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Kyoto University
LIMIT PERIOD FORMULA FOR SPECIAL CYCLES ON REAL HYPERBOLIC SPACES

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1. PRELIMINARY

1.1. Let $G$ be a connected semisimple Lie group with finite center of non-compact type. We fix a Haar measure $dg$ of $G$. Given a uniform lattice $\Gamma \subset G$ i.e., discrete subgroup such that $\Gamma \backslash G$ is compact, let $L^2(\Gamma \backslash G)$ be the Hilbert space of all the measurable functions $\phi: G \to \mathbb{C}$ such that $\phi(\gamma g) = \phi(g)$ for any $\gamma \in \Gamma$ with the finite $L^2$-norm

$$\int_{\Gamma \backslash G} |\phi(g)|^2 \, dg < +\infty.$$ 

Then, the right regular action of $G$ on $L^2(\Gamma \backslash G)$ yields a unitary representation of $(R_{\Gamma}, L^2(\Gamma \backslash G))$, which, by a fundamental theorem of Gelfand, Graev and Piatetsuki-Shapiro, is discretely decomposable to irreducible unitary representations of $G$ with finite multiplicities:

there exists a function $\hat{G} \ni \pi \mapsto m_{\Gamma}(\pi) \in \mathbb{N}$ s.t.

$$L^2(\Gamma \backslash G) = \bigoplus_{\pi \in \hat{G}} L^2(\Gamma \backslash G)_\pi,$$

$$L^2(\Gamma \backslash G)_\pi \cong \pi^{\oplus m_{\Gamma}(\pi)}$$

($\pi$-isotypic part)

Let $K$ be a maximal compact subgroup of $G$ and $(\tau, F_{\tau})$ an irreducible unitary representation of $K$. Then, the space of $F_{\tau}$-valued $\pi$-automorphic forms on $\Gamma$ defined by

$$L^2_{\tau}(\Gamma \backslash G)_{\pi} \overset{def}{=} \text{Hom}_K(F_{\tau}^\vee, L^2(\Gamma \backslash G)_\pi)$$

$$\cong \{L^2(\Gamma \backslash G)_\pi \otimes_{\mathbb{C}} F_{\tau}\}^K$$

becomes a Hilbert space in a natural way; it is of finite dimension

$$\dim_{\mathbb{C}} L^2_{\tau}(\Gamma \backslash G)_{\pi} = m_{\Gamma}(\pi) \text{ mult}_K(\tau^\vee, \pi).$$

1.2. Let $H$ be a connected symmetric subgroup of $G$. Thus, there exists an involutive automorphism $\sigma$ of $G$ such that $H = (G^\sigma)^c$. We assume that $\sigma$ is taken so that $\sigma(K) = K$. Then, $K_H = K \cap H$ is a maximal compact subgroup of $H$. Let $(\tau, F_{\tau})$ be an irreducible unitary representation of $K$. Since $K_H$ is a symmetric subgroup of $K$, the trivial representation of $K_H$ occurs in $\tau|K_H$ at most once, i.e., $\dim F_{\tau}^{K_H} \leq 1$.

Let $\mathfrak{L}_G^H$ be the set of uniform lattices $\Gamma \subset G$ such that $\sigma(\Gamma) = \Gamma$. For each $\Gamma \in \mathfrak{L}_G^H$, the intersection $\Gamma_H = \Gamma \cap H$ is a uniform lattice of $H$.

Fix a Haar measure $dh$ of $H$. Given $\Gamma \in \mathfrak{L}_G^H$, $\pi \in \hat{G}$ and $\tau \in \hat{K}$, consider the map

$$L^2_{\tau}(\Gamma \backslash G)_\pi \ni \phi \mapsto \phi^H \overset{def}{=} \int_{\Gamma_H \backslash H} \phi(h) \, dh \in F_{\tau}^{K_H}.$$
and set
\[ P_\tau(\Gamma)_\pi \overset{\text{def}}{=} \sum_{\phi \in B_\tau(\Gamma)_\pi} \|\phi^H\|^2, \]
where \( B_\tau(\Gamma)_\pi \) is an orthonormal basis of \( L^2_\tau(\Gamma \backslash G)_\pi \). It is easy to see that \( P_\tau(\Gamma)_\pi \) is independent of the choice of \( B_\tau(\Gamma)_\pi \).

1.3. In this note, we are interested in the asymptotic behavior of \( P_\tau(\Gamma)_\pi \) (with fixed \( \pi \) and \( \tau \)) when \( \Gamma \to \{e\} \). To make the meaning of \( \Gamma \to \{e\} \) more exact, we introduce the notion of a tower of lattices. A sequence \( \{\Gamma_n\} \) is called a tower if

1. \( \Gamma_n \) is uniform lattice in \( G \)
2. \( \Gamma_{n+1} \subset \Gamma_n \), \( [\Gamma_n : \Gamma_{n+1}] < +\infty \)
3. \( \Gamma_n \) is normal in \( \Gamma_0 \)
4. \( \bigcap \Gamma_n = \{e\} \)

A tower \( \{\Gamma_n\} \) in \( G \) is said to be \( H \)-admissible if \( \Gamma_n \in \mathcal{L}_G^H \) for all \( n \). Then, for a given tower of \( H \)-admissible uniform lattices in \( G \), we have some speculation on the limiting behaviour of \( P_\tau(\Gamma_n)_\pi \) as \( n \to \infty \); we report a partial result obtained for a particular symmetric pair \( (G, H) \).

2. Speculations

2.0.1. Group case. Let \( G_0 \) be a connected semisimple Lie group with finite center, and \( \{\Gamma_{0,n}\} \) a tower of uniform lattices in \( G_0 \). Let \( \hat{G}_{0,d} \) be the equivalence classes of irreducible unitary representations with square integrable matrix coefficients. Then, for any \( \pi_0 \in \hat{G}_{0,d} \), the formal degree of \( \pi_0 \) is the number \( d(\pi_0) \) such that
\[
\int_G (\pi_0(g)v_1|v_2)(\pi(g)w_1|w_2)\,dg = \frac{(v_1|w_1)(v_2|w_2)}{d(\pi_0)} \text{ for any } v_1, v_2, w_1, w_2 \in \mathcal{H}_{\pi_0}.
\]
For convenience, set \( d(\pi_0) = 0 \) for \( \pi_0 \in \hat{G}_0 - \hat{G}_{0,d} \). Then, the limit multiplicity formula proved by DeGeorge-Wallach [5] asserts
\[
\lim_{n \to \infty} \frac{m_{\Gamma_{0,n}}(\pi_0)}{\text{vol}(\Gamma_{0,n} \backslash G_0)} = d(\pi_0), \quad \pi_0 \in \hat{G},
\]
which was extended to a tower of non-uniform lattices by L.Clozel and G. Savin.

This result is reformulated in our framework as follows. Fix a maximal compact subgroup \( K_0 \subset G_0 \). Then, \( K = K_0 \times K_0 \) is a maximal compact subgroup of \( G = G_0 \times G_0 \). For \( \pi_0 \in \hat{G}_0 \) and \( \tau_0 \in \hat{K}_0 \), set \( \pi = \pi_0 \boxtimes \hat{\pi}_0 \) and \( \tau = \tau_0 \boxtimes \tau_0 \).

If \( \Gamma \subset G \) is of the form \( \Gamma_0 \times \Gamma_0 \) with \( \Gamma_0 \subset G_0 \) a uniform lattice, then
\[
L^2_\tau(\Gamma \backslash G)_\pi \cong L^2_{\tau_0}(\Gamma_0 \backslash G_0)_{\pi_0} \boxtimes L^2_{\tau_0}(\Gamma_0 \backslash G_0)_{\pi_0}.
\]
If \( H = \Delta G_0 \) is the diagonal subgroup of \( G \), then,
\[
P_\tau(\Gamma)_\pi = \frac{\text{mult}_{K_0}(\tau'_0, \pi_0)}{\dim \tau_0} m_{\Gamma_0}(\pi_0).
\]
Given a tower of uniform lattices \( \{ \Gamma_{0,n} \} \) in \( G_0 \), the direct products \( \Gamma_n = \Gