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Formal degrees of supercuspidal representations of ramified $U(3)$

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Abstract

Formal degrees of supercuspidal representations of p-adic unramified $U(3)$ are obtained as a part of the explicit Plancherel formula by Jabon-Keys-Moy. In this note, we compute those of ramified $U(3)$ in terms of supercuspidal types. As a corollary, we give a new proof of stability of very cuspidal representations of $U(3)$.

1 Introduction

Let F_0 be a non-archimedean local field. Let $\mathfrak{o}_0$ denote the ring of integers in F_0, $\mathfrak{p}_0 = \varpi_0\mathfrak{o}_0$ the maximal ideal in $\mathfrak{o}_0$, and $k_0 = \mathfrak{o}_0/\mathfrak{p}_0$ the residue field. Throughout this paper, we will always assume that the characteristic p of k_0 is not 2. We denote by q the cardinality of k_0.

Let F be a quadratic extension over F_0. We write $\mathfrak{o}_F$, $\mathfrak{p}_F$ and k_F for the analogous objects for F. Let $^- \in \mathrm{Gal}(F/F_0)$. We choose a uniformizer ϖ_F of F so that $\overline{\varpi_F} = \pm\varpi_F$.

Let $V = F^3$ be the space of three dimensional column vectors and let h denote the hermitian form on V defined by

$$h(v,w) = {}^t\overline{v}Hw,\ v,w \in V, \tag{1.1}$$

where

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.2}$$

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Put $G = U(3)(F/F_0) = \{g \in GL_3(F) \mid {}^t\overline{g}Hg = H\}$. Then G is the F_0-points of a unitary group in three variables defined over F_0.

Jabon, Keys and Moy [8] gave an explicit Plancherel formula of G. In particular, they computed formal degrees of the discrete series representations of G. But they assumed that F is unramified over F_0, when they calculated formal degrees of supercuspidal representations of G. The aim of this note is to determine formal degrees of the supercuspidal representations of G when F is ramified over F_0. This result completes the explicit Plancherel formula of G by Jabon-Keys-Moy.

The result in [8] is based on Moy's classification of the irreducible admissible representations of unramified G in [9], and formal degrees of the supercuspidal representations of unramified G are given in terms of nondegenerate representations in *loc. cit.* After Moy's work [9], Blasco [2] constructed the supercuspidal representations of G via compact induction from representations of open compact subgroups of G. Moreover Stevens [12] proved that all supercuspidal representations of a p-adic classical group come via compact induction from maximal simple types. In this note, we will use Stevens' construction to describe the supercuspidal representations of ramified G.

Let π be an irreducible supercuspidal representation of G. Then it follows from [2] and [12] that there is an irreducible representation λ of an open compact subgroup J of G such that π is isomorphic to $\mathrm{ind}_J^G\lambda$. By the well-known fact on formal degrees, the formal degree $d(\pi)$ of π is given by

$$d(\pi) = \frac{\deg\lambda}{\mathrm{vol}(J)}. \tag{1.3}$$

The formal degree $d(\pi)$ depends on the choice of Haar measure on G. In [8], Jabon, Keys and Moy chose the Haar measure on G normalized so that the volume of a special maximal compact subgroup $G \cap GL_3(\mathfrak{o}_F)$ equals to 1. We however use another normalization.

Let p be an odd prime and let q be a positive power of p. Put $\mathsf{G} = U(3)(\mathbf{F}_{q^2}/\mathbf{F}_q)$. Let τ be an irreducible cuspidal representation of G. It is well known that

$$\dim\tau = (q-1)(q+1)^2,\ (q-1)(q^2-q+1),\ \text{or}\ q(q-1). \tag{1.4}$$

Let U be a maximal unipotent subgroup of G. Then we have

$$\frac{|\mathsf{U}|\dim\tau}{|\mathsf{G}|} = \frac{1}{q^3+1},\ \frac{1}{(q+1)^3},\ \text{or}\ \frac{q}{(q^3+1)(q+1)^2}. \tag{1.5}$$

