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Some asymptotic expansions of the Eisenstein series

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2 Definition of the Eisenstein series

Let $i=\sqrt{-1}$, $s=\sigma+it\in\mathbb{C}$ and H be the upper half plane. The non-holomorphic Eisenstein series for $SL_2(\mathbb{Z})$ with weight 0 is

$$E(z,s)=y^s\sum_{\{c,d\}}|cz+d|^{-2s}. \tag{1}$$

Here $z=x+iy\in H$, and the summation is taken over $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$, a complete system of representation of $\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\in SL_2(\mathbb{Z})\right\}\backslash SL_2(\mathbb{Z})$. The Fourier expansion is as follows:

$$\begin{aligned}\zeta(2s)E(z,s) &= \zeta(2s)y^s+\sqrt{\pi}\zeta(2s-1)\tfrac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s} \\ &\quad +\tfrac{4\pi^s}{\Gamma(s)}\sqrt{y}\sum_{n=1}^{\infty}n^{s-\frac{1}{2}}\sigma_{1-2s}(n)K_{s-\frac{1}{2}}(2\pi ny)\cos(2\pi nx),\end{aligned} \tag{2}$$

where $K_\nu(\tau)$ is the modified Bessel function and $\sigma_s(n)$ is the sum of s-th powers of positive divisors of n. We call the first two terms of (2) are the constant term of $E(z,s)$.

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3 y-aspect of the Eisenstein series

It is well-known that **the constant term represents the y-aspect of $E(z,s)$ as $y \to \infty$.** Because the Bessel function in (2) decays exponentially. Therefore there exist positive constants A_1 and A_2 depending only on s such that (except on the poles)

$$|E(z,s)| \leqq A_1 y^{\mathrm{Re}(s)} + A_2 y^{1-\mathrm{Re}(s)} \qquad (y \to \infty). \qquad (3)$$

The invariance of $E(z,s)$ under the action of $SL_2(\mathbb{Z})$ gives the asymptotic behavior when $y \to 0$. For every $y > 0$, except on the poles,

$$|E(z,s)| \leqq \begin{cases} A_1(y^{-\mathrm{Re}(s)} + y^{\mathrm{Re}(s)}) & (\mathrm{Re}(s) > \frac{1}{2}) \\ A_2(y^{-1+\mathrm{Re}(s)} + y^{1-\mathrm{Re}(s)}) & (\mathrm{Re}(s) \leqq \frac{1}{2}). \end{cases} \qquad (4)$$

The t-aspect of $E(z,s)$ is not simple. **The non-constant terms in (2) are not negligible when $t \to \infty$.**

Empirically, the behavior of $E(z,s)$ respect to $\mathrm{Im}(s) = t$ is similar to the behavior of $\zeta(s)^2$,

$$E(z,s) \quad \Longleftarrow ? \Longrightarrow \quad \zeta(s)^2.$$

Problem.

Investigate the asymptotic behavior $E(z,s)$ with respect to $\mathrm{Im}(s) = t$.

4 Definition of the Airy Function

The modified Bessel function $K_\nu(\tau)$ (ν, $\tau \in \mathbb{C}$) is defined by the integral

$$K_\nu(\tau) = \frac{1}{2}\int_0^\infty u^{\nu-1} \exp\left(-\frac{1}{2}\tau(u+\frac{1}{u})\right) du,$$

which satisfies the modified Bessel equation:

$$\frac{d^2w}{d\tau^2} + \frac{1}{\tau}\frac{dw}{d\tau} - \left(1+\frac{\nu^2}{\tau^2}\right) w = 0.$$

For $\tau \in \mathbb{R}$, the **Airy function** is defined by

$$\mathrm{Ai}(\tau) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}u^3 + \tau u\right) du = \frac{1}{\sqrt{3}\pi}\tau^{\frac{1}{2}} K_{\frac{1}{3}}\left(\frac{2}{3}\tau^{\frac{3}{2}}\right),$$

which satisfies the differential equation

$$\frac{d^2w}{d\tau^2} = \tau w.$$

The representation of $\mathrm{Ai}(\tau)$ for $\tau \in \mathbb{C}$ ($|\arg \tau| < \pi$) is

$$\mathrm{Ai}(\tau) = \frac{\exp\left(-\frac{2}{3}\tau^{\frac{3}{2}}\right)}{2\pi} \int_0^\infty \exp\left(-\tau^{\frac{1}{2}}u\right) \cos\left(\frac{1}{3}u^{\frac{3}{2}}\right) u^{-\frac{1}{2}} du.$$

In which fractional powers take their principal values.

