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Formal degrees of supercuspidal representations of ramified $U(3)$ (Automorphic representations, automorphic $L$-functions and arithmetic)

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Formal degrees of supercuspidal representations of ramified $U(3)$

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Abstract

Formal degrees of supercuspidal representations of $p$-adic unramified $U(3)$ are obtained as a part of the explicit Plancherel formula by Jabon-Keys-Moy. In this note, we compute those of ramified $U(3)$ in terms of supercuspidal types. As a corollary, we give a new proof of stability of very cuspidal representations of $U(3)$.

1 Introduction

Let $F_0$ be a non-archimedean local field. Let $\mathfrak{o}_0$ denote the ring of integers in $F_0$, $\mathfrak{p}_0 = \mathfrak{w}_0 \mathfrak{o}_0$ the maximal ideal in $\mathfrak{o}_0$, and $k_0 = \mathfrak{o}_0/\mathfrak{p}_0$ the residue field. Throughout this paper, we will always assume that the characteristic $p$ of $k_0$ is not 2. We denote by $q$ the cardinality of $k_0$.

Let $F$ be a quadratic extension over $F_0$. We write $\mathfrak{o}_F$, $\mathfrak{p}_F$ and $k_F$ for the analogous objects for $F$. Let $- \in \text{Gal}(F/F_0)$. We choose a uniformizer $\mathfrak{w}_F$ of $F$ so that $\overline{\mathfrak{w}_F} = \pm \mathfrak{w}_F$.

Let $V = F^3$ be the space of three dimensional column vectors and let $h$ denote the hermitian form on $V$ defined by

$$h(v,w) = {}^t\overline{v}Hw, \ v,w \in V,$$

where

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.2}$$

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Put $G = U(3)(F/F_0) = \{ g \in GL_3(F) \mid {}^t\overline{g}Hg = H \}$. Then $G$ is the $F_0$-points of a unitary group in three variables defined over $F_0$.

Jabon, Keys and Moy [8] gave an explicit Plancherel formula of $G$. In particular, they computed formal degrees of the discrete series representations of $G$. But they assumed that $F$ is unramified over $F_0$, when they calculated formal degrees of supercuspidal representations of $G$. The aim of this note is to determine formal degrees of the supercuspidal representations of $G$ when $F$ is ramified over $F_0$. This result completes the explicit Plancherel formula of $G$ by Jabon-Keys-Moy.

The result in [8] is based on Moy's classification of the irreducible admissible representations of unramified $G$ in [9], and formal degrees of the supercuspidal representations of unramified $G$ are given in terms of nondegenerate representations in loc. cit. After Moy's work [9], Blasco [2] constructed the supercuspidal representations of $G$ via compact induction from representations of open compact subgroups of $G$. Moreover Stevens [12] proved that all supercuspidal representations of a $p$-adic classical group come via compact induction from maximal simple types. In this note, we will use Stevens' construction to describe the supercuspidal representations of ramified $G$.

Let $\pi$ be an irreducible supercuspidal representation of $G$. Then it follows from [2] and [12] that there is an irreducible representation $\lambda$ of an open compact subgroup $J$ of $G$ such that $\pi$ is isomorphic to $\text{ind}^G_J \lambda$. By the well-known fact on formal degrees, the formal degree $d(\pi)$ of $\pi$ is given by

$$d(\pi) = \frac{\deg \lambda}{\text{vol}(J)}. \quad (1.3)$$

The formal degree $d(\pi)$ depends on the choice of Haar measure on $G$. In [8], Jabon, Keys and Moy chose the Haar measure on $G$ normalized so that the volume of a special maximal compact subgroup $G \cap GL_3(\mathfrak{o}_F)$ equals to 1. We however use another normalization.

Let $p$ be an odd prime and let $q$ be a positive power of $p$. Put $G = U(3)(\mathbf{F}_{q^2}/\mathbf{F}_q)$. Let $\tau$ be an irreducible cuspidal representation of $G$. It is well known that

$$\dim \tau = (q - 1)(q + 1)^2, \ (q - 1)(q^2 - q + 1), \text{ or } q(q - 1). \quad (1.4)$$

