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Kyoto University
On the principal series representation of $SU(2, 2)$

G. Bayarmagnai

1 Introduction

Let $G$ denote the the special unitary group $SU(2, 2)$. In the paper, we will deal with the principal series representations of $G$ which are parabolically induced by the minimal parabolic subgroup $P_{\text{min}}$ with Langlands decomposition $P_{\text{min}} = MAN$:

$$\pi_{\sigma, \nu} = \text{Ind}_{P_{\text{min}}}^{G} (\sigma \otimes e^{\nu+\rho} \otimes 1_{N}),$$

where $\rho$ is the half sum associated to the root system of the pair $(G, A)$, $\nu$ is a complex valued real linear form on $a = \text{Lie}(A)$, $\sigma$ is a unitary character of $M$.

Let $\sigma$ be a continuous unitary character of $N$. We then have the Jacquet functional $J_{\sigma, \nu}$ on the space of differentiable functions of $L^{2}(K)$, the representation space of $\pi_{\sigma, \nu}$, such that $J_{\sigma, \nu}(\pi_{\sigma, \nu}(n)f) = \eta(n)J_{\sigma, \nu}(f)$ for any $n \in N$. The functional defines an intertwiner $J$ from $\pi_{\sigma, \nu}|_{K}$ to $A_{\eta}(N \setminus G)$ by sending any $v \in \pi_{\sigma, \nu}|_{K}$ to the function $J_{\sigma, \nu}(g) := J_{\sigma, \nu}(\pi_{\sigma, \nu}(g)v)$. $(g \in G)$. Here the subspace of all $K$-finite vectors of $\pi_{\sigma, \nu}$ is denoted by $\pi_{\sigma, \nu}|_{K}$ and $A_{\eta}(N \setminus G)$ is the subspace of $C^{\infty}(G)$ consisting of all moderate growth functions $f(g)$ such that $f(ng) = \eta(n)f(g)$ for $n \in N$ and $g \in G$. In fact, $J$ is an intertwiner of $K$ and $g$-equivariant, and hence the study of the image of $J$ (the Whittaker model) leads us to the problem of the investigations of the $(g, K)$-module structure and the functions $J_{\sigma, \nu}(g)$ for certain $K$-types of $\pi_{\sigma, \nu}$.

The main goal of this paper is to describe the above mentioned objects in terms of parameters of the principal series representation $\pi_{\sigma, \nu}$ explicitly. Note that our results are quite similar to that of Ishii [4] and Oda [5], for both $Sp(2, \mathbb{R})$ and $SU(2, 2)$ have the same restricted root system.

We also consider a matrix representations of the Knapp-Stein intertwinning operator which have been motivated by a result of Goodman-Wallach [2].

2 Preliminaries

Let $K$ be the compact group $SU(2) \times U(2)$. Then $K$ is the maximal compact subgroup of $G$ fixed by the Cartan involution $\theta$ for $G$ given by

$$\theta(g) = ^{t}g^{-1}, \quad g \in G.$$

We fix the following basis for the 7 dimensional Lie algebra $\mathfrak{k}$, the complexification of $\mathfrak{k} = \text{Lie}(K)$:

$$h^{1} = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}, \quad h^{2} = \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}, \quad I_{2, 2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$e_{\pm}^{1} = \begin{pmatrix} e_{\pm} & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{\pm}^{2} = \begin{pmatrix} 0 & 0 \\ 0 & e_{\pm} \end{pmatrix},$$

where $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $e_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. 
For every $K$-module $V$, it is clear that $I_{2,2} \in \mathfrak{g}_{C}$ commutes with the action of $K$ on $V$. If $V$ is irreducible, then by Schur’s lemma, the operator is a scalar of the identity map.

**Lemma 2.1** Let $m_1, m_2$ be positive integers and $l$ be an integer. If $m_1 + m_2 + l$ is an even integer, then there is an irreducible $K$-module $(π_{m_1, m_2}; \iota), (V_{m_1, m_2})$ with a basis $\{f_{pq} \mid 0 \leq p \leq m_1, 0 \leq q \leq m_2\}$ of $V_{m_1, m_2}$ such that $I_{2,2}f_{pq} = lf_{pq}$ and

$$
\begin{align*}
    h^1(f_{pq}) &= (2p - m_1)f_{pq}, \\
    e^1_{+}(f_{pq}) &= (m_1 - p)f_{p+1,q}, \\
    e^1_{-}(f_{pq}) &= pf_{p-1,q},
\end{align*}
$$

$$
\begin{align*}
    h^2(f_{pq}) &= (2q - m_2)f_{pq}, \\
    e^2_{+}(f_{pq}) &= (m_2 - q)f_{p,q+1}, \\
    e^2_{-}(f_{pq}) &= qf_{p,q-1}.
\end{align*}
$$

It follows from the fact that $SU(2) \times SU(2) \times \mathbb{C}^{(1)}$ is a twofold covering of $K$ with the projection given by

$$
pr(g_1, g_2; u) = \text{diag}(ug_1, u^{-1}g_2), \quad g_1, g_2 \in SU(2), \quad u \in \mathbb{C}^{(1)}.
$$

## 3 $K$-finite vectors in the principal series

In this section, for each simple $K$-module $τ \in \hat{K}$, we associate a matrix function $S^{(τ)}_\nu(k), \quad k \in K$, whose entries give a basis for the $τ$-isotypic component of $π_{τ,ν}$. The main feature of this basis is that the both $\mathfrak{g}$ and $K$-actions on $π_{τ,ν}|_K$ have simple expressions in terms of parameters of given representation. For more details about this theme, we refer to [5] which is our main reference.

**Proposition 3.1** Let $H(τ)$ be the $τ$-isotypic component of $L^2(K)$, and put $\dim(τ) = n$. There exists a unique square matrix function $S^{(τ)}(k), \quad k \in K$, of size $n$ with entries in $H(τ)$,

$$
S^{(τ)}(k) = \begin{bmatrix}
f_{11}(k) & \cdots & f_{n1}(k) \\
\vdots & \ddots & \vdots \\
f_{1n}(k) & \cdots & f_{nn}(k)
\end{bmatrix}
$$

satisfying the following two conditions:

1. $S^{(τ)}(1_K) = \text{diag}(1, \ldots, 1) \in M_n(\mathbb{C})$,
2. For each $α (1 \leq α \leq n)$, the set $\{f_{α1}(k), \ldots, f_{αn}(k)\}$ is a basis for $τ$ as in Lemma 2.1. Moreover, we have

$$
H(τ) = \bigoplus_α W_α,
$$

where $W_α$ denotes the space spanned by $f_{α1}(k), \ldots, f_{αn}(k)$.

**Proof.** The existence of the matrix function is similar to that of [5]. We consider the uniqueness. Assume that there exist two matrices $F^{(τ)}(k) = \{f_{ij}(k)\}$ and $G^{(τ)}(k) = \{g_{ij}(k)\}$ as required. Denote by $F_α$ the isomorphism between $τ$ and the space spanned by $\{f_{αj}(k), \ldots, f_{αn}(k)\}$. Similarly, we define $G_α$ for the $α$-th column of $G^{(τ)}(k)$. As a result, we obtain two ordered bases $\{F_α\}_α$ and $\{G_α\}_α$ for the $n$-dimensional vector space $\text{Hom}_K(τ, H(τ))$. Then we have the $n$ by $n$ matrix $A = \{a_{αβ}\}$, the change of coordinate matrix, such that

$$
F_α = \sum_β a_{αβ}G_β.
$$

For a basis $\{f_γ\}$ of $τ$, one obtains

$$
f_{αγ}(k) = F_α(f_γ) = \sum_β a_{αβ}G_β(f_γ) = \sum_β a_{αβ}f_βγ(k).
$$
Evaluation at the point $1_K$ shows that
\[ a_{\alpha\gamma} = \delta_{\alpha\gamma}. \]
If $v \neq 0 \in W_{\alpha} \cap W_{\beta}$, then $Kv = W_{\alpha} = W_{\beta}$. Schur's lemma and second condition imply that $\alpha = \beta$. Assume there is a matrix $S(\tau)(k)$ as required, we then have the direct sum decomposition of $H(\tau)$. □

For each $\tau_m = \tau_{[m_1, m_2;l]} \in \hat{K}$, define a finite set $I(\tau_m)$ to be the collection of indices $\alpha$ such that $W_{\alpha}$ occurs in $\pi_{\sigma, \nu} |_{K}$ as a $K$-module. Thus, the cardinality of $I(\tau_m)$ is the $K$-multiplicity of $\tau_m$ in $\pi_{\sigma, \nu}$. Let $s$ be the integer parameter corresponding to $\sigma \in \hat{M}$. By setting $n = (m_1 + m_2 + s)/2$, one can see that $p + q = n$ if $\alpha \in I(\tau_m)$ with $\alpha = (m_2 + 1)p + q + 1, (q \leq m_2)$. We identify the index $\alpha$ with the pair $(p, q)$ defined by $\alpha$.

We define a matrix function $S_{\tau_{m}}^{(\tau_{m})}(k)$ attached to the $\tau$-isotypic component of $\pi_{\sigma, \nu}$ by eliminating all the $\alpha$-th columns of $S(\tau_m)(k)$ when $p + q \neq n$ and change the $\alpha$-th columns by $0$ if $\alpha \notin I(\tau_m)$ and $p + q = n$.

4 The $(g, K)$-module structure on $\pi_{\sigma, \nu}$

Let $g = \mathfrak{t} + \mathfrak{p}$ be the Cartan decomposition of $g = \text{Lie}(G)$ corresponding to $\theta$. In this section, we explicitly describe $p_{\mathbb{C}}$-action on the space
\[ \pi_{\sigma, \nu} |_{K} \cong \bigoplus_{\tau_m \in K} \bigoplus_{\alpha \in I(\tau_m)} W_{\alpha}. \]
Since the adjoint representation of $K$ on $p_{\mathbb{C}}$ splits into two irreducible components, the antiholomorphic part $p_{-}$ and the holomorphic part $p_{+}$, it is enough to investigate the $p_{+}$-action for our purpose. Let $E_{ij}$ be the matrix unit of $M_4(\mathbb{R})$ with $1$ in the $(i, j)$-entry and zero elsewhere. Then the set $\{E_{ij} \mid i, j = 1, 2; j = 3, 4\}$ forms a basis for $p_{+}$. For a fixed pair $(e_1, e_2)$, $e_j \in \{\pm 1\}$ with $j = 1, 2$, we define $c^{j}_{t}$ by
\[ c^{j}_{t} = \frac{t}{m_j + 1} (0 \leq t \leq m_j + e_j). \]
Let $(\tau_m, V_m)$ be an irreducible representation of $K$ with parametrization $m = [m_1, m_2; l]$. By the well known Clebsch-Gordan theorem, the irreducible components in the $K$-module $p_{+} \otimes_{\mathbb{C}} \tau_m$ are precisely the $K$-representations
\[ T = \{ \tau_{[m_1+e_1, m_2+e_2; l+2]} \mid e_1, e_2 \in \{\pm 1\} \}, \]
and we will denote these by $\tau_{[\pm, \pm; \pm]}$ or $\tau_{[e_1, e_2; \pm]}$.

For each $K$-isomorphism between $\tau_m$ and $W_{\alpha}$ in Proposition 3.1, we have the following surjective homomorphism $p_{+} \otimes_{\mathbb{C}} \tau_m \rightarrow p_{+} W_{\alpha}$ of $K$-modules. Therefore, we obtain an injection
\[ p_{+} H_{\sigma, \nu}(\tau_m) \hookrightarrow \bigoplus_{\tau_{m'} \in T} H_{\sigma, \nu}(\tau_{m'}) \]
which implies the following theorem. Here $H_{\sigma, \nu}(\tau_m)$ stands for the $\tau_m$-isotypic component of $\pi_{\sigma, \nu}$.

**Theorem 4.1** Let $\tau_{[e_1, e_2; \pm]}$ be a simple $K$-submodule of the $K$-module $p_{+} \otimes_{\mathbb{C}} \tau_m$ for a given simple $K$-module $\tau_m$ and the $K$-module $(\text{Ad}, p_{+})$. Then we have that
\[ C_{[e_1, e_2; \pm]} S^{(\tau_m)}(k) = S_{[e_1, e_2; \pm]}^{(\tau_{[e_1, e_2; \pm]})}(k) \Gamma_{[e_1, e_2; \pm]}, \]
where the product of matrices of the left hand side is the differential operation. Here, $r = (s + l)/2$ and

\[ \text{and otherwise, } C_{[e_1, e_2; \pm]} S^{(\tau_m)}(k) = 0. \]
1. $\Gamma_{[-,-,+]} = \{a_{ij}\}_{0 \leq i \leq n-1, 0 \leq j \leq n}$ is a matrix whose all non zero entries are given by

$$a_{t-1,t} = \frac{1}{2}(\nu_2 + 1 + m_1 + r - 2t)$$

if $(t, n-t) \in I(\tau_m)$, $(t-1, n-t) \in I(\tau_m')$, 

$$a_{t,t} = -\frac{1}{2}(\nu_1 - 1 - m_2 + r - 2t)$$

if $(t, n-t) \in I(\tau_m)$, $(t, n-t-1) \in I(\tau_m')$.

and $C_{[-,-,+]} = \{C_{ij}\}$ is a matrix of size $(m_1m_2) \times (m_1+1)(m_2+1)$ with entries given by

$$C_{m_2p+q+1, (m_2+1)p+q+1} = -E_{14},$$

$$C_{m_2p+q+1, (m_2+1)p+q+2} = -E_{13},$$

$$C_{m_2p+q+1, (m_2+1)(p+1)+q+1} = E_{24},$$

$$C_{m_2p+q+1, (m_2+1)(p+1)+q+2} = E_{23},$$

for each $0 \leq p \leq m_1 - 1$ and $0 \leq q \leq m_2 - 1$, but all other entries are 0.