The modified Bessel function $K_{it}(\tau)$ decays exponentially for large τ. The asymptotic expansion of $K_{it}(\tau)$ for the cases $\tau/t \not\sim 1$ or $\tau - t = o(\tau^{1/3})$ are obtainable by using saddle-point method.

However, in the transitional regions, namely τ/t is nearly equal to 1 while $|\tau - t|$ is large, the investigation becomes much more involved. As an another approach, the theory of asymptotic solutions of differential equations are employed, where the Airy function plays a fundamental role (cf. [4], 7.4.3, [14]).

5 Main Theorem and Corollaries

Theorem 1 *Let $z = x + iy \in H$ and $t > 289$.Assume that $y < t^{\frac{1}{3}-\delta}$ for any positive constant δ, and $N \geqq 0$ is any integer satisfying $t - (4\log t)^{\frac{2}{3}} t^{\frac{1}{3}} \leqq 2\pi y N < t$. Define*

$$\frac{2}{3}\tau_n = t \log \frac{t + \{t^2 - (2\pi y)^2\}^{\frac{1}{2}}}{2\pi y} - \{t^2 - (2\pi n y)^2\}^{\frac{1}{2}}.$$

Then for every $\varepsilon > 0$,

$$\begin{aligned} E(z, \tfrac{1}{2} + it) &= \tfrac{4\sqrt{2}\pi^{\frac{1}{2}+it} y^{\frac{1}{2}}}{\zeta(1+2it)} \sum_{n=1}^{N} n^{-it} \sigma_{2it}(n) \{t^2 - (2\pi n y)^2\}^{-\frac{1}{4}} \cos(2\pi n x) \tau_n^{\frac{1}{4}} \mathrm{Ai}(-\tau_n^{\frac{2}{3}}) \\ &\quad + y^{\frac{1}{2}+it} + y^{\frac{1}{2}-it} e^{i\theta} + O\left(y^{-\frac{3}{2}} t^{-\frac{1}{3}} (\log t)^{\frac{1}{2}+\varepsilon} + y^{-\frac{1}{2}} (\log t)^{\frac{4}{3}+\varepsilon} \log(t/y)\right). \end{aligned}$$

Here $e^{i\theta} = \pi^{-2it} \zeta(2it)\Gamma(it) / \overline{\zeta(2it)\Gamma(it)}$. The implied O-constant depends at most on ε and δ.

Corollary 1 *Suppose $t - t^{\frac{1}{3}+\frac{1}{4}} \leqq 2\pi y M \leqq t$. For every $\varepsilon > 0$,*

$$\begin{aligned} E(z, \tfrac{1}{2} + it) &= \tfrac{4\sqrt{2}\pi^{it} y^{\frac{1}{2}}}{\zeta(1+2it)} \sum_{n=1}^{M} n^{-it} \sigma_{2it}(n) \{t^2 - (2\pi n y)^2\}^{-\frac{1}{4}} \cos(2\pi n x) \cos(\tfrac{2}{3}\tau_n - \tfrac{\pi}{4}) \\ &\quad + y^{\frac{1}{2}+it} + y^{\frac{1}{2}-it} e^{i\theta} + O\left(y^{-\frac{3}{2}} t^{-\frac{1}{3}} (\log t)^{\frac{1}{2}+\varepsilon} + y^{-\frac{1}{2}} t^{\frac{1}{4}} (\log t)^{\frac{4}{3}+\varepsilon} \log(t/y)\right). \end{aligned}$$

Corollary 2 *Let $z = x + iy \in H$. Assume that $c_0 < y < t^{\frac{1}{3}-c_1}$ for some positive constants c_0 and c_1. Then for every $\varepsilon > 0$,*

$$E(z, \tfrac{1}{2} + it) = O\left(y^{-\frac{1}{2}}\, t^{\frac{1}{2}+\varepsilon}\right) \qquad \text{as } t \to \infty.$$

Remark. Corollary 2 is a convexity bound for the Eisenstein series which includes the y-factor.

