<table>
<thead>
<tr>
<th>日時</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理者</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>受理者</th>
<th>未定</th>
</tr>
</thead>
<tbody>
<tr>
<td>受理機関</td>
<td>数理解析研究所講究録  数理解析研究所講究録</td>
</tr>
<tr>
<td>受理日</td>
<td>未定</td>
</tr>
<tr>
<td>受理番号</td>
<td>未定</td>
</tr>
<tr>
<td>受理者名</td>
<td>未定</td>
</tr>
<tr>
<td>受理機関名</td>
<td>未定</td>
</tr>
</tbody>
</table>
Generalized Whittaker functions of degenerate principal series

Kazuki Hiroe*

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan.

Abstract

In the theory of modular forms, modular forms with weights are important objects. For automorphic forms on $SL(2, \mathbb{R})$, the notion of weights are translated to characters of $SO(2)$. Hence for general cases, $K$-types of admissible representations can be seen as a generalization of weights of corresponding automorphic forms. In this paper, we consider degenerate principal series representations and define a class of their $K$-types which are called strongly spherical (Definition 3.2). And we give a characterization of generalized Whittaker functions with strongly spherical $K$-types of degenerate principal series representation (Theorem 5.2). The contents in this paper will appear with concrete proofs in [2].

1 Notation and preliminaries

In this section we give a quick review of some definitions and well known facts in the representation theory of Lie groups.

Let $G$ be a connected real semisimple Lie group, $K$ a maximal compact subgroup and $\theta$ the associated Cartan involution. Throughout this paper we assume that $G$ is split over $\mathbb{R}$ and has a complexification $G_{\mathbb{C}}$. The differentiation of $\theta$ is also written by same symbol. The associated Cartan decomposition of Lie algebra $\mathfrak{g}$ of $G$ is denoted by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Here $\mathfrak{k}$ and $\mathfrak{s}$ are eigenspaces of $\theta$ with eigenvalues 1 and $-1$ respectively.

*E-mail:kazuki@ms.u-tokyo.ac.jp
Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{s}$ and $\Sigma$ the root system of $(\mathfrak{g}, \mathfrak{a})$. Its Weyl group $W$ is isomorphic with $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. Fix a positive system $\Sigma^+$ of $\Sigma$ and denote the set of simple roots by $\Pi = \{\alpha_1, \ldots, \alpha_r\}$. Let $\mathfrak{n}$ be the sum of the root space $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for any } H \in \mathfrak{a}\}$ for $\alpha \in \Sigma^+$, i.e., $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. Then we have an Iwasawa decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $G = KAN$ where $A = \exp \mathfrak{a}$ and $N = \exp \mathfrak{n}$. Also we define $\bar{\mathfrak{n}} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}$ and $\bar{N} = \exp \bar{\mathfrak{n}}$. Let us denote the Killing form on $\mathfrak{g}$ by $B$. For $\lambda \in \mathfrak{a}^*$, we take $H_\lambda \in \mathfrak{a}$ satisfying the equations $\lambda(H) = B(H_\lambda, H)$ for any $H \in \mathfrak{a}$. We introduce an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}^*$ defined by $\langle \mu, \nu \rangle = B(H_\mu, H_\nu)$ for $\mu, \nu \in \mathfrak{a}^*$.