This is the usual normalization of dimensions of irreducible representations in the representation theory of finite groups of Lie type. We can identify

$$\frac{|\mathsf{U}|\deg\tau}{|\mathsf{G}|} = \mathrm{vol}(\mathsf{U})d(\tau).$$

To obtain an analog for p-adic $U(3)$, we normalize Haar measure on G so that the volume of the first congruence subgroup B_1 of the standard Iwahori subgroup of G is 1:

$$B_1 = \left(1 + \begin{pmatrix} \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix}\right) \cap G. \tag{1.6}$$

Then the following proposition holds:

Proposition 1.1. *Suppose that F is ramified over F_0. Let π be an irreducible supercuspidal representation of G. Then we have*

$$d(\pi) = \frac{q^a}{(q+1)^b 2^c},$$

for some $a, b, c \geq 0$.

Remark 1.2. Suppose that F is unramified over F_0. Then by [8], for a supercuspidal representation π of G, we have

$$d(\pi) = \frac{q^a}{(q^3+1)^b (q+1)^c},$$

for some $a, b, c \geq 0$.

This research has an application to the local Langlands correspondence for G. Recently, by investigating the local theta correspondence, Blasco [3] proved that a very cuspidal representation π of G is stable, that is, π forms a singleton L-packet on G. She also described the base change for very cuspidal representations of G in terms of theory of types. We give a new proof of stability of very cuspidal representations of G by showing that very cuspidal representations are characterized by their formal degrees and they are all generic. Our proof is also valid for depth zero supercuspidal representations of unramified G.

2 The supercuspidal representations

2.1 Construction

We begin by recalling Stevens' construction of the supercuspidal representations of p-adic classical groups. For more details, one should consult [11] and [12].

Let F be a non-archimedean local field. Let $\mathfrak{o}_F$ denote the ring of integers in F, $\mathfrak{p}_F$ the maximal ideal in $\mathfrak{o}_F$ and $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ the residue field. We always assume the characteristic p of k_F is not equal to 2. For any arbitrary non-archimedean local field E, we write $\mathfrak{o}_E$, $\mathfrak{p}_E$ and k_E for the analogous objects for E.

Let $^-$ be a galois involution of F. We allow the possibility $^-$ is trivial. Let F_0 denote the subfield of F consisting of the $^-$-fixed elements. We write $\mathfrak{o}_0$, $\mathfrak{p}_0$ and k_0 for the analogous objects for F_0 and put $q = \mathrm{Card}(k_0)$.

Let h be a nondegenerate hermitian or skew hermitian form on a finite dimensional F-vector space V. We also denote by $^-$ the involution on A induced by h. We write $A = \mathrm{End}_F(V)$ and $A_- = \{X \in A \mid X + \overline{X} = 0\}$. Let G^+ denote the group of isometries of (V, h) and G the connected component of G^+. Then G is the F_0-points of a unitary, symplectic, or special orthogonal group, and the Lie algebra of G is isomorphic to A_-.

Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum in A (see [11] Definition 3.2). Then β is a semisimple element in A_-. We write $E = F[\beta]$, $B = \mathrm{End}_E(V)$, and G_E for the G-centralizer of β. Note that G_E is not contained in any proper parabolic subgroup of G. The self-dual $\mathfrak{o}_E$-lattice sequence Λ in V gives rise to a kind of valuation ν_Λ on A, and the non-negative integer n is equal to $-\nu_\Lambda(\beta)$. The sequence Λ defines a decreasing filtration $\{\mathfrak{a}_k(\Lambda)\}_{k\in\mathbf{Z}}$ on A by its $^-$-stable open compact $\mathfrak{o}_F$-lattices . We get a filtration $\{P_k(\Lambda)\}_{k\geq 0}$ of a parahoric subgroup $P_0(\Lambda) = G \cap \mathfrak{a}_0(\Lambda)$ of G by its open normal subgroups, where $P_k(\Lambda) = G \cap (1 + \mathfrak{a}_k(\Lambda))$, $k \geq 1$. Put $P_k(\Lambda_{\mathfrak{o}_E}) = G_E \cap P_k(\Lambda)$, for $k \geq 0$. Then $\{P_k(\Lambda_{\mathfrak{o}_E})\}_{k\geq 0}$ is a filtration of a parahoric subgroup $P_0(\Lambda_{\mathfrak{o}_E})$ of G_E by its open normal subgroups.