2. $\Gamma_{[+,-,+]} = \{a_{ij}\}_{0 \leq i \leq n+1, 0 \leq j \leq n}$ is a matrix whose all non zero entries are given by

$$a_{t,t} = \frac{1}{2}(\nu_2 + 1 + m_1 + r - 2t)c_{\nu-t}^{2}$$

if $(t, n-t) \in I(\tau_m)$, $(t, n-t+1) \in I(\tau_m')$,

$$a_{t+1,t} = \frac{1}{2}(\nu_1 + 3 + 2m_1 + m_2 + r - 2t)c_{t+1}^{1}(c_{\nu-t}^{2}-1)$$

if $(t, n-t) \in I(\tau_m)$, $(t+1, n-t) \in I(\tau_m')$.

and $C_{[+,-,+]} = \{C_{ij}\}$ is a matrix of size $(m_1+2)(m_2+2) \times (m_1+1)(m_2+1)$ with entries given by

$$C_{(m_2+2)p+q+1, (m_2+1)p+q+1} = -(1-c_p^{1})(1-c_q^{2})E_{23},$$

$$C_{(m_2+2)p+q+1, (m_2+1)p+q} = -c_q^{2}E_{14},$$

$$C_{(m_2+2)p+q+1, (m_2+1)(p-1)+q+1} = -(1-c_q^{2})E_{23},$$

$$C_{(m_2+2)p+q+1, (m_2+1)(p-1)+q} = c_q^{2}E_{14},$$

for each $0 \leq p \leq m_1 + 1$ and $0 \leq q \leq m_2 + 1$, but all other entries are 0.

3. $\Gamma_{[-,+,+]} = \{a_{ij}\}_{0 \leq i \leq n, 0 \leq j \leq n}$ is a square matrix whose all non zero entries are given by

$$a_{t-1,t} = \frac{1}{2}(\nu_2 + 1 + m_1 + r - 2t)c_{\nu-t-1}^{2}$$

if $(t, n-t) \in I(\tau_m)$, $(t-1, n-t+1) \in I(\tau_m')$,

$$a_{t,t} = \frac{1}{2}(\nu_1 + 1 + 2m_1 - m_2 + r - 2t)c_{\nu-t}^{1}$$

if $(t, n-t) \in I(\tau_m)$, $(t+1, n-t) \in I(\tau_m')$.

and $C_{[-,+,+]} = \{C_{ij}\}$ is a matrix of size $m_1(m_2+2) \times (m_1+1)(m_2+1)$ with entries given by

$$C_{(m_2+2)p+q+1, (m_2+1)p+q+1} = -(1-c_q^{2})E_{13},$$

$$C_{(m_2+2)p+q+1, (m_2+1)p+q} = c_q^{2}E_{14},$$

$$C_{(m_2+2)p+q+1, (m_2+1)(p+1)+q+1} = -(1-c_q^{2})E_{23},$$

$$C_{(m_2+2)p+q+1, (m_2+1)(p+1)+q} = c_q^{2}E_{24},$$

for $0 \leq p \leq m_1 + 1$ and $0 \leq q \leq m_2 - 1$, but all other entries are 0.

4. $\Gamma_{[+,-,-]} = \{a_{ij}\}_{0 \leq i \leq n, 0 \leq j \leq n}$ is a square matrix whose all non zero entries are given by

$$a_{t,t} = \frac{1}{2}(\nu_2 + 1 + m_1 + r - 2t)(1-c_t^{1})$$

if $(t, n-t) \in I(\tau_m)$, $(t, n-t) \in I(\tau_m')$,

$$a_{t+1,t} = \frac{1}{2}(\nu_1 + 1 + 2m_1 - m_2 + r - 2t)c_{t+1}^{1}$$

if $(t, n-t) \in I(\tau_m)$, $(t+1, n-t-1) \in I(\tau_m')$. 

and $C_{[+,-;+]} = \{C_{ij}\}$ is a matrix of size $(m_1 + 2)m_2 \times (m_1 + 1)(m_2 + 1)$ with entries given by

\[
\begin{align*}
C_{m_2p+q+1,(m_2+1)p+q+1} &= (1-c_p^1)E_{24}, \\
C_{m_2p+q+1,(m_2+1)p+q+2} &= (1-c_p^1)E_{23}, \\
C_{m_2p+q+1,(m_2+1)(p-1)+q+1} &= c_p^1 E_{14}, \\
C_{m_2p+q+1,(m_2+1)(p-1)+q+2} &= c_p^1 E_{13},
\end{align*}
\]

for $0 \leq p \leq m_1 + 1$ and $0 \leq q \leq m_2 - 1$, but all other entries are 0.

4.0.1 The Knapp-Stein operator

In this subsection, we consider a matrix representation of the Knapp-Stein operator with respect to the basis for $\pi_{\sigma,\nu}|_K$. This is motivated by Theorem 6.7 in the paper of Goodman-Wallach [2].

Let us recall the Knapp-Stein intertwining operator $A_{\sigma,\nu}^{s}$ from the space of all $C^\infty$-vectors of $\pi_{\sigma,\nu}$ to that of $\pi_{s(\sigma),s(\nu)}$ defined by

\[
(A_{\sigma,\nu}^{s}f)(k) = \int_{\overline{N}_{\theta}} a(n_{\partial}s^{*}k)^{\nu+\rho} f(k(n_{\epsilon}s^{*}k))dn_{s}, \quad (f \in \pi_{\sigma,\nu}^\infty)
\]

Here $s^{*} \in K$ such that $s := Ad(s^{*}) \in W(A), \overline{N}_{s} = N \cap s^{*}Ns^{*-1}$ and $s(\sigma)$ is a character of $M$ given by $s(\sigma)(m) = \sigma(s^{*}ms^{*-1}), m \in M$. Since it is a linear map from $\pi_{\sigma,\nu}$ to $\pi_{s(\sigma),s(\nu)}$ satisfying

\[
A_{\sigma,\nu}^{s}\pi_{\sigma,\nu}(x)f = \pi_{s(\sigma),s(\nu)}(x)A_{\sigma,\nu}^{s}f, \quad x \in G \ (or \ U(\mathfrak{g}))
\]

we have a linear map

\[
A^{s}(\tau) : \text{Hom}_{K}(\tau, \pi_{\sigma,\nu}|_K) \rightarrow \text{Hom}_{K}(\tau, \pi_{s(\sigma),s(\nu)}|_K)
\]

for any $\tau \in \hat{K}$.

Let $[\alpha_i]$ be the $K$-isomorphism from $\tau$ to $W_{\alpha_i}$ for $\alpha_i \in I(\tau)$. We equip the space $\text{Hom}_{K}(\tau, \pi_{\sigma,\nu}|_K)$ with the basis consisting of the $K$-homomorphisms $[\alpha_i]$. Similarly, we choose a basis for the space $\text{Hom}_{K}(\tau, \pi_{s(\sigma),s(\nu)}|_K)$

Then we want to compute all entries $a_{ij}$ of the matrix $A^{s}(\tau) = (a_{ij})$ such that

\[
A^{s}(\tau)[\alpha_i] = \sum_{\alpha_j \in I} a_{ij} \cdot [\alpha_j^s]
\]

where $I = \{\alpha^s \mid W_{\alpha^s} \leftrightarrow \pi_{s(\sigma),s(\nu)}|_K\}$. For each basis vector $f_{pq}$ of $\tau$ as in Lemma 2.1, we have that

\[
(A^{s}(\tau)[\alpha_i])(f_{pq}) = \sum_{\alpha_j \in I} a_{ij} \cdot [\alpha_j^s](f_{pq}) = \sum_{\alpha_j \in I} a_{ij} \cdot f_{\alpha_j}(k).
\]

On the other hand, by definition of the map $A^{s}(\tau)$, one has

\[
(A^{s}(\tau)[\alpha_i])(f_{pq}) = (A_{\sigma,\nu}^{s}f_{\alpha_i}(k), \quad \alpha_i \in I(\pi_{\sigma,\nu}, \tau).
\]

Thus we have the following formula for the coefficients $a_{ij}$ of the matrix $A^{s}(\tau)$ for each $\tau \in \hat{K}$.

**Lemma 4.2** Let $\alpha_i$ be in $I(\pi_{\sigma,\nu}, \tau)$ and $\alpha_j^s$ be in $I(\pi_{s(\sigma),s(\nu)}, \tau)$. Then the $(i,j)$-th coefficient of $A^{s}(\tau)$

\[
a_{ij} = (A_{\sigma,\nu}^{s}f_{\alpha_i}(m))(14).
\]

**Example 4.3**
Let $s$ be a generator of $W(A)$ whose image is the matrix diag$(1,-1)$ under the representation of $W(A)$ on $a^*$. Then we choose the corresponding $s^* \in K$ as the matrix diag$(1,-i,1,i)$ and hence

$$\overline{N}_s = \exp(\mathfrak{g}_{-2\lambda_2}) = \left\{ n_s(t) = \kappa^{-1} \begin{pmatrix} 1 & 1 \\ t & 1 \end{pmatrix} \kappa : t \in \mathbb{R} \right\}.$$ 

Since $n_s \in \overline{N}_s$, one has $n_s I_{2,2} n_s = I_{2,2}$ and hence $n_s s^* = I_{2,2} t n_s^{-1} I_{2,2} s^*$. Thus, we have the following.