Let $U$ be a maximal unipotent subgroup of $G$. Then we have

$$\frac{|U| \dim \tau}{|G|} = \frac{1}{q^3 + 1}, \ \frac{1}{(q + 1)^3}, \text{ or } \frac{q}{(q^3 + 1)(q + 1)^2}. \quad (1.5)$$
This is the usual normalization of dimensions of irreducible representations in the representation theory of finite groups of Lie type. We can identify

$$\frac{|U| \deg \tau}{|G|} = \text{vol}(U)d(\tau).$$

To obtain an analog for $p$-adic $U(3)$, we normalize Haar measure on $G$ so that the volume of the first congruence subgroup $B_1$ of the standard Iwahori subgroup of $G$ is 1:

$$B_1 = \left( 1 + \begin{pmatrix} p_F & o_F & o_F \\ p_F & p_F & o_F \\ p_F & p_F & p_F \end{pmatrix} \right) \cap G. \quad (1.6)$$

Then the following proposition holds:

**Proposition 1.1.** Suppose that $F$ is ramified over $F_0$. Let $\pi$ be an irreducible supercuspidal representation of $G$. Then we have

$$d(\pi) = \frac{q^a}{(q+1)^b2^c},$$

for some $a, b, c \geq 0$.

**Remark 1.2.** Suppose that $F$ is unramified over $F_0$. Then by [8], for a supercuspidal representation $\pi$ of $G$, we have

$$d(\pi) = \frac{q^a}{(q^3+1)^b(q+1)^c},$$

for some $a, b, c \geq 0$.

This research has an application to the local Langlands correspondence for $G$. Recently, by investigating the local theta correspondence, Blasco [3] proved that a very cuspidal representation $\pi$ of $G$ is stable, that is, $\pi$ forms a singleton $L$-packet on $G$. She also described the base change for very cuspidal representations of $G$ in terms of theory of types. We give a new proof of stability of very cuspidal representations of $G$ by showing that very cuspidal representations are characterized by their formal degrees and they are all generic. Our proof is also valid for depth zero supercuspidal representations of unramified $G$.  

2 The supercuspidal representations

2.1 Construction

We begin by recalling Stevens' construction of the supercuspidal representations of $p$-adic classical groups. For more details, one should consult [11] and [12].

Let $F$ be a non-archimedean local field. Let $\mathfrak{o}_F$ denote the ring of integers in $F$, $\mathfrak{p}_F$ the maximal ideal in $\mathfrak{o}_F$ and $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ the residue field. We always assume the characteristic $p$ of $k_F$ is not equal to 2. For any arbitrary non-archimedean local field $E$, we write $\mathfrak{o}_E$, $\mathfrak{p}_E$ and $k_E$ for the analogous objects for $E$.

Let $-\$ be a galois involution of $F$. We allow the possibility $-\$ is trivial. Let $F_0$ denote the subfield of $F$ consisting of the $-\$-fixed elements. We write $\mathfrak{o}_0$, $\mathfrak{p}_0$ and $k_0$ for the analogous objects for $F_0$ and put $q = \text{Card}(k_0)$.

Let $h$ be a nondegenerate hermitian or skew hermitian form on a finite dimensional $F$-vector space $V$. We also denote by $-\$ the involution on $A$ induced by $h$. We write $A = \text{End}_F(V)$ and $A_- = \{X \in A \mid X + \overline{X} = 0\}$. Let $G^+$ denote the group of isometries of $(V, h)$ and $G$ the connected component of $G^+$. Then $G$ is the $F_0$-points of a unitary, symplectic, or special orthogonal group, and the Lie algebra of $G$ is isomorphic to $A_-$. Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum in $A$ (see [11] Definition 3.2). Then $\beta$ is a semisimple element in $A_-$. We write $E = F[\beta]$, $B = \text{End}_E(V)$, and $G_E$ for the $G$-centralizer of $\beta$. Note that $G_E$ is not contained in any proper parabolic subgroup of $G$. The self-dual $\mathfrak{o}_E$-lattice sequence $\Lambda$ in $V$ gives rise to a kind of valuation $\nu_\Lambda$ on $A$, and the non-negative integer $n$ is equal to $-\nu_\Lambda(\beta)$. The sequence $\Lambda$ defines a decreasing filtration $\{a_k(\Lambda)\}_{k \in \mathbb{Z}}$ on $A$ by its $-\$-stable open compact $\mathfrak{o}_F$-lattices. We get a filtration $\{P_k(\Lambda)\}_{k \geq 0}$ of a parahoric subgroup $P_0(\Lambda) = G \cap a_0(\Lambda)$ of $G$ by its open normal subgroups, where $P_k(\Lambda) = G \cap (1 + a_k(\Lambda))$, $k \geq 1$. Put $P_k(\Lambda_{o_E}) = G_E \cap P_k(\Lambda)$, for $k \geq 0$. Then $\{P_k(\Lambda_{o_E})\}_{k \geq 0}$ is a filtration of a parahoric subgroup $P_0(\Lambda_{o_E})$ of $G_E$ by its open normal subgroups.