From a skew semisimple stratum $[\Lambda, n, 0, \beta]$, we obtain open compact subgroups

$$H^1 \subset J^1 \subset J \tag{2.1}$$

of G (see [11] §3.2). The groups H^1 and J^1 are both pro-q subgroups of G. The group J is given by $J = P_0(\Lambda_{\mathfrak{o}_E})J^1$ and the quotient J/J^1 is isomorphic to $P_0(\Lambda_{\mathfrak{o}_E})/P_1(\Lambda_{\mathfrak{o}_E})$.

Let θ be a semisimple character associated to $[\Lambda, n, 0, \beta]$ (see [11] Definition 3.13). Then θ is an abelian character of H^1. By [11] Corollary 3.29, there exists a unique irreducible representation η of J^1 such that $\mathrm{Hom}_{H^1}(\eta|_{H^1}, \theta) \neq \{0\}$. The degree $\deg(\eta)$ of η is given by $\deg(\eta) = [J^1 : H^1]^{1/2}$.

Suppose that $B \cap \mathfrak{a}_0(\Lambda)$ is a maximal $^-$-stable $\mathfrak{o}_E$-order in B. Then J/J^1 is isomorphic to a product of classical groups defined over extensions over k_0. Note that the group J/J^1 is not always connected. Let κ be a β-extension of η (see [12] §4.1). Then κ is an extension of η to J. Let τ be an irreducible cuspidal representation of J/J^1, that is, an irreducible representation of J/J^1 whose restriction to the connected component of J/J^1 is irreducible and cuspidal. Then $\pi = \mathrm{ind}_J^G \kappa \otimes \tau$ is an irreducible supercuspidal representation of G. It follows from [12] Theorem 7.14 that every irreducible supercuspidal representation is obtained in this way.

2.2 Formal degrees

Let $\pi = \mathrm{ind}_J^G \kappa \otimes \tau$ be an irreducible supercuspidal representation of G with underlying skew semisimple stratum $[\Lambda, n, 0, \beta]$. It follows from (1.3), the formal degree $d(\pi)$ of π is given by

$$d(\pi) = \frac{\deg(\kappa \otimes \tau)}{\mathrm{vol}(J)}. \tag{2.2}$$

By [12] Corollary 2.9 and [6] (2.10), there exists a self-dual $\mathfrak{o}_E$-lattice sequence Λ^{m} in V such that $\mathfrak{a}_0(\Lambda^{\mathrm{m}}) \cap B$ is a minimal $^-$-stable $\mathfrak{o}_E$-order in B and $\mathfrak{a}_1(\Lambda^{\mathrm{m}}) \supset \mathfrak{a}_1(\Lambda)$.

We normalize Haar measure on G so that the volume of the first congruence subgroup B_1 of an Iwahori subgroup is 1. Then we obtain the following proposition:

Proposition 2.1. *Let $\pi = \mathrm{ind}_J^G \kappa \otimes \tau$ be an irreducible supercuspidal representation of G with underlying skew semisimple stratum $[\Lambda, n, 0, \beta]$. Then we have*

$$d(\pi) = \frac{[B_1 : J^1][J^1 : H^1]^{1/2}}{[P_1(\Lambda^{\mathrm{m}}_{\mathfrak{o}_E}) : P_1(\Lambda_{\mathfrak{o}_E})]} \frac{\deg(\tau)}{[P_0(\Lambda_{\mathfrak{o}_E}) : P_1(\Lambda^{\mathrm{m}}_{\mathfrak{o}_E})]}. \tag{2.3}$$

Note that $P_1(\Lambda^{\mathrm{m}}_{\mathfrak{o}_E})$ is the first congruence subgroup of the Iwahori subgroup $P_0(\Lambda^{\mathrm{m}}_{\mathfrak{o}_E})$ of G_E. Put $\mathsf{G} = P_0(\Lambda_{\mathfrak{o}_E})/P_1(\Lambda_{\mathfrak{o}_E})$ and $\mathsf{U} = P_1(\Lambda^{\mathrm{m}}_{\mathfrak{o}_E})/P_1(\Lambda_{\mathfrak{o}_E})$.