Assume $n_s = n_s(t) \in \overline{N}_s$. Let $n' \in N$, $a(n_s s^*) \in A$ and $k(n_s s^*) \in K$ be so that $n_s s^* = n'a(n_s s^*) k(n_s s^*)$.

Then

$$a(n_s s^*)^{\nu+\rho} = (1+t^2)^{-\frac{m_1}{2}-1},$$

$$k(n_s s^*) = \text{diag}(1,-iu,-1,-iu^{-1})$$

where $u = ((1-it)/(1+it))^\frac12$.

For a fixed $\tau_m \in \hat{K}$, therefore

$$f_{\gamma_i,\beta_j}(k(n_s s^*)) = 0 \text{ when } \gamma_i \neq \beta_j$$

If $\tau = \tau_{[m_1,m_2;1]}$ then we have

$$A^\ast(\tau) = 2\pi 2^{-\nu} \Gamma(\nu_2) \text{ diag} \left[ \frac{(-1)^{(m_1+m_2)/2-p+1}i^{m_2+r}}{\Gamma(\frac12 \nu_2 + \frac12 + d) \Gamma(\frac12 \nu_2 + \frac12 - d)} \right]_p$$

where $d = \frac12(m_1 + r - 2p)$ for $(p, (m_1 + m_2)/2 - p) \in I(\pi_{\sigma,\nu}, \tau_m)$.

## 5 Whittaker functions

The main focus of this section is on the integral expressions of Whittaker functions on $G$ related to certain principal series. The results of the section 4.1 lead us to the study of Whittaker functions related to some $K$-types. For this purpose, we focus our investigation on the principal series representations which contain one dimensional $K$-types and apply the method used in [4] to evaluate such Whittaker functions. More precisely, in this setting, the character $\sigma$ of $M$ factors through a character $\chi$ of $\mu_2$. Let $(\pi_{\chi,\nu}, L^2(\mu_2))$ denote the principal representation series corresponding to such character $\sigma$.

For an integer $u$, define a function $f_u(k)$ on $K$ by $f_u(k) := \det(k_{22})^u$, $k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in K$.

**Lemma 5.1** Let $f_u(k)$ be as above. Then $\tau_{[0,0,2u]} \cong \mathbb{C} f_u(k)$ as $K$-modules. Moreover, if $\chi(-1) = (-1)^u$ then $f_u(k) \in L^2(\mu_2)$ and $[\pi_{\chi,\nu} : \tau_{[0,0,2u]}] = 1$.

### 5.1 The Jacquet integral.

Let $J_{\chi,\nu}$ be the Jacquet functional on the subspace of differentiable functions of $L^2(\mu_2)$ given by

$$J_{\chi,\nu}(f) = \int_{\mathcal{N}} \eta(n)^{-1} a(s^* n)^{\nu+\rho} f(k(s^* n)) dn$$

for a differentiable function $f$ in $L^2(\mu_2)$ and the longest element $s \in W(A)$. Here $W(A)$ is the Weyl group defined as the quotient of $M^* = N_{\mathfrak{k}}(a)$, the normalizer of $a$ in $K$, by $M$ and $s^*$ is an element of $M^*$ mapping to the longest element $s \in W(A)$. 


Then one has \( J_{\chi,\nu}(\pi(n)f) = \eta(n)J_{\chi,\nu}(f) \) and hence
\[
J \in \text{Hom}_{(g, K)}(\pi_{\chi,\nu}|_{K}, \mathcal{A}_{\eta}(N \backslash G)),
\]
where \( J \) associates \( v \in \pi_{\chi,\nu}|_{K} \) to the function \( J_{v}(g) := J_{\chi,\nu}(\pi_{\chi,\nu}(g)v) \), \( (g \in G) \). We want to have an explicit formula for the \( A \)-radial part:
\[
J_{f_{u}}(a) = J_{\chi,\nu}(\pi_{\chi,\nu}(a)f_{u}) = a^{-\nu+\rho} \int_{N} \eta(ana)^{-1}a(s^{*}n)^{\nu+\rho}f_{u}(k(s^{*}n))dn.
\]
In our case, we can choose \( I_{2,2} \in K \) for \( s^{*} \in K \).