From a skew semisimple stratum $[\Lambda, n, 0, \beta]$, we obtain open compact subgroups

$$H^1 \subset J^1 \subset J$$ \hspace{1cm} (2.1)

of $G$ (see [11] §3.2). The groups $H^1$ and $J^1$ are both pro-$q$ subgroups of $G$. The group $J$ is given by $J = P_0(\Lambda_{o_E})J^1$ and the quotient $J/J^1$ is isomorphic to $P_0(\Lambda_{o_E})/P_1(\Lambda_{o_E})$. 

Let $\theta$ be a semisimple character associated to $[\Lambda, n, 0, \beta]$ (see [11] Definition 3.13). Then $\theta$ is an abelian character of $H^1$. By [11] Corollary 3.29, there exists a unique irreducible representation $\eta$ of $J^1$ such that $\text{Hom}_{H^1}(\eta|_{H^1}, \theta) \neq \{0\}$. The degree $\deg(\eta)$ of $\eta$ is given by $\deg(\eta) = [J^1 : H^1]^{1/2}$.

Suppose that $B \cap a_0(\Lambda)$ is a maximal $-\beta$-stable $0_E$-order in $B$. Then $J/J^1$ is isomorphic to a product of classical groups defined over extensions over $k_0$. Note that the group $J/J^1$ is not always connected. Let $\kappa$ be a $\beta$-extension of $\eta$ (see [12] §4.1). Then $\kappa$ is an extension of $\eta$ to $J$. Let $\tau$ be an irreducible cuspidal representation of $J/J^1$, that is, an irreducible representation of $J/J^1$ whose restriction to the connected component of $J/J^1$ is irreducible and cuspidal. Then $\pi = \text{ind}_J^G \kappa \otimes \tau$ is an irreducible supercuspidal representation of $G$. It follows from [12] Theorem 7.14 that every irreducible supercuspidal representation is obtained in this way.

### 2.2 Formal degrees

Let $\pi = \text{ind}_J^G \kappa \otimes \tau$ be an irreducible supercuspidal representation of $G$ with underlying skew semisimple stratum $[\Lambda, n, 0, \beta]$. It follows from (1.3), the formal degree $d(\pi)$ of $\pi$ is given by

$$d(\pi) = \frac{\deg(\kappa \otimes \tau)}{\text{vol}(J)}.$$

(2.2)

By [12] Corollary 2.9 and [6] (2.10), there exists a self-dual $\mathfrak{o}_E$-lattice sequence $\Lambda^m$ in $V$ such that $a_0(\Lambda^m) \cap B$ is a minimal $-\beta$-stable $\mathfrak{o}_E$-order in $B$ and $a_1(\Lambda^m) \supset a_1(\Lambda)$.

We normalize Haar measure on $G$ so that the volume of the first congruence subgroup $B_1$ of an Iwahori subgroup is 1. Then we obtain the following proposition:

**Proposition 2.1.** Let $\pi = \text{ind}_J^G \kappa \otimes \tau$ be an irreducible supercuspidal representation of $G$ with underlying skew semisimple stratum $[\Lambda, n, 0, \beta]$. Then we have

$$d(\pi) = \frac{[B_1 : J^1][J^1 : H^1]^{1/2}}{[P_1(\Lambda^m_{\sigma_E}) : P_1(\Lambda_{\sigma_E})][P_0(\Lambda_{\sigma_E}) : P_1(\Lambda^m_{\sigma_E})]} \frac{\deg(\tau)}{[P_1(\Lambda^m_{\sigma_E}) : P_1(\Lambda_{\sigma_E})]}.$$

(2.3)

Note that $P_1(\Lambda^m_{\sigma_E})$ is the first congruence subgroup of the Iwahori subgroup $P_0(\Lambda_{\sigma_E})$ of $G_E$. Put $G = P_0(\Lambda_{\sigma_E})/P_1(\Lambda_{\sigma_E})$ and $U = P_1(\Lambda^m_{\sigma_E})/P_1(\Lambda_{\sigma_E})$. 
Then $U$ is a maximal unipotent subgroup of $G$. We put

$$d(\pi)_{p'} = \frac{\deg(\tau)}{[P_0(\Lambda_{0_E}) : P_1(\Lambda_{0_E})]}.$$  \hspace{1cm} (2.4)

Then we have

$$d(\pi)_{p'} = \frac{|U|\deg(\tau)}{|G|}.$$  \hspace{1cm} (2.5)

Therefore, we can reduce the computation of $d(\pi)_{p'}$ to the representation theory of finite groups of Lie type.