6 t-aspect of the Eisenstein series

Known fact 1. A convexity bound is known (see [16] (p. 258)), which is a consequence of the Phragmén-Lindelöf convexity principle;

$$E(z, \tfrac{1}{2} + it) = O_y(t^{\frac{1}{2}+\varepsilon}).$$

Known fact 2. The spectral theory of automorphic forms gives the following estimate:

$$\sum_{0<t_j<T} |u_j(z)|^2 + \frac{1}{2\pi}\int_0^T |E(z, \tfrac{1}{2} + it)|^2 dt = O\left(T^2 + Ty\right).$$

Here $\{u_j\}$ is an orthonormal system of cusp forms for $SL_2(\mathbb{Z})$ with $\Delta u_j = (\frac{1}{4} + t_j^2)u_j$. (See for example [6], (13.1).)
A convexity bound for the Riemann zeta-function is

$$\zeta(\tfrac{1}{2} + it) = O(t^{\frac{1}{4}+\varepsilon}).$$

Further, the sub-convexity bound, the classical result for the Riemann zeta-function due to Hardy-Littlewood is

$$\zeta(\tfrac{1}{2} + it) = O(t^{\frac{1}{6}+\varepsilon}).$$

The mean value theorem of the Riemann zeta-function on the critical line is

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^2 dt = T \log T + (\text{error}).$$

These facts for the Riemann zeta-function support the following conjecture.

The Lindelöf hypothesis For any positive ε,

$$\zeta(\tfrac{1}{2} + it) = O(t^{\varepsilon}).$$

(This is true if the Riemann hypothesis is true.)

From the standpoint of the similarity between $E(z,s)$ and $\zeta(s)^2$, we set up the following

Conjecture. Assume $y \geqq c_0$ for a positive constant c_0. For any positive ε,

$$E(z,\tfrac{1}{2}+it) = O(t^{\varepsilon}+y^{\frac{1}{2}}) \qquad as\ \ t\to\infty.$$

Here the y-factor comes from the constant term.

7 Jutila's formula

Theorem (Jutila 1984) *Let $t \geqq 12\pi$, δ be a positive number, $t^{\delta} \leqq N \leqq t/12\pi$, and*

$$N' = t/2\pi + N/2 - (N^2/4 + Nt/2\pi)^{\frac{1}{2}}.$$

Define

$$f(t,n) = 2t\operatorname{arcsinh}\sqrt{\pi n/2t} + (\pi^2 n^2 + 2\pi n t)^{\frac{1}{2}} + \pi/4.$$

Then,

$$\begin{aligned}|\zeta(\tfrac{1}{2}+it)|^2 &= 2^{\frac{1}{2}}\sum_{n=1}^{N}(-1)^n d(n) n^{-\frac{1}{2}}\left(\tfrac{1}{4}+\tfrac{t}{2\pi n}\right)^{-\frac{1}{4}}\cos(f(t,n))\\ &\quad + 2\sum_{n=1}^{N'} d(n) n^{-\frac{1}{2}}\cos\left(t\log(t/2\pi n) - t - \tfrac{\pi}{4}\right) + O\left(N^{\frac{1}{4}}t^{-\frac{1}{4}}(\log t)^2 + \log t\right).\end{aligned}$$

Remark 1. Jutila's formula is a differentiated version of Atkinson's formula. In the proofs of these formulas, Voronoi's summation formula and the saddle-point method are used as the main instruments.

Remark 2. There are some differences between Theorem 1 and the formulas on the square of the Riemann zeta-function. Atkinson type formulas usually have two summations, whereas Theorem 1 and Corollary 1 consist of one summation $\sum_{1\leqq n\leqq N}$. This difference is explained by each approximate functional equations;

$$\zeta^2(s) = \sum_{n=1}^{N} d(n) n^{-s} + \pi^{2s-1}\frac{\Gamma^2(\frac{1}{2}-\frac{s}{2})}{\Gamma^2(\frac{s}{2})}\sum_{n=1}^{N'} d(n) n^{s-1} + O(N^{\frac{1}{2}-\sigma}\log t),$$

where $0 \leqq \sigma \leqq 1$, $NN' = (t/2\pi)^2$, $N \geqq 1$, $N' \geqq 1$.

For the case of $E(z,s)$, the Fourier expansion (2) itself may be regarded as one self dual (approximate) functional equation except the constant term. Originally Voronoi's summation formula consists of one summation.

8 Other asymptotic expansions

Define the holomorphic Eisenstein series for $SL_2(\mathbb{Z})$ as

$$E_s(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} (m+nz)^{-s}.$$

Theorem 2 *(K. Matsumoto [9]) Assume* $0 < |\arg(z)| < \pi$ *and* $\mathrm{Re}(s) > -N+1$ *for any positive integer N, then for*

$|z| \geq 1$,

$$E_s(z) = (1+e^{\pi is})\zeta(s) + O(|z|^{-\mathrm{Re}(s)-N}).$$

For

$|z| \leq 1$,

$$\begin{aligned} E_s(z) &= (1+z^{-s})(1+e^{\pi is})\zeta(s) + (e^{\pi is}-e^{-\pi is})\frac{\zeta(s-1)}{s-1}z^{-1} - \left(1+\frac{e^{\pi is}+e^{-\pi is}}{2}\right)\zeta(s) \\ &+ \sum_{1\leq k\leq N-1, k:\mathrm{odd}} (e^{\pi is}-e^{-\pi is})\binom{-s}{k}\zeta(s+k)\zeta(-k)z^k + O(|z|^N). \end{aligned}$$

Define the non-holomorphic Eisenstein series of weight k attached to $SL_2(\mathbb{Z})$ as

$$E_k(z,s) = \frac{1}{2}\sum_{\substack{c,d=-\infty \\ (c,d)=1}}^{\infty} (cz+d)^{-k}|cz+d|^{-2s}, \tag{5}$$

and define Ramanujan's Φ-function

$$\Phi_{s_1,s_2}(e(z)) = \sum_{l_1,l_2=1}^{\infty} l_1^{s_1} l_2^{s_2} e(l_1 l_2 z) = \sum_{l=1}^{\infty} \sigma_{s_1-s_2}(l) l^{s_2} e(lz).$$

Theorem 3 *(M. Katsurada [8]) For any* $z \in H$ *and any integer* $N \geq 0$ *the following formula holds in* $-N < \mathrm{Re}(s) < 1+N$ *except at* $s = 1$.

$$E_0(z,s) = 1 + \frac{\sqrt{\pi}\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}y^{1-2s} + \frac{(2\pi)^{2s}}{\Gamma(s)\zeta(2s)}\{S_N(s;z)+R_N(s;z)\}.$$

Here

$$S_N(s;z) = \sum_{n=0}^{N-1} \frac{(-1)^n (s)_n (1-s)_n}{n!} \Phi^*_{s-n-1,-s-n}(e(z))(4\pi y)^{-s-n}$$

with

$$\Phi^*_{s_1,s_2}(e(z)) = \Phi_{s_1,s_2}(e(z)) + \Phi_{s_1,s_2}(\overline{e(z)}).$$

The remainder term $R_N(s;z)$ *is estimated as*

$$R_N(s;z) = O\left\{(|t|+1)^{2N} e^{-2\pi y} y^{-\mathrm{Re}(s)-N}\right\}.$$

Remark. Theorem 3 yields various known results on $E_0(z,s)$, including its functional properties and its asymptotic aspects as $z \to 0$. Especially the Mellin-Barnes integral transformation shows that the functional equation of $E_0(z,s)$ reduces eventually into the simple property

$$\Phi_{s_1,s_2}(e(z)) = \Phi_{s_2,s_1}(e(z)).$$

Theorem 4 *(M. Katsurada, T. Noda (to appear))*

$$\begin{aligned} E_k(z,s) &= 1+(-1)^{k/2}2\pi\frac{\Gamma(2s+k-1)}{\Gamma(s)\Gamma(s+k)}\frac{\zeta(2s+k-1)}{\zeta(2s+k)}(2y)^{1-2s-k} \\ &+\frac{(-1)^{k/2}(2\pi)^{2s+k}}{\zeta(2s+k)\Gamma(s+k)}\{S_{N+k/2}(s,2s+k;z)+R_{N+k/2}(s,2s+k;z)\} \\ &+\frac{(-1)^{k/2}(2\pi)^{2s+k}}{\zeta(2s+k)\Gamma(s)}\{S_{N-k/2}(s+k,2s+k;-\bar{z})+R_{N-k/2}(s+k,2s+k;-\bar{z})\} \end{aligned}$$

