) \in \widetilde{G}$.

Let f be a non-zero cusp form in $S(k+1/2, N, \chi)$, where χ is a quadratic character. And we denote $\varpi_f := \mathbb{C}[\widetilde{G}]f$, i.e., the $\mathbb{C}[\widetilde{G}]$-module generated by f.

Put $\widetilde{B} := \widetilde{B}(N) = \widetilde{\Gamma}_0(N)/\underline{\underline{\Gamma}}(N)$. Using relations $f \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \chi(d)f$ for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we see that $\mathbb{C}f$ becomes a $\mathbb{C}[\widetilde{B}]$-module via ϖ.

Then we have a following natural surjective $\mathbb{C}[\widetilde{G}]$-homomorphism

$$\Phi_f : \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\,\mathbb{C}f \cong \mathbb{C}[\widetilde{G}] \otimes_{\mathbb{C}[\widetilde{B}]} \mathbb{C}f \ \rightarrow\ \mathbb{C}[\widetilde{G}]f = \varpi_f\ , \qquad \eta \otimes f \mapsto \varpi(\eta)f\ .$$

Hence ϖ_f can be considered as a subrepresentation of $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\,\mathbb{C}f$. Therefore, it is enough to study the induced representation $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\,\mathbb{C}f$ in order to study ϖ_f.

In a usual way, we can decompose $\widetilde{G}$ and $\widetilde{B}$ into local components as follows:

$$\widetilde{G}(N) = \widetilde{G}(2^{\mu}) \times \prod_{p|M} SL_2(\mathbb{Z}/p\mathbb{Z})\ , \qquad \widetilde{B}(N) = \widetilde{B}(2^{\mu}) \times \prod_{p|M} B(p)\ .$$

Here $B(p) := \left\{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}/p\mathbb{Z})\right\}$.

Hence the $\mathbb{C}[\widetilde{B}]$-module $\mathbb{C}f$ can be decomposed into local components and therefore, we can also decompose $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\,\mathbb{C}f$ into local components

$$\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\,\mathbb{C}f\ \cong\ \rho_2 \otimes \left(\bigotimes_{p|M} \rho_p\right)\ ,$$

where ρ_2 (resp. ρ_p) is a certain representation of $\widetilde{G}(2^{\mu})$ (resp. $SL_2(\mathbb{Z}/p\mathbb{Z})$).

We can give an explicit description of these local components ρ_2 and ρ_p $(p \mid M)$, i.e., those irreducible decompositions and explicit basis of those irreducible components, etc.. However, the complete results are too complicated to describe here. Hence we skip the details. Please see the forthcoming paper for the details.

Instead of that, we will express partial results in the next section for the simplest two cases: (i) $N = 4M$ and $\chi = \mathbf{1}$ and (ii) $N = 8M$ and $\chi = \left(\frac{2}{*}\right)$. And we give two application of those.

3. In order to establish a theory of newforms, we must obtain linear independence of spaces of oldforms. And it can be derived by using non-vanishing property of Fourier coefficients of cusp forms. We can get such properties by studying ϖ_f.

Now, it is well-known that Fourier coefficients relate to representations of the unipotent subgroup $U = \underline{\underline{\Gamma}}_1(2^\mu)/\underline{\underline{\Gamma}}(2^\mu) \cong \mathbb{Z}/2^\mu\mathbb{Z}$. Hence, for our purpose, we must find the irreducible decomposition of 2-primary component ρ_2 and moreover decompose those irreducible components as $\mathbb{C}[U]$-modules.

We denote by $\widehat{U}$ the character group of U. Then $\widehat{U}$ is given by the following:

$$\widehat{U} = \{\psi_a \mid a \in \mathbb{Z}/2^\mu\mathbb{Z}\}\,, \qquad \psi_a\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = e(ax/2^\mu)\,.$$

Under the above notation, we have the following results.

$\boxed{\text{The case of } N = 4M \text{ and } \chi = \mathbf{1}}$

$$\rho_2 \cong \mathcal{A}_0 \oplus \mathcal{A}_1\,. \qquad (\text{as } \mathbb{C}[\widetilde{G}]\text{-modules})$$

$$\begin{aligned} \mathrm{Res}_U\,\mathcal{A}_0 &\cong \psi_0 \oplus \psi_{-(-1)^k}\,, \\ \mathrm{Res}_U\,\mathcal{A}_1 &\cong \psi_0 \oplus \psi_1 \oplus \psi_2 \oplus \psi_3\,. \qquad (\text{as } \mathbb{C}[U]\text{-modules}) \end{aligned}$$

$\boxed{\text{The case of } N = 8M \text{ and } \chi = \left(\tfrac{2}{*}\right)}$

$$\rho_2 \cong \mathcal{B}_0 \oplus \mathcal{B}_1 \oplus \mathcal{B}_2\,. \qquad (\text{as } \mathbb{C}[\widetilde{G}]\text{-modules})$$

$$\begin{aligned} \mathrm{Res}_U\,\mathcal{B}_0 &\cong \psi_0 \oplus \psi_4 \oplus \psi_{-(-1)^k}\,, \\ \mathrm{Res}_U\,\mathcal{B}_1 &\cong \psi_0 \oplus \psi_4 \oplus \psi_{-5(-1)^k}\,, \end{aligned}$$

$$\begin{aligned}\mathrm{Res}_U\,\mathcal{B}_2 \;&\cong\; \psi_0 \oplus \psi_2 \oplus \psi_4 \oplus \psi_6 \oplus \psi_{(-1)^k} \oplus \psi_{5(-1)^k} \,. \\ &(\text{as } \mathbb{C}[U]\text{-modules})\end{aligned}$$

Remark 3. We have the complete results for arbitrary level N and arbitrary quadratic character χ.

Here, we give two applications of the above results.

The first application is a characterization of plus spaces via the representation ϖ. Put $f = \sum_{n\geq 1} a(n)e(nz)$. Then each component ψ_α occurred in $\mathrm{Res}_U\,\mathcal{A}_i$ and $\mathrm{Res}_U\,\mathcal{B}_j$ corresponds to a family of Fourier coefficients $\{a(n) \mid n \equiv n_\alpha \bmod 4\}$, where n_α is a constant depending only on α. Hence, the above decompositions suggest the representations $\mathcal{A}_0$, $\mathcal{B}_0$, and $\mathcal{B}_1$ correspond to the plus spaces of level $4M$ and $8M$. In fact, we can obtain the following characterization of the plus spaces.

Let us prepare one more notation.
As we mentioned above, $\widetilde{G}(N) = \widetilde{G}(2^\mu) \times \prod_{p|M} SL_2(\mathbb{Z}/p\mathbb{Z})$. In particular, $\widetilde{G}(2^\mu)$ can be considered as a subgroup of $\widetilde{G} = \widetilde{G}(N)$. Then we put $\varpi_2(f) := \mathbb{C}[\widetilde{G}(2^\mu)]f$.

Theorem 4 (Skoruppa (the case of $\mu = 2$), Ueda). Let the notation as above. And put $\sigma_k = 1 + e\big((2k-1)/4\big)$.

(1) For a non-zero $f \in S(k+1/2, 4M)$, we have the following characterization.

$$f \in S(k+1/2, 4M)_{pl} \quad\Leftrightarrow\quad f \mid Y_4 \;=\; 2\sigma_k\, f \quad\Leftrightarrow\quad \varpi_2(g) \;\cong\; \mathcal{A}_0\,,$$

where $g := f \mid \widetilde{W}(4)^{-1}$.

(2) For a non-zero $f \in S(k+1/2, 8M)$, we have the following characterization.

$$f \in \begin{cases} S(k+1/2, 8M)_{pl,+} \\ S(k+1/2, 8M)_{pl,-} \end{cases} \Leftrightarrow\; f \mid Y_8 \;=\; \begin{cases} 2\sqrt{2}\,\sigma_k\, f \\ -2\sqrt{2}\,\sigma_k\, f \end{cases} \Leftrightarrow\; \varpi_2(g) \;\cong\; \begin{cases} \mathcal{B}_0 \\ \mathcal{B}_1 \end{cases}$$

where $g := f \mid \widetilde{W}(8)^{-1}$ and moreover

$$S(k+1/2, 8M)_{pl,+} := \left\{ \begin{array}{l} f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S(k+1/2, 8M)\;; \\ a(n) = 0 \text{ if } (-1)^k n \equiv 2,3,5,6,7 \pmod 8 \end{array} \right\},$$

$$S(k+1/2,8M)_{pl,-} := \left\{ \begin{array}{l} f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S(k+1/2,8M)\ ; \\ a(n) = 0 \text{ if } (-1)^k n \equiv 1,2,3,6,7 \pmod 8 \end{array} \right\} .$$

□

Next we will consider linear independence of oldforms in $S(k+1/2,8M)$ as the second application. For the sake of simplicity, we treat only the simplest case $M=1$.

First, we note the following. If eigenforms have different systems of eigenvalues on Hecke operators, then they are linearly independent. Hence it is enough to consider eigenforms which belong the same system of eigenvalues.

Now, let $f = \sum_{n\geq 1} a(n)e(nz) \in S^{\mathrm{new}}(k+1/2,4)$ be common eigenform of $\tilde{T}(p^2)$ for all primes p. Then, we can see $\varpi_2(f \mid \tilde{W}(4)^{-1}) \cong \mathcal{A}_1$ by using a similar argument to those of characterizations of plus spaces. Therefore, for any $t \in \mathbb{Z}/4\mathbb{Z}$, there exists a positive integer m_t such that $m_t \equiv t \pmod 4$ and that $a(m_t) \neq 0$.

On the other hand, since Y_8 satisfies the relation ${Y_8}^3 = Y_8$, we have $f \mid Y_8 \in S(k+1/2,8)_{pl}$ by using the characterization of plus space. Hence the m_2-th Fourier coefficient of $f \mid Y_8$ vanishes. Therefore, f and $f \mid Y_8$ are linearly independent.

Next, let $f \in S^{\mathrm{new}}(k+1/2,4)_{pl}$ be a non-zero common eigenform of $\tilde{T}(p^2)$ for all prime numbers p. Then we have the following two fact ([K])

(1) $\qquad f$ and $f \mid U(4)$ are linearly independent.

(2) $\qquad f \mid U(4) \notin S(k+1/2,4)_{pl}$.

Moreover, we can prove that f and $f \mid Y_8$ are linearly independent as follows:

First, we get from direct calculations

$$f \mid Y_8 = c_0 f(4z) + c_1 \Bigg(\sum_{(-1)^k n \equiv 1 \pmod 8} a(n)e(nz) - \sum_{(-1)^k n \equiv 5 \pmod 8} a(n)e(nz) \Bigg) ,$$

where $f = \sum_{n\geq 1} a(n)e(nz)$ and c_0, c_1 are non-zero constants.

For simplicity, we denote by h the second term of the right-hand side.

Then, if $f \mid Y_8 = \alpha f$ for some $\alpha \in \mathbb{C}$,

$$\alpha f \ = \ c_0 f(4z) + h\ .$$

Apply a shift operator $U(4)$ to the both sides

$$\alpha f \mid U(4) \;=\; c_0 f(4z) \mid U(4) + h \mid U(4) \;=\; c_0 f + h \mid U(4)\;.$$

Observing the shape of Fourier coefficients of h, we can see $h \mid U(4) = 0$.

Hence we get

$$\alpha f \mid U(4) = c_0 f\;.$$

This is a contradiction to the above statement (1).

Combining this, the statement (2), and the characterization of $S(k+1/2, 8)_{pl}$, we get linear independence of $f, f \mid Y_8$, and $f \mid U(4)$.

Thus we obtain linear independence of spaces of oldforms and a theory of newforms for the case of level $8M$ and weight $k+1/2$.

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