We denote the centralizer of $A$ in $K$ by $M$. Then a minimal parabolic subgroup $P$ is defined by $P = MAN$. Let $\Theta \subset \Pi$ be a finite subset and define the parabolic subgroup $P_{\Theta}$ associated to $\Theta$ as follows. Let $\mathfrak{a}_{\Theta} = \{H \in \mathfrak{a} \mid \alpha(H) = 0 \text{ for any } \alpha \in \Theta\}$ and $\mathfrak{a}^\perp_{\Theta}$ the orthogonal complement of $\mathfrak{a}_{\Theta}$ in $\mathfrak{a}$ with respect to the Killing form. Furthermore let $\mathfrak{n}_\Theta = \bigoplus_{\alpha \in \Sigma^+ \setminus \text{span}(\Theta)} \mathfrak{g}_\alpha$ and $\mathfrak{m}_\Theta = \mathfrak{a}^\perp_{\Theta} \oplus \bigoplus_{\alpha \in \Sigma^+ \cap \text{span}(\Theta)} \mathfrak{g}_\alpha$. Then we can define the parabolic subalgebra associated to $\Theta$ by $\mathfrak{p}_{\Theta} = \mathfrak{m}_{\Theta} \oplus \mathfrak{a}_{\Theta} \oplus \mathfrak{n}_{\Theta}$. Let $L_{\Theta} = Z_G(\mathfrak{a}_{\Theta})$, $K_{\Theta} = L_{\Theta} \cap K$ and $M_{\Theta} = K_{\Theta} \exp(\mathfrak{m}_{\Theta} \cap \mathfrak{s})$. Then we can define the parabolic subgroup associated to $\Theta$ by $P_{\Theta} = M_{\Theta}A_{\Theta}N_{\Theta}$. If $\Theta = \emptyset$, the parabolic subgroup $P_\emptyset = M_\emptysetA_\emptysetN_\emptyset$ equals to the minimal parabolic subgroup $P = MAN$ defined above.

We write $\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}$ etc. as the complexifications of $\mathfrak{g}, \mathfrak{k}$ etc. Let $U(\mathfrak{g}), U(\mathfrak{k})$ etc. be the universal enveloping algebras of complexifications of $\mathfrak{g}, \mathfrak{k}$ etc. Also let $Z(\mathfrak{g}), Z(\mathfrak{k})$ be the centers of universal enveloping algebras $U(\mathfrak{g}), U(\mathfrak{k})$ respectively. As it is well-known, there is an inclusion

$$Z(\mathfrak{g}) \subset U(\mathfrak{a}) \oplus \bar{\mathfrak{n}}_{\mathbb{C}}U(\mathfrak{g}).$$

Let $\sigma: Z(\mathfrak{g}) \to U(\mathfrak{a})$ be the projection map along this decomposition. Put $\rho = \text{tr}(\text{Ad}|_{\mathfrak{n}}) \in \mathfrak{a}_{\mathbb{C}}^*$, then we can define the $\rho$-shifted map $\sigma': Z(\mathfrak{g}) \to U(\mathfrak{a})$ by $\sigma'(X)(\lambda) = \sigma(X)(\lambda - \rho)$ for $X \in Z(\mathfrak{g})$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. It is well known that this map gives an algebra isomorphism

$$\sigma': Z(\mathfrak{g}) \to U(\mathfrak{a})^W,$$

which is called Harish-Chandra isomorphism. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we can define a character of $Z(\mathfrak{g})$ by

$$\chi_\lambda: Z(\mathfrak{g}) \to \mathbb{C}$$

$$X \mapsto \sigma'(X)(\lambda).$$

For $C^\infty(G, E)$, the space of smooth functions from $G$ to a finite dimensional vector space $E$, we can consider natural actions of $G$ and $\mathfrak{g}$ by left (right)
translations and left (right) derivations, i.e.,

\begin{align}
L_g f(x) &= f(g^{-1}x), & R_g f(x) &= f(xg), \\ L_X f(x) &= \frac{d}{dt} L_{(\exp tX)} f(x)\big|_{t=0}, & R_X &= \frac{d}{dt} R_{(\exp tX)} f(x)\big|_{t=0},
\end{align}

(1.1)

where \( x, g \in G \), \( X \in \mathfrak{g} \) and \( f \in C^\infty(G, E) \).

Let \((\pi, E)\) be a continuous representation of \( G \) where \( E \) is a Hausdorff locally convex complete topological vector space. We write the space of \( K \)-finite vectors of \( E \) by \( E_K \).

\section{Poisson transform on vector bundle.}

The Poisson transform is a continuous \( G \)-homomorphism from a spherical principal series representation to the space of right \( K \)-invariant functions on \( G \). As a generalization of this, we will define a vector-valued Poisson transform and determine its image.

Let \((\tau, V_\tau)\) be an irreducible unitary representation of \( K \) and \( \lambda \) an element of \( \mathfrak{a}_C^* \). Then we consider the induced representation \( \pi_{\tau,\lambda} \) realized as follows. The representation space is

\[ \mathcal{H}_{\tau,\lambda}^\infty = \{ f \in C^\infty(G, V_\tau) \mid f(gm) = \tau(m)^{-1}a^{\lambda-\rho}f(g) \text{ for } (m, a, n, g) \in M \times A \times N \times G \} \]

and \( G \) acts on this space by left translation, i.e., \( \pi_{\tau,\lambda}(g)f(x) = L_g f(x) = f(g^{-1}x) \) for \( f \in \mathcal{H}_{\tau,\lambda}^\infty \) and \( g \in G \). This is an admissible representation of \( G \) with infinitesimal character \( \chi_\lambda \). Also we denote the space of \( K \)-finite vectors of \( \mathcal{H}_{\tau,\lambda}^\infty \) by \( H_{\tau,\lambda} \) which becomes a \((\mathfrak{g}_C, K)\)-module naturally.

Also we consider another induced representation. The representation space is

\[ C_{\tau}^\infty(G/K; \chi_\lambda) = \{ f \in C^\infty(G, V_\tau) \mid f(gk) = \tau(k)^{-1}f(g), (k, g) \in K \times G, R_X f = \chi_\lambda(X)f \text{ for } X \in Z(\mathfrak{g}) \} \]

and \( G \) acts on this space by left translation. We denote the space of its \( K \)-finite vectors by \( C_{\tau}^\infty(G/K; \chi_\lambda)_K \).

We define the generalized Harish-Chandra \( C \)-function as follows,

\[ C(\lambda, \tau) = \int_{\overline{N}} \tau(k(\overline{n}))e^{-(\lambda+\rho)H(\overline{n})}d\overline{n}. \]
Here $g = k(g) \exp H(g)n(g)$ for $k(g) \in K, H(g) \in \mathfrak{a}$ and $n(g) \in N$. It is known that this integral is absolutely convergent by the operator norm of $\text{End}(V_\tau)$ in \{\(\lambda \in a^*_C \mid \text{Re}\langle\lambda, \alpha\rangle > 0\) for any $\alpha \in \Sigma^+\}. \) It is meromorphically continued in all $a^*_C$ (cf. [4]).

Since $M$ is the finite abelian group, $V_\tau$ can be decomposed as the direct sum of 1-dimentional representations of $M$. Therefore we can take a basis \{\(v_1, \ldots, v_l\)\} of $V_\tau$ so that there exist 1-dimentional representation $\sigma_i (i = 1, \ldots, l)$ of $M$ such that $\tau(m)v_i = \sigma_i(m)v_i (i = 1, \ldots, l)$ for $m \in M$. Also we take the dual basis \{\(v_1^*, \ldots, v_l^*\)\} of $V_{\tau}^* = \text{Hom}_\mathbb{C}(V_{\tau}, \mathbb{C})$, i.e., each $v_i$ satisfies $v_i^*(v_j) = \delta_{ij}$ for $i, j = 1, \ldots, l$. We regard $V_{\tau}^*$ as a representation space of $M$ by the contragradient representation.

**Definition 2.1** (Poisson transform). We define the $G$-homomorphism $\mathcal{P}_{\tau, \lambda}$ from $\mathcal{H}_{\tau, \lambda}^\infty$ to $C_\tau^\infty(G/K; \chi_\lambda)$ by

\[
\mathcal{P}_{\tau, \lambda}: \mathcal{H}_{\tau, \lambda}^\infty \to C_\tau^\infty(G/K; \chi_\lambda)
\]

\[
f \mapsto \int_K \tau(k)f(gk)dk
\]

This is called the Poisson transform.

We see that $\mathcal{P}_{\tau, \lambda}$ gives a bijection between the $K$-finite subspaces for generic $\lambda \in a^*_C$.

**Theorem 2.2.** We put following assumptions.

1. $\lambda \in a^*_C$ is regular and dominant, i.e.,

\[
2\frac{\langle \lambda, \beta \rangle}{\langle \beta, \beta \rangle} \notin \{0, -1, -2, \ldots\} \text{ for any } \beta \in \Sigma^+.
\]

2. The determinant of $C(\tau, \lambda) \in \text{End}(V_\tau)$ is nonzero.

Then $\mathcal{P}_{\tau, \lambda}$ gives a $(\mathfrak{g}_C, K)$-isomorphism,

\[
\mathcal{P}_{\tau, \lambda}: \mathcal{H}_{\tau, \lambda} \cong C_\tau^\infty(G/K; \chi_\lambda)K.
\]