Then U is a maximal unipotent subgroup of G. We put

$$d(\pi)_{p'} = \frac{\deg(\tau)}{[P_0(\Lambda_{\mathfrak{o}_E}) : P_1(\Lambda^{\mathfrak{m}}_{\mathfrak{o}_E})]}. \tag{2.4}$$

Then we have

$$d(\pi)_{p'} = \frac{|\mathsf{U}| \deg(\tau)}{|\mathsf{G}|}. \tag{2.5}$$

Therefore, we can reduce the computation of $d(\pi)_{p'}$ to the representation theory of finite groups of Lie type.

Remark 2.2. Although all supercuspidal representations of p-adic classical groups are constructed, they have not been classified. So the term $d(\pi)_{p'}$ depends on the way of construction of π.

Next, the term $d(\pi)/d(\pi)_{p'} = [B_1 : J^1][J^1 : H^1]^{1/2}[P_1(\Lambda^{\mathfrak{m}}_{\mathfrak{o}_E}) : P_1(\Lambda_{\mathfrak{o}_E})]^{-1}$ is a non-negative power of $q = \mathrm{Card}(k_0)$ because all groups in this term are pro-q subgroups of G or G_E.

To compute $d(\pi)/d(\pi)_{p'}$, we recall the definition of the groups H^1 and J^1. For a skew semisimple stratum $[\Lambda, n, 0, \beta]$, we get a sequence of skew semisimple strata $\{[\Lambda, n, r_i, \gamma_i]\}_{i=0,\ldots,k}$ such that

(i) $0 = r_0 < r_1 < \ldots < r_k = n$;

(ii) $\gamma_0 = \beta$ and $\gamma_n = 0$;

(iii) $[\Lambda, n, r_{i+1}, \gamma_i]$ is equivalent to $[\Lambda, n, r_{i+1}, \gamma_{i+1}]$, that is, $\nu_\Lambda(\gamma_i - \gamma_{i+1}) \geq -r_{i+1}$.

Put $G_i = C_G(\gamma_i)$. Then we have

$$\begin{aligned} H^1 &= (G_0 \cap P_1)(G_1 \cap P_{[r_1/2]+1}) \cdots (G_{k-1} \cap P_{[r_{k-1}/2]+1})P_{[n/2]+1}, \\ J^1 &= (G_0 \cap P_1)(G_1 \cap P_{[(r_1+1)/2]}) \cdots (G_{k-1} \cap P_{[(r_{k-1}+1)/2]})P_{[(n+1)/2]+1}. \end{aligned}$$

So we get

$$\begin{aligned} d(\pi)/d(\pi)_{p'} &= \frac{[B_1 : P_1]}{[P_1(\Lambda^{\mathfrak{m}}_{\mathfrak{o}_E}) : P_1(\Lambda_{\mathfrak{o}_E})]} \\ &\quad \times \Pi_{i=1}^{k} x_i(1, [\frac{r_i+1}{2}]) \cdot x_i([\frac{r_i+1}{2}], [\frac{r_i}{2}] + 1)^{1/2}, \end{aligned}$$

where $x_i(s,t) = \dfrac{[G_i \cap P_s : G_i \cap P_t]}{[G_{i-1} \cap P_s : G_{i-1} \cap P_t]}$.

Suppose that F is quadratic ramified over F_0. Let $G = U(3)(F/F_0)$. Let e denote the F_0-period of Λ. Then we can check that $x_i(s,t)$ is e-periodic.

Write $r_i = et_i - s_i$, $0 \le s_i < e$. We obtain

$$d(\pi)/d(\pi)_{p'} = q^m,$$

where $m = \sum_{i=1}^{k} t_i \dfrac{\dim_{F_0} \mathrm{Lie}(G_i) - \dim_{F_0} \mathrm{Lie}(G_{i-1})}{2} - j$ for some j.

3 Ramified $U(3)$ case

We shall return to the case of ramified $U(3)$. We let $G = U(3)(F/F_0)$, where F is ramified over F_0. Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum for G. Then the G-centralizer of β has one of the following forms. In the table below, we write $U(1,1)$ for the quasi-split unitary group in two variables, $U(2)$ for the anisotropic unitary group in two variables, and $U(1)$ for the norm-1 subgroup of the multiplicative group of F.