**Remark 2.2.** Although all supercuspidal representations of $p$-adic classical groups are constructed, they have not been classified. So the term $d(\pi)_{p'}$ depends on the way of construction of $\pi$.

Next, the term $d(\pi)/d(\pi)_{p'} = [B_1 : J^1][J^1 : H^1]^{1/2}[P_1(\Lambda_{0_E}^m) : P_1(\Lambda_{0_E})]^{-1}$ is a non-negative power of $q = \text{Card}(k_0)$ because all groups in this term are pro-$q$ subgroups of $G$ or $G_E$.

To compute $d(\pi)/d(\pi)_{p'}$, we recall the definition of the groups $H^1$ and $J^1$. For a skew semisimple stratum $[\Lambda, n, 0, \beta]$, we get a sequence of skew semisimple strata $\{[\Lambda, n, r_i, \gamma_i]\}_{i=0,\ldots,k}$ such that

(i) $0 = r_0 < r_1 < \ldots < r_k = n$;

(ii) $\gamma_0 = \beta$ and $\gamma_n = 0$;

(iii) $[\Lambda, n, r_{i+1}, \gamma_i]$ is equivalent to $[\Lambda, n, r_{i+1}, \gamma_{i+1}]$, that is, $\nu_\Lambda(\gamma_i - \gamma_{i+1}) \geq -r_{i+1}$.

Put $G_i = C_G(\gamma_i)$. Then we have

$$H^1 = (G_0 \cap P_1)(G_1 \cap P_{[r_1/2]+1}) \cdots (G_{k-1} \cap P_{[r_{k-1}/2]+1})P_{[n/2]+1},$$

$$J^1 = (G_0 \cap P_1)(G_1 \cap P_{(r_1+1)/2}) \cdots (G_{k-1} \cap P_{(r_{k-1}+1)/2})P_{(n+1)/2}+1.$$  \hspace{1cm} (2.6)

So we get

$$d(\pi)/d(\pi)_{p'} = \frac{[B_1 : P_1]}{[P_1(\Lambda_{0_E}^m) : P_1(\Lambda_{0_E})]} x_i(1, \left[\frac{r_i + 1}{2}\right]) \cdot x_i(\left[\frac{r_i + 1}{2}\right], \left[\frac{r_i}{2}\right] + 1)^{1/2},$$

where $i = 1, \ldots, k.$
where \(x_i(s, t) = \frac{[G_i \cap P_s : G_i \cap P_t]}{[G_{i-1} \cap P_s : G_{i-1} \cap P_t]}\).

Suppose that \(F\) is quadratic ramified over \(F_0\). Let \(G = U(3)(F/F_0)\). Let \(e\) denote the \(F_0\)-period of \(\Lambda\). Then we can check that \(x_i(s, t)\) is \(e\)-periodic.

Write \(r_i = et_i - s_i, 0 \leq s_i < e\). We obtain

\[d(\pi)/d(\pi)_{p'} = q^m,\]

where \(m = \sum_{i=1}^{k} t_i \frac{\dim_{F_0} \text{Lie}(G_i) - \dim_{F_0} \text{Lie}(G_{i-1})}{2} - j\) for some \(j\).

### 3 Ramified \(U(3)\) case

We shall return to the case of ramified \(U(3)\). We let \(G = U(3)(F/F_0)\), where \(F\) is ramified over \(F_0\). Let \([\Lambda, n, 0, \beta]\) be a skew semisimple stratum for \(G\). Then the \(G\)-centralizer of \(\beta\) has one of the following forms. In the table below, we write \(U(1, 1)\) for the quasi-split unitary group in two variables, \(U(2)\) for the anisotropic unitary group in two variables, and \(U(1)\) for the norm-1 subgroup of the multiplicative group of \(F\).