holds in the region $-N-k/2 < \mathrm{Re}(s) < N-k/2+1$ except at the complex zeros of $\zeta(2s+k)$ and at the real poles of $E_k(s;z)$. Here the remainder terms are estimated as

$$R_{N+k/2}(s;2s+k;z) = O\{(|t|+1)^{2N+k}e^{-2\pi y}y^{-\sigma-N-k/2}\}$$

and

$$R_{N-k/2}(s+k;2s+k;-\bar{z}) = O\{(|t|+1)^{2N-k}e^{-2\pi y}y^{-\sigma-N-k/2}\}.$$

Remark 1. The asymptotic expansion of $E_k(s;z)$ established by transferring from the derived asymptotic expansion of $E_0(s;z)$ (Theorem 3) to that of $E_k(s;z)$ through successive use of Maass' weight change operators.

Remark 2. Theorem 4 also gives a new alternative proof of the Fourier expansion of $E_k(z,s)$, consequently gives new proofs of various results on $E_k(z,s)$, for example, functional equation, special values, the Kronecker limit formula, the eigenfunction equation for the non-Euclidean Laplacian and so on.

9 Outline of the proof of the main theorem

Balogh [3] gave one uniform asymptotic expansion of the modified Bessel function by using Airy functions. Balogh's result is based on Olver's works. The following proposition (Olver [14], Chap.11, p. 425) is the uniform asymptotic expansion of the modified Bessel function of imaginary order, which is crucial in this report.

Proposition 1 *([3], Olver [14] p.425) For $t \in \mathbb{R}_{>0}$, $m \geqq 0$ and $u \in \mathbb{C}$ with $|\arg(u)| < \pi$,*

$$\begin{aligned} K_{it}(tu) = \frac{\pi}{t^{\frac{1}{3}}}\exp\left(-\frac{\pi}{2}t\right)\left(\frac{4\xi}{1-u^2}\right)^{\frac{1}{4}}\Bigg\{\mathrm{Ai}(-t^{\frac{2}{3}}\xi)\sum_{k=0}^{m}\frac{A_k(\xi)}{t^{2k}} \\ +t^{-\frac{4}{3}}\mathrm{Ai}'(-t^{\frac{2}{3}}\xi)\sum_{k=0}^{m-1}\frac{B_k(\xi)}{t^{2k}}+\varepsilon_{2m+1}(t,\xi)\Bigg\}. \end{aligned}$$

The Airy function $\mathrm{Ai}(\tau)$ ($\tau \in \mathbb{R}$) decays rapidly as $\tau \to \infty$, and decays slowly (with oscillation) as $\tau \to -\infty$. More precisely, we have following

Proposition 2 *(I) For $\tau \in \mathbb{C}$ with $|\arg \tau| < \pi$, we have*

$$\mathrm{Ai}(\tau) = \frac{\exp\left(-\frac{2}{3}\tau^{\frac{3}{2}}\right)}{2\pi^{\frac{1}{2}}\tau^{\frac{1}{4}}} \sum_{l=0}^{n-1} a_l \left(-\frac{2}{3}\tau^{\frac{3}{2}}\right)^{-l} + (error).$$

(II) For $\tau \in \mathbb{R}_{>0}$, we have

$$\mathrm{Ai}(-\tau) = \frac{1}{\pi^{\frac{1}{2}}\tau^{\frac{1}{4}}} \sum_{l=0}^{n-1} a_l \cos\left(\frac{2}{3}\tau^{\frac{3}{2}} - \frac{1}{4}\pi - \frac{\pi}{2}l\right) \left(\frac{2}{3}\tau^{\frac{3}{2}}\right)^{-l} + (error).$$

In (I) and (II), fractional powers of τ take their principal values.