For each type of G_E, the quotient $\mathsf{G} = J/J^1$ has one of the following forms:

G_E	G
$U(3)$	$O(2,1)$ $SL(2) \times O(1)$
$U(1,1) \times U(1)$	$SL(2) \times O(1)$
$U(2) \times U(1)$	$O(2) \times O(1)$
$U(1)^3$	$O(1)^3$
$U(1)(E_1/E_{1,0}) \times U(1)$ E_1/F : quadratic	$O(1)^2$, $E_1/E_{1,0}$: ramified $U(1)(k_{E_1}/k_{E_{1,0}}) \times O(1)$, $E_1/E_{1,0}$: unramified
$U(1)(E/E_0)$ E/F : cubic	$O(1)$

Fortunately, we know degrees of all irreducible cuspidal representations of G. We therefore get the term $d(\pi)_{p'}$ for all supercuspidal representations π of G. Recall that $d(\pi)/d(\pi)_{p'}$ is a non-negative power of q. So we obtain the following proposition:

Proposition 3.1. *Let π be an irreducible supercuspidal representation of G. Then we have*

$$d(\pi) = \frac{q^a}{(q+1)^b 2^c},$$

for some $a, b, c \geq 0$.

In the computation of $d(\pi)/d(\pi)_{p'}$, we can ignore an element $[\Lambda, n, r_i, \gamma_i]$ in a sequence $\{[\Lambda, n, r_i, \gamma_i]\}_{i=0,\ldots,k}$ such that $G_i = G_{i+1}$. Therefore we need only sequences of semisimple strata with $k \leq 3$, that is, $\{[\Lambda, n, 0, \beta], \ [\Lambda, n, n, 0]\}$ or $\{[\Lambda, n, 0, \beta], \ [\Lambda, n, r, \gamma], \ [\Lambda, n, n, 0]\}$. For each type of β, there exist at most two choices of Λ because $\mathfrak{a}_0(\Lambda) \cap B$ is a maximal $^-$-stable $\mathfrak{o}_E$-order.

Now we obtain the following table of formal degrees of the supercuspidal representations of ramified $U(3)$:

n/e	r/e	a	b	c
0		0	1	1
		0	1	2
m		$3m$	1	1
$m-1/2$		$3m-2$	0	1
$m-1/2$		$3m-2$	0	2
$m-1/2$		$3m-2$	0	3
$m-1/2$		$2m-1$	1	1
		$2m-1$	1	2
	k	$2m+k-1$	1	1
	$k-1/2$	$2m+k-2$	0	2
	$k-1/2$	$2m+k-2$	0	3
$m-1/6$		$3m-1$	0	1
$m-5/6$		$3m-3$	0	1

A special representation of G is a discrete series representation of G which is not supercuspidal. By [8], the formal degree of a special representation π of G is given by

$$d(\pi) = \begin{cases} \dfrac{1}{q+1}, & \text{if } \pi \text{ is a twist of the Steiberg representation;} \\ \dfrac{q^m}{(q+1)2}, \quad m \geq 0, & \text{otherwise.} \end{cases}$$

4 An application to the LLC

4.1 Discrete L-packets on G

From now on, we further assume that $\mathrm{ch}(F_0) = 0$. Suppose F is ramified over F_0. Let $G = U(3)(F/F_0)$. Let $\Pi(G)$ denote the discrete L-packets on G. By [7] and [10], $\Pi(G)$ has the following properties:

(i) $\Pi(G)$ is a partition of the discrete series representations of G by finite subsets;

(ii) Let $\Pi \in \Pi(G)$. Then $d(\pi_1) = d(\pi_2)$, for $\pi_1, \pi_2 \in \Pi$;

(iii) Every discrete L-packet $\Pi \in \Pi(G)$ contains exactly one generic representation;

(iv) A discrete series representation π of G is stable if and only if $\{\pi\} \in \Pi(G)$.

4.2 Stable discrete series

We know the following representations of G are stable:

(i) a twist of the Steinberg representation of G([10]);

(ii) a very cuspidal representation of G, that is an irreducible supercuspidal representation π of G with underlying skew stratum $[\Lambda, n, 0, \beta]$ such that $E = F[\beta]$ is a cubic extension over F([3]).