For each type of \(G_E\), the quotient \(G = J/J^1\) has one of the following forms:

<table>
<thead>
<tr>
<th>(G_E)</th>
<th>(G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(U(3))</td>
<td>(O(2, 1))</td>
</tr>
<tr>
<td>(SL(2) \times O(1))</td>
<td></td>
</tr>
<tr>
<td>(U(1, 1) \times U(1))</td>
<td>(SL(2) \times O(1))</td>
</tr>
<tr>
<td>(U(2) \times U(1))</td>
<td>(O(2) \times O(1))</td>
</tr>
<tr>
<td>(U(1)^3)</td>
<td>(O(1)^3)</td>
</tr>
<tr>
<td>(U(1)(E_1/E_{1,0}) \times U(1))</td>
<td>(O(1)^2, E_1/E_{1,0} : \text{ramified})</td>
</tr>
<tr>
<td>(E_1/F : \text{quadratic})</td>
<td>(U(1)(k_{E_1}/k_{E_{1,0}}) \times O(1), E_1/E_{1,0} : \text{unramified})</td>
</tr>
<tr>
<td>(U(1)(E/E_0))</td>
<td>(O(1))</td>
</tr>
<tr>
<td>(E/F : \text{cubic})</td>
<td></td>
</tr>
</tbody>
</table>

Fortunately, we know degrees of all irreducible cuspidal representations of \(G\). We therefore get the term \(d(\pi)_{p'}\) for all supercuspidal representations \(\pi\) of \(G\). Recall that \(d(\pi)/d(\pi)_{p'}\) is a non-negative power of \(q\). So we obtain the following proposition:
Proposition 3.1. Let $\pi$ be an irreducible supercuspidal representation of $G$. Then we have

$$d(\pi) = \frac{q^a}{(q+1)^b2^c},$$

for some $a, b, c \geq 0$.

In the computation of $d(\pi)/d(\pi)_{p'}$, we can ignore an element $[\Lambda, n, r_i, \gamma_i]$ in a sequence $\{[\Lambda, n, r_i, \gamma_i]\}_{i=0,\ldots,k}$ such that $G_i = G_{i+1}$. Therefore we need only sequences of semisimple strata with $k \leq 3$, that is, $\{[\Lambda, n, 0, \beta], [\Lambda, n, n, 0]\}$ or $\{[\Lambda, n, 0, \beta], [\Lambda, n, r, \gamma], [\Lambda, n, n, 0]\}$. For each type of $\beta$, there exist at most two choices of $\Lambda$ because $a_0(\Lambda) \cap B$ is a maximal $-\,$-stable $\sigma_E$-order.

Now we obtain the following table of formal degrees of the supercuspidal representations of ramified $U(3)$:

<table>
<thead>
<tr>
<th>$n/e$</th>
<th>$r/e$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$m$</td>
<td>$3m$</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$m - 1/2$</td>
<td>$3m - 2$</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$m - 1/2$</td>
<td>$3m - 2$</td>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$m - 1/2$</td>
<td>$3m - 2$</td>
<td>0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$m - 1/2$</td>
<td>$2m - 1$</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$k$</td>
<td>$2m + k - 1$</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$k - 1/2$</td>
<td>$2m + k - 2$</td>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$k - 1/2$</td>
<td>$2m + k - 2$</td>
<td>0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$m - 1/6$</td>
<td>$3m - 1$</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$m - 1/6$</td>
<td>$3m - 3$</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

A special representation of $G$ is a discrete series representation of $G$ which is not supercuspidal. By [8], the formal degree of a special representation $\pi$ of $G$ is given by

$$d(\pi) = \begin{cases} 
\frac{1}{q + 1}, & \text{if } \pi \text{ is a twist of the Steiberg representation;} \\
\frac{q^m}{(q+1)^2}, & m \geq 0, \text{ otherwise.}
\end{cases}$$
4 An application to the LLC

4.1 Discrete $L$-packets on $G$

From now on, we further assume that $\text{ch}(F_0) = 0$. Suppose $F$ is ramified over $F_0$. Let $G = U(3)(F/F_0)$. Let $\Pi(G)$ denote the discrete $L$-packets on $G$. By [7] and [10], $\Pi(G)$ has the following properties:

(i) $\Pi(G)$ is a partition of the discrete series representations of $G$ by finite subsets;

(ii) Let $\Pi \in \Pi(G)$. Then $d(\pi_1) = d(\pi_2)$, for $\pi_1, \pi_2 \in \Pi$;

(iii) Every discrete $L$-packet $\Pi \in \Pi(G)$ contains exactly one generic representation;

(iv) A discrete series representation $\pi$ of $G$ is stable if and only if $\{\pi\} \in \Pi(G)$.