On the critical line $s = \frac{1}{2} + it$, we divide the summation of (2) into five segments:

$$\zeta(1+2it)E(z, \tfrac{1}{2}+it) = S_0(z,t) + S_1(z,t) + S_2(z,t) + S_3(z,t) + S_\infty(z,t).$$

Here

$$S_0(z,t) = \zeta(1+2it)y^{\frac{1}{2}+it} + \sqrt{\pi}\zeta(2it)\frac{\Gamma(it)}{\Gamma(\frac{1}{2}+it)}y^{\frac{1}{2}-it}.$$

For $j = 1,2,3$,

$$S_j(z,t) = 4\pi^{\frac{1}{2}+it}\Gamma(\tfrac{1}{2}+it)^{-1}\sqrt{y} \sum_{N_{j-1} \leqq n < N_j} n^{it}\sigma_{-2it}(n)K_{it}(2\pi ny)\cos(2\pi nx),$$

and

$$S_\infty(z,t) = 4\pi^{\frac{1}{2}+it}\Gamma(\tfrac{1}{2}+it)^{-1}\sqrt{y} \sum_{n=N_3}^{\infty} n^{it}\sigma_{-2it}(n)K_{it}(2\pi ny)\cos(2\pi nx).$$

Applying the estimations Proposition 1, Proposition 2 and

$$\zeta(1+it)^{-1} = O((\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}}) \qquad (t \geqq 2)$$

to $S_j(z,t)$, we obtain the the proof of Theorem 1.

References

[1] F. V. Atkinson, *The mean-value of the Riemann zeta-function*, Acta Math. 81, 1949, 353-376.

[2] C. B. Balogh, *Uniform asymptotic expansions of the modified Bessel function of the third kind of large imaginary order*, Bull.Amer. Math. Soc., 72, 1966, 40-43.

[3] C. B. Balogh, *Asymptotic expansions of the modified Bessel function of the third kind of imaginary order*, SIAM-J.-Appl.-Math. 15, 1967, 1315-1323.

[4] A. Erdélyi, et al. *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, 1953.

[5] A. Ivić, *The Riemann zeta-function*, A Wiley-Intersince Publication, 1985.

[6] H. Iwaniec, *Spectral Methods of Automorphic Forms*, 2nd ed., Graduate Studies in Mathematics vol. 53, A.M.S., Revista Matemática Iberoamericana, 1995.

[7] M. Jutila, *Transformation formulae for Dirichlet polynomials*, J. Number Theory, 18, 1984, 135-156.

[8] M. Katsurada, *Complete asymptotic expansions associated with Epstein zeta-functions*, Ramanujan J. (2007), 14, 249-275

[9] K. Matsumoto, *Asymptotic expansions of double zeta-functions of Barnes, of Shintani, and Eisenstein series*, Nagoya Math. J. 172, 2003, 59-102

[10] F. W. J. Olver, *The asymptotic solutions of linear differential equations of the second order for large values of a parameter*, Philos. Trans. Roy. Soc. London, Ser. A, 247, 1954, 307-327.

[11] F. W. J. Olver, *The asymptotic expansions of Bessel functions of large order*, Ibid., 247, 1954, 328-368.

[12] F. W. J. Olver, *Error bounds for first approximations in turning-point problems*, SIAM-J.-Appl.-Math. 11, 1963, 748-772.

[13] F. W. J. Olver, *Error bounds for asymptotic expansions in turning-point problems*, Ibid., 12, 1964, 200-214.

[14] F. W. J. Olver, *Asymptotics and special functions*, A K Peters, 1997. (Academic Press, 1974.)

[15] P. Shiu, *A Brun-Titchmarsh theorem for multiplicative functions*, J. Reine Angew. Math., 313, 1980, 161-170.

[16] A. Terras, *Harmonic analysis on symmetric spaces and applications I*, Springer-Verlag 1985.

[17] E. C. Titchmarsh and D. R. Heath-Brown, *The Theory of the Riemann zeta-function (2nd edition)*, Oxford science publications, 1986.

[18] G. F. Voronoi, *Sur une fonction transcendante et ses applications à la sommation de quelques séries*, Ann. École Normale, 21, (3), 1904, 207-268, 459-534.