**Remark 2.3.** This theorem is first proved by An Yang [5] in more general settings. However Yang put a stronger assumption

\[
2\frac{\langle \lambda, \beta \rangle}{\langle \beta, \beta \rangle} \notin \mathbb{Z} \text{ for any } \beta \in \Sigma.
\]

This is too strong for our purpose in this paper. Therefore we need a refined theorem under the weaker condition as above.
3 Strongly spherical $K$-types and vector valued Poisson transforms of degenerate principal series representations

Our purpose of this note is to give a characterization of the vector-valued generalized Whittaker functions of degenerate principal series. To do this, we need the Poisson transforms on degenerate principal series representations. Hence we need to restrict the vector-valued Poisson transform to degenerate principal series representations and determine their images.

Take a finite subset $\Theta \subset \Pi$ and let $P_{\Theta}$ be the corresponding parabolic subgroup of $G$. For $\lambda \in (a_{\Theta})_{\mathbb{C}}$, we define a character $\lambda_{\Theta}$ of $p_{\Theta}$ by

$$\lambda_{\Theta}: p_{\Theta} \rightarrow \mathbb{C}$$

$$X + H \mapsto \lambda(H),$$

where $X \in m_{\Theta} + n_{\Theta}$ and $H \in a_{\Theta}$. We take a character $\Lambda_{\Theta}$ of $P_{\Theta}$ whose differentiation is $\lambda_{\Theta}$. Then we define a degenerate principal series representation of $G$ as follows. The representation space is $C^\infty(G/P_{\Theta}; \Lambda_{\Theta}) = \{ f \in C^\infty(G) \mid f(gp) = \Lambda_{\Theta}(p)f(g) \text{ for } p \in P_{\Theta}, g \in G \}$. The action of $G$ on this space is defined by left translation. We denote the space of $K$-finite vectors of $C^\infty(G/P_{\Theta}; \Lambda_{\Theta})$ by $H_{\Theta, \lambda}$.

**Definition 3.1** (annihilator ideal). *We define a left ideal of $U(g)$ by*

$$J_{\Theta}(\lambda) = \sum_{X \in (p_{\Theta})_{\mathbb{C}}} U(g)(X - \lambda_{\Theta}(X))$$

*and also define a two-sided ideal*

$$I_{\Theta}(\lambda) = \bigcap_{g \in G} Ad(g)J_{\Theta}(\lambda).$$

This two-sided ideal $I_{\Theta}(\lambda)$ is studied by H. Oda and T. Oshima in [3] and they give explicit generators of $I_{\Theta}(\lambda)$. This ideal is very important tool to investigate $C^\infty(G/P_{\Theta}; \Lambda_{\Theta})$, because we can show that for any $X \in I_{\Theta}(\lambda)$ and $f \in C^\infty(G/P_{\Theta}; \Lambda_{\Theta})$, we have $R_{X}f = 0$, i.e., $I_{\Theta}(\lambda)$ is the annihilator ideal of $C^\infty(G/P_{\Theta}; \Lambda_{\Theta})$. Also it is known that $I_{\Theta}(\lambda)$ is the annihilator of the generalized Verma module $U(g)/J_{\Theta}(\lambda)$.

We define the notion of strongly spherical $K$-types.
Definition 3.2 (Strongly spherical $K$-type). Let $(\tau, V_\tau)$ be a irreducible unitary representation of $K$ such that $\dim \text{Hom}_K(V_\tau, H_{\Theta, \lambda}) \neq 0$. We call this representation $\tau$ a strongly spherical $K$-type of $H_{\Theta, \lambda}$ if the dimension of $V_\tau^{m_\Theta \cap \mathfrak{k}} = \{v \in V_\tau \mid \tau(X)v = 0 \text{ for } X \in m_\Theta \cap \mathfrak{k}\}$ is equal to 1.

Remark 3.3. If $\Theta = \emptyset$, i.e., $P_\Theta$ is minimal parabolic subgroup, this condition says $V_\tau$ is 1-dimensional because $m_\Theta$ is trivial. On the other hand, if $(K, M_\Theta \cap K)$ is a symmetric pair, it is easy to see that every irreducible unitary representation of $K$ is strongly spherical.

For these strongly spherical $K$-types, we can consider vector valued Poisson transform of degenerate principal series. And we can determine its image. For an irreducible representation $(\tau, V_\tau)$ of $K$, we define a space

$$C_\tau^\infty(G/K; I_\Theta(\lambda)) = \{f \in C^\infty(G, V_\tau) \mid f(gk) = \tau(k^{-1})f(g), R_Xf = 0 \text{ for } g \in G, k \in K, X \in I_\Theta(\lambda)\}.$$ 

This is a $G$-representation by the left translation.

Theorem 3.4. We use the notations as above. For $\lambda \in (a_\Theta^*)_\mathbb{C}$, we assume that

1. $\lambda + \rho$ is regular and dominant.

2. $\det C(\tau, \lambda + \rho) \neq 0$.

Let $(\tau, V_\tau)$ be a strongly spherical $K$-type of $H_{\Theta, \lambda}$. Then the restriction of $\mathcal{P}_{\tau, \lambda}$ to $H_{\Theta, \lambda}$ gives a following $(\mathfrak{g}_\mathbb{C}, K)$-isomorphism,

$$\mathcal{P}_{\Theta, \lambda}: H_{\Theta, \lambda} \rightarrow C_\tau^\infty(G/K; I_\Theta(\lambda))_K, \phi \mapsto \int_K \tau(k)\phi(gk)dk.$$ 

Here we note that we can see $a_\Theta^* \subset a^*$ by the Killing form $B$.

Proof. By the assumption, we have the $(\mathfrak{g}_\mathbb{C}, K)$-isomorphism

$$\mathcal{P}_{\tau, \lambda}: H_{\tau, \lambda} \rightarrow C_\tau^\infty(G/K; \chi_\lambda)_K, \phi \mapsto \int_K \tau(k)\phi(gk)dk.$$ 

Since $H_{\Theta, \lambda}$ is a $(\mathfrak{g}_\mathbb{C}, K)$-submodule of $H_{\tau, \lambda}$, we have

$$\mathcal{P}_{\tau, \lambda}(H_{\Theta, \lambda}) \subset C_\tau^\infty(G/K; I_\Theta(\lambda))_K.$$
Here we notice that since it is easy to show that \( \sum_{X \in Z(\mathfrak{g})} U(\mathfrak{g})(X - \chi_{\lambda}(X)) \subset I_{\Theta}(\lambda) \), we have \( C_{\tau}^\infty(G/K; I_{\Theta}(\lambda)) \subset C_{\tau}^\infty(G/K; \chi_{\lambda}). \) It remains to show that \( H_{\Theta, \lambda} \supset \mathcal{P}_{\tau, \lambda}^{-1}(C_{\tau}^\infty(G/K; I_{\Theta}(\lambda))_{K}) \). To show this, we take an arbitrary element \( u \in C_{\tau}^\infty(G/K; I_{\Theta}(\lambda)) \). We can see \( \lambda \in (a_{\mathbb{C}}^*)_{\mathbb{C}} \) as an element of \( a_{\mathbb{C}}^* \), hence we denote this by \( \lambda_{\Theta} \in a_{\mathbb{C}}^* \). We define a character of the Borel subalgebra of \( \mathfrak{g}_{\mathbb{C}} \), \( b = a_{\mathbb{C}} + n_{\mathbb{C}} \) as follows,

\[
\lambda_{b} : \quad b \longrightarrow \mathbb{C} \\
H + X \longmapsto \lambda(H)
\]