4.2 Stable discrete series

We know the following representations of $G$ are stable:

(i) a twist of the Steinberg representation of $G([10])$;

(ii) a very cuspidal representation of $G$, that is an irreducible supercuspidal representation $\pi$ of $G$ with underlying skew stratum $[\Lambda, n, 0, \beta]$ such that $E = F[\beta]$ is a cubic extension over $F([3])$.


We can characterize these stable discrete series representations by formal degrees.

Proposition 4.2. Let $\pi$ be a discrete series representation of $G$. Then

(i) $\pi$ is a twist of the Steinberg representation if and only if $d(\pi) = \frac{1}{q+1}$,

(ii) $\pi$ is a very cuspidal representation if and only if $d(\pi) = \frac{q^m}{2}$, for $m \geq 0$. 

Now we get a new proof of stability of very cuspidal representations of $G$. By basic properties of discrete $L$-packets on $G$ and Proposition 4.2, it is enough to prove the following lemma:

**Lemma 4.3.** A very cuspidal representation of $G$ is generic.

### 4.3 Genericity of very cuspidal representations

We shall prove Lemma 4.3. This proof is based on results by Blondel-Stevens [4] for $Sp(4)$.

Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum for $G$ such that $E = F[\beta]$ is a cubic extension over $F_0$. Let $\pi = \text{ind}_J^G \lambda$ be an irreducible supercuspidal representation of $G$ with underlying skew semisimple stratum $[\Lambda, n, 0, \beta]$.

It follows from [5] Proposition 1.6 that $\pi$ is generic if and only if there exists a nondegenerate character $\chi$ of a maximal unipotent subgroup $U$ of $G$ such that

$$\text{Hom}_{J \cap U} (\lambda|_{J \cap U}, \chi|_{J \cap U}) \neq \{0\}.$$

Note that a maximal unipotent subgroup $U$ of $G$ corresponds to a flag $\{0\} \subset V_1 \subset V_1^\perp \subset V$, where $V_1^\perp$ denotes the orthogonal complement of $V_1$.

Let $\psi_0$ be an additive character of $F_0$ with conductor $p_0$. We define a map $\psi_\beta : M_3(F) \to \mathbb{C}$ by

$$\psi_\beta(x) = \psi_0(\text{tr}_{F/F_0} \circ \text{tr}_{M_3(F)/F}(\beta(x - 1))), \ x \in M_3(F). \quad (4.6)$$

Let $U$ be a maximal unipotent subgroup of $G$ corresponding to a flag $\{0\} \subset V_1 \subset V_1^\perp \subset V$. Then it follows from [4] Proposition 3.1 that $\psi_\beta|_U$ is a character of $U$ if and only if $\beta V_1 \subset V_1^\perp$.

By the assumption that $E$ is cubic over $F$, we can find such a flag of $V$, and hence we get a maximal unipotent subgroup $U$ of $G$ such that $\psi_\beta|_U$ is a character of $U$.

By the construction of $J$ and $\lambda$, the restriction of $\lambda$ to $J \cap U$ contains $\psi_\beta|_{J \cap U}$. This completes the proof of Lemma 4.3.

### 4.4 Unramified case

Suppose $F$ is unramified over $F_0$. In this case, we know the following discrete series representations of $G = U(3)(F/F_0)$ are stable:
(i) a twist of the Steinberg representation of $G([10])$;

(ii) a twist of a depth 0 supercuspidal representation, that is, a twist of $\text{ind}^G_J \tau$ where $J$ is a conjugate of a special maximal compact subgroup $G \cap GL_3(o_F)$ and $\tau$ is an inflation of a cubic cuspidal representation of $U(3)(k_F/k_0)([1])$;

(iii) a very cuspidal representation of $G([3])$.

We note that our proof of stability is valid for supercuspidal representations in cases (ii) and (iii). In fact, we can characterize these representations by their formal degrees. By [8], for a discrete series representation $\pi$ of $G$, we have

\[ \pi \text{ is a twist of the Steinberg representation } \iff d(\pi) = \frac{q^2 + 1}{(q^3 + 1)(q + 1)^2}, \]

\[ \pi \text{ is a twist of a depth 0 supercuspidal representation } \iff d(\pi) = \frac{1}{q^3 + 1}. \]

\[ \pi \text{ is a very cuspidal representation } \iff d(\pi) = \frac{q^{m-1}}{q + 1} \text{ or } \frac{q^m}{q^3 + 1}, \text{ for } m > 0. \]

Moreover, we can prove genericity of representations in cases (ii) and (iii) along with the lines of [4].

References