Remark 4.1. Blasco [3] proved stability of very cuspidal representations of (ramified and unramified) $U(3)$ by investigating the local theta correspondence.

We can characterize these stable discrete series representations by formal degrees.

Proposition 4.2. *Let π be a discrete series representation of G. Then*

(i) π is a twist of the Steinberg representation if and only if $d(\pi) = \dfrac{1}{q+1}$,

(ii) π is a very cuspidal representation if and only if $d(\pi) = \dfrac{q^m}{2}$, for $m \geq 0$.

Now we get a new proof of stability of very cuspidal representations of G. By basic properties of discrete L-packets on G and Proposition 4.2, it is enough to prove the following lemma:

Lemma 4.3. *A very cuspidal representation of G is generic.*

4.3 Genericity of very cuspidal representations

We shall prove Lemma 4.3. This proof is based on results by Blondel-Stevens [4] for $Sp(4)$.

Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum for G such that $E = F[\beta]$ is a cubic extension over F_0. Let $\pi = \mathrm{ind}_J^G \lambda$ be an irreducible supercuspidal representation of G with underlying skew semisimple stratum $[\Lambda, n, 0, \beta]$.

It follows from [5] Proposition 1.6 that π is generic if and only if there exists a nondegenerate character χ of a maximal unipotent subgroup U of G such that

$$\mathrm{Hom}_{J\cap U}(\lambda|_{J\cap U}, \chi|_{J\cap U}) \neq \{0\}.$$

Note that a maximal unipotent subgroup U of G corresponds to a flag $\{0\} \subsetneq V_1 \subsetneq V_1^{\perp} \subsetneq V$, where $V_1^{\perp}$ denotes the orthogonal complement of V_1.

Let ψ_0 be an additive character of F_0 with conductor $\mathfrak{p}_0$. We define a map $\psi_\beta : M_3(F) \to \mathbf{C}$ by

$$\psi_\beta(x) = \psi_0(\mathrm{tr}_{F/F_0} \circ \mathrm{tr}_{M_3(F)/F}(\beta(x-1))), \ x \in M_3(F). \tag{4.6}$$

Let U be a maximal unipotent subgroup of G corresponding to a flag $\{0\} \subsetneq V_1 \subsetneq V_1^{\perp} \subsetneq V$. Then it follows from [4] Proposition 3.1 that $\psi_\beta|_U$ is a character of U if and only if $\beta V_1 \subset V_1^{\perp}$.

By the assumption that E is cubic over F, we can find such a flag of V, and hence we get a maximal unipotent subgroup U of G such that $\psi_\beta|_U$ is a character of U.

By the construction of J and λ, the restriction of λ to $J \cap U$ contains $\psi_\beta|_{J\cap U}$. This completes the proof of Lemma 4.3.

4.4 Unramified case

Suppose F is unramified over F_0. In this case, we know the following discrete series representations of $G = U(3)(F/F_0)$ are stable:

(i) a twist of the Steinberg representation of G([10]);

(ii) a twist of a depth 0 supercuspidal representation, that is, a twist of $\mathrm{ind}_J^G \tau$ where J is a conjugate of a special maximal compact subgroup $G \cap GL_3(\mathfrak{o}_F)$ and τ is an inflation of a cubic cuspidal representation of $U(3)(k_F/k_0)$([1]);

(iii) a very cuspidal representation of G([3]).

We note that our proof of stability is valid for supercuspidal representations in cases (ii) and (iii). In fact, we can characterize these representations by their formal degrees. By [8], for a discrete series representation π of G, we have

$$\pi \text{ is a twist of the Steinberg representation} \iff d(\pi) = \frac{q^2+1}{(q^3+1)(q+1)^2},$$

$$\pi \text{ is a twist of a depth 0 supercuspidal representation} \iff d(\pi) = \frac{1}{q^3+1}.$$

$$\pi \text{ is a very cuspidal representation} \iff d(\pi) = \frac{q^{m-1}}{q+1} \text{ or } \frac{q^m}{q^3+1}, \text{ for } m > 0.$$

Moreover, we can prove genericity of representations in cases (ii) and (iii) along with the lines of [4].

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