where \( H \in a_{\mathbb{C}} \) and \( X \in n_{\mathbb{C}} \). We define a left ideal of \( U(\mathfrak{g}) \) by \( J(\lambda_{b}) = \sum_{X \in b} U(\mathfrak{g})(X - \lambda_{b}(X)) \). Then for any \( X \in J(\lambda_{b}) \) and \( f \in H_{\tau, \lambda} \) we have \( R_{X} f = 0 \). Hence \( \mathcal{P}_{\tau, \lambda}^{-1}u \) satisfies that \( R_{X} \mathcal{P}_{\tau, \lambda}^{-1}u = 0 \) for any \( J_{\Theta}(\lambda) \) because \( J_{\Theta}(\lambda) = I_{\Theta}(\lambda) + J(\lambda_{b}) \) by the result of Oda and Oshima (Theorem 3.12 in [3]). This implies that there exists a representation \( \sigma \) of \( M_{\Theta} \) which satisfies that \( \text{Hom}_{M_{\Theta} \cap K}(\sigma, \tau) \neq \{0\} \) and differentiation of \( \sigma \) is trivial. And \( \mathcal{P}_{\tau, \lambda}^{-1}u \in C_{\tau}^\infty - \text{Ind}_{P_{\Theta}}^{G}(\sigma \otimes e^{-\lambda} \otimes 1_{N_{\Theta}}) \). However since \( \dim V_{\tau}^{m_{\Theta} \cap t} = 1 \), \( \sigma \) must be equal to \( \Lambda_{\Theta}|_{M_{\Theta}} \). \( \square \)

### 4 Maximal globalization

The vector-valued Poisson transform gives a \((\mathfrak{g}_{\mathbb{C}}, K)\)-isomorphism from the degenerate principal series \( H_{\Theta, \lambda} \) to \( C_{\tau}^\infty(G/K; I_{\Theta}(\lambda))_{K} \) if \( \tau \) is a strongly spherical \( K \)-type of \( H_{\Theta, \lambda} \). Furthermore, we see that this \((\mathfrak{g}_{\mathbb{C}}, K)\)-isomorphism extends to the continuous \( G \)-isomorphism.

Let \( X \) be an admissible \((\mathfrak{g}_{\mathbb{C}}, K)\)-module with finite length. We consider the space of \((\mathfrak{g}_{\mathbb{C}}, K)\)-homomorphisms from the dual \((\mathfrak{g}_{\mathbb{C}}, K)\)-module \( X^* \) to \( C_{\tau}^\infty(G), \text{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(X^*, C_{\tau}^\infty(G)) \) where \( G \) acts on \( C_{\tau}^\infty(G) \) by left translation. Since \( C_{\tau}^\infty(G) \) has a uniformly covergent topology and \( X^* \) has a countably many basis, we can define the complete locally convex topology on \( \text{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(X^*, C_{\tau}^\infty(G)) \). On the other hand, \( G \) can also act on \( C_{\tau}^\infty(G) \) by right translation. This action is continuous on the topology of \( \text{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(X^*, C_{\tau}^\infty(G)). \) the space of \( K \)-finite elements of \( \text{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(X^*, C_{\tau}^\infty(G)) \) can be identified with \( (X^*)^* \cong X \) by the evaluation at the origin, i.e., for \( I \in \text{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(X^*, C_{\tau}^\infty(G)), X^* \ni v \mapsto I(v)(e) \in \mathbb{C} \) is a linear form of \( X^* \). Hence \( \text{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(X^*, C_{\tau}^\infty(G)) \) is a continuous \( G \) representation and its \( K \)-finite subspace is \( X \), i.e., \( \text{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(X^*, C_{\tau}^\infty(G)) \) is a globalization of \( X \). This is called the maximal globalization [1].

Let us return to our setting. In the previous section we see that there is a
\( (\mathfrak{g}_C, K) \)-isomorphism

\[
\mathcal{P}_{\Theta, \lambda}: \quad H_{\Theta, \lambda} \mapsto C^\infty_\tau(G/K; I_{\Theta}(\lambda))_K
\]

This \((\mathfrak{g}_C, K)\)-isomorphism can be extend to \(G\)-isomorphism as follows. If \((\tau, V_\tau)\) is a strongly spherical \(K\)-type of \((\mathfrak{g}_C, K)\)-module \(H_{\Theta, \lambda}\), it is multiplicity free by definition. We fix a \(K\)-projection \(p_\tau: H_{\Theta, \lambda} \to V_\tau\). We define a \(K\)-embedding \(\iota_\tau: V_\tau^* \to H_{\Theta, \lambda}^*\) as the dual map of \(p_\tau\).

**Theorem 4.1.** We assume that

1. \(\lambda_\Theta + \rho\) is regular and dominant.

2. \(\det C(\lambda + \rho, \tau) \neq 0\).