5.1.1 The first modification

Let \( E_{ij} \) be the usual matrix with 1 in \((i,j)\)-entry and zero elsewhere. Put
\[
E_{0} = \kappa^{-1}(E_{12} - E_{43})\kappa, \quad E_{1} = i\kappa^{-1}(E_{12} + E_{43})\kappa, \quad E_{2} = \kappa^{-1}E_{24}\kappa,
\]
\[
F_{0} = \kappa^{-1}(E_{14} + E_{23})\kappa, \quad F_{1} = i\kappa^{-1}(E_{14} - E_{23})\kappa, \quad F_{2} = \kappa^{-1}E_{13K}\kappa,
\]
by setting \( i = \sqrt{-1} \) and
\[
\kappa = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -i & 0 & i & 0 \\ 0 & -i & 0 & i \end{pmatrix}.
\]

Then the corresponding root spaces of positive roots in \( \Phi(g, a) \) are given by
\[
\mathfrak{g}_{\lambda_{1}-\lambda_{2}} = E_{0}\cdot \mathbb{R} \oplus E_{1}\cdot \mathbb{R}, \quad \mathfrak{g}_{2\lambda_{2}} = E_{2}\cdot \mathbb{R},
\]
\[
\mathfrak{g}_{\lambda_{1}+\lambda_{2}} = F_{0}\cdot \mathbb{R} \oplus F_{1}\cdot \mathbb{R}, \quad \mathfrak{g}_{2\lambda_{1}} = F_{2}\cdot \mathbb{R}.
\]

Let \( \mathfrak{n} \) be a subalgebra defined by \( \mathfrak{n} = \sum_{\alpha \in \Phi^{+}} \mathfrak{g}_{\alpha} \). We now describe elements of a maximal nilpotent subgroup \( N \) of \( G \) given by \( N = \exp(\mathfrak{n}) \).

The Killing form \( B(X, Y) = \text{tr}(\text{ad}X \cdot \text{ad}Y), \ (X, Y \in \mathfrak{g}) \) and Cartan involution \( \theta \) of \( \mathfrak{g} \) induce an inner product \( \langle , \rangle \) of \( \mathfrak{g} \) via
\[
\langle X, Y \rangle = -B(X, Y^{\theta}), \ (X, Y \in \mathfrak{g}).
\]

Then one has that \( \langle \mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta} \rangle = 0 \) if \( \alpha \neq \beta \), because of the involution \( \theta \). Moreover, one can see that the set \( \{E_{i}, F_{i} \mid i = 0,1,2\} \) is an \( \langle , \rangle \)-orthogonal basis for \( \mathfrak{n} \) such that a each element \( \mathfrak{n} = n(n_{0}, n_{1}, n_{2}, n_{3}) \) in the maximal unipotent group \( N = \exp(\mathfrak{n}) \) is expressed in the form:
\[
\kappa^{-1} \begin{pmatrix} 1 & n_{0} & 1 \\ & 1 & n_{1} \\ & & 1 -n_{2} \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & n_{1} & n_{2} \\ 1 & n_{2} & n_{3} \\ 1 \end{pmatrix} \kappa.
\]
for \( n_{1}, n_{3} \in \mathbb{R}, \ n_{0}, n_{2} \in \mathbb{C} \).

Lemma 5.2 We have

1. Set \( N_{1} = \begin{pmatrix} n_{1} & n_{2} \\ n_{2} & n_{3} \end{pmatrix} \) for \( n = n(n_{0}, n_{1}, n_{2}, n_{3}) \in N \). Then
\[
f_{u}(k(I_{2,2}n)) = \left( \frac{\det(1 - \sqrt{-1}N_{1})}{\det(1 + \sqrt{-1}N_{1})} \right)^{\#}.
\]
2. Let $\eta$ be a character of $N$ determined by a real number $c_2$ and $c = c_0 + \sqrt{-1}c_1 \in \mathbb{C}$. Then
\[
\eta(\alpha a^{-1}) = \exp(2\sqrt{-1}\left( \frac{a_1}{a_2} \Re(\overline{c}n_0) + c_2a_2^2n_3 \right)),
\]
where $a_i = \exp(t_i)$, $(i = 1, 2)$ for $a = (a_1, a_2) \in A$.

3. For $\nu = (\nu_1, \nu_2) \in \text{Hom}(k, \mathbb{R})$, we have that $a(I_{2, 2}n)^{\nu + \rho} = \Delta_1^{-\nu_1 + \frac{3}{2}n_2} \Delta_2^{-\frac{3}{2}n_2 - 1}$ where
\[
\Delta_1 = 1 + n_1^2 + n_2n_2 + (n_0n_2 + \overline{n}_0n_2)(n_1 + n_3) + \overline{n}_0n_0(1 + \overline{n}_2n_2 + n_3^2),
\]
\[
\Delta_2 = 1 + n_1^2 + 2n_2n_2 + n_3^2 + (n_1n_3 - n_2n_2)^2
\]
for $n = (n_0, n_1, n_2, n_3) \in N$.

For future convenience, we choose a new coordinate for $A$ by
\[
y = (y_1, y_2) = \left( \frac{a_1}{a_2}, a_2^2 \right).
\]
Since $f \rightarrow J_f(a)$ is the Whittaker realization of $\pi_{\chi, \nu}$, $J_f(a)$ is the radial part of a Whittaker function on $G$ belonging to $\pi_\nu$. Thus, in the new coordinate system, we can summarize that the radial part of Whittaker function associated with the $K$-type $\tau_u$ can be written in the form
\[
y_1^{-\nu_1 + 3}y_2^{-\nu_2 + 3} \int_N \Delta_1^{-\nu_1 + \frac{3}{2}n_2} \Delta_2^{-\frac{3}{2}n_2 - 1} \exp(-2\sqrt{-1}(y_1Re(\overline{c}n_0) + c_2y_2n_3))f_u(k(I_{2, 2}n))dn,
\]
where $dn$ is a multiplicative Haar measure on $N$. Now we shall give a normalization of Haar measure of $N$.

Since the exponential map of $n$ onto $N$ is an analytic isomorphism, there exists a unique Haar measure $dn$ on $N$ that corresponds to Lebesgue measure on $n$.

Lemma 5.3 The radial part of the moderate growth Whittaker function $W_{(\nu_1, \nu_2)}(y_1, y_2; u)$ (up to constant) associated with the $K$-type $\tau_u$ can be written in the form
\[
y_1^{-\nu_1 + 3}y_2^{-\nu_2 + 3} \int_{\mathbb{R}^4} \Delta_1^{-\nu_1 + \frac{3}{2}n_2} \Delta_2^{-\frac{3}{2}n_2 - 1} \exp(-2\sqrt{-1}(c_0z_0y_1 + c_1t_0y_1 - n_3y_2))f_u(k(I_{2, 2}n))dn,
\]
with respect to $dz_0dt_0dn_1dz_2dt_2dn_3$. Here $n_i = z_i + \sqrt{-1}t_i$ $(i = 0, 2)$.

In fact, it suffices to consider the cases $u = 0$ and $1$ for our purposes. Let $K_\mu(z)$ be the Bessel function.

Theorem 5.4 Let $\pi_{\chi, \nu}$ be irreducible and $\eta$ be a nondegenerated unitary character $N$. Then we have the following assertions on the $A$-radial part of the primary Whittaker function $W_{(\nu_1, \nu_2)}(y_1, y_2; u)$.

If $\chi$ is trivial then the Whittaker function $W_{(\nu_1, \nu_2)}(y_1, y_2; 0)$ is identified with $y_1^3y_2^5$ times
\[
\int_0^\infty \int_0^\infty T_{\nu_1, \nu_2}(y_1, y_2, t_1, t_2) \frac{dt_1}{t_1} \frac{dt_2}{t_2}.
\]

If $\chi$ is non-trivial then the Whittaker function $\tilde{W}_{(\nu_1, \nu_2)}(y_1, y_2; 1)$ is identified with $y_1^7y_2^5/4$ times
\[
\int_0^\infty \int_0^\infty T_{\nu_1, \nu_2}(y_1, y_2, t_1, t_2)(\sqrt{t_1/t_2} - 1/\sqrt{t_1t_2}) \frac{dt_1}{t_1} \frac{dt_2}{t_2}
\]
where $T_{\nu_1, \nu_2}(y_1, y_2, t_1, t_2)$ is the function
\[
K_{\frac{\nu_1 + \nu_2}{2}}(2\sqrt{t_1t_2} / \sqrt{t_1 + t_2})K_{\frac{\nu_1 - \nu_2}{2}}(2\sqrt{t_1t_2}) \exp\left(-|c_2|y_2t_1 - \frac{|c_2|y_2}{t_1} - \frac{t_2}{|c_2|y_2} - (c_0^2 + c_1^2)|c_2|y_2/t_2 \right)
\]
Note here that, we need the following formula to reduce the number of integral symbols corresponding to the root spaces $\mathfrak{h}_{\lambda_1 - \lambda_2}$ and $\mathfrak{h}_{\lambda_1 + \lambda_2}$.

Formula 5.5 Let $a, c \in \mathbb{R}^*_+$ and $b, \alpha, \beta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 = 1$. Then we have
\[
\int_\mathbb{R} \int_\mathbb{R} \exp(-c(x^2 + y^2) - a(\alpha x + \beta y)^2 + 2\sqrt{-1}b(\alpha x + \beta y))dxdy = \frac{\pi}{(c^2 + ac)^{\frac{1}{2}}} \exp\left(-\frac{b^2}{a + c}\right).
\]
References


[5] T. Oda, The standard $(g, K)$-modules of $Sp(2, \mathbb{R})$, Preprint Series, UTMS 2007-3, Graduate School of Mathematical Sciences, University of Tokyo.