Let \((\tau, V_\tau)\) be a strongly spherical \(K\)-type of \(H_{\Theta, \lambda}\). Then we have the following topological \(G\)-isomorphism.

\[
\Phi: \quad \text{Hom}_{(\mathfrak{g}_C, K)}(H_{\Theta, \lambda}^*, C^\infty(G)) \mapsto C^\infty_\tau(G/K; I_{\Theta}(\lambda))
\]

\[
I \mapsto \sum_{i=1}^{l} I(\iota_\tau(v_i^*))(g)v_i
\]

## 5 Generalized Whittaker models

Finally we give the main theorem of this note. We can give a characterization of vector-valued generalized Whittaker functions as solutions of system of differential equations which comes from \(I_{\Theta}(\lambda)\).

Let \(U\) be a closed subgroup of \(N\) and \( (\eta, V_\eta) \) an irreducible unitary representation of \(U\). We consider a representation of \(G\) induced from \(\eta\). The representation space is

\[
C^\infty_\eta(U \backslash G) = \{ f: G \to V_\eta^\infty \text{ smooth} \mid f(ug) = \eta(u)f(g) \text{ for all } u \in U, g \in G \}
\]

Here \(V_\eta^\infty\) stands for the space of smooth vectors of \(V_\eta\). We note that \(V_\eta^\infty\) has a Hausdorff complete locally convex topology and we can define the derivation of \(f: G \to V_\eta^\infty\) by the convergence on the topology of \(V_\eta^\infty\).

**Definition 5.1** (Generalized Whittaker model). Let \(X\) be an admissible \((\mathfrak{g}_C, K)\)-module with finite length. Let \(U\) be a closed subgroup of \(N\) and \((\eta, V_\eta)\) an
irreducible unitary representation of $U$. We consider the space of $(\mathfrak{g}_C,K)$-homorphisms from $X$ to $C_\eta^\infty(U\backslash G)$,

$$\text{Hom}_{(\mathfrak{g}_C,K)}(X, C_\eta^\infty(U\backslash G)).$$

If $\text{Hom}_{(\mathfrak{g}_C,K)}(X, C_\eta^\infty(U\backslash G)) \neq \{0\}$, we say $X$ has generalized Whittaker models.

We consider generalized Whittaker models of $H_{\Theta,\lambda}$. Let $(\tau, V_\tau)$ be a strongly spherical $K$-type of $H_{\Theta,\lambda}$. Take an irreducible unitary representation $(\eta, V_\eta)$ of $N$. For the algebraic tensor product $V_\eta^\infty \otimes V_\tau$, we can define a natural topology comes from $V_\eta^\infty$ because $V_\tau$ is finite dimensional. Hence we can consider the following space of smooth functions from $G$ to $V_\eta^\infty \otimes V_\tau$,

$$C^\infty_{\eta,\tau}(U\backslash G/K) = \{ f: G \rightarrow V_\eta^\infty \otimes V_\tau \text{ smooth} \mid f(ugk) = \eta(u) \otimes \tau(k^{-1})f(g) \text{ for } u \in U, g \in G, k \in K, \}.$$

Also we define

$$C_{\eta,\tau}(U\backslash G/K; I_{\Theta}(\lambda)) = \{ f \in C^\infty_{\eta,\tau}(U\backslash G/K) \mid R_Xf = 0 \text{ for } X \in I_{\Theta}(\lambda) \}.$$

As a corollary of Theorem 4.1, we have the following characterization of generalized Whittaker models.

**Theorem 5.2.** We use the same notations as Theorem 4.1. We assume that

1. $\lambda_{\Theta} + \rho$ is regular and dominant.

2. $\det C(\lambda + \rho, \tau) \neq 0$.

Let $(\tau, V_\tau)$ be a strongly spherical $K$-type of $H_{\Theta,\lambda}$. Then we have the following linear isomorphism.

$$\Phi: \text{Hom}_{(\mathfrak{g}_C,K)}(H_{\Theta,\lambda}^*, C_\eta^\infty(U\backslash G)) \rightarrow C^\infty_{\eta,\tau}(U\backslash G/K; I_{\Theta}(\lambda))$$

$$I \mapsto \sum_{i=1}^{l} I(\iota_{\tau}(v_i^*))g(v_i).$$

**Acknowledgement**

The author would like to thank Professor Yoshihiro Ishikawa for his many advice to make this article more readable.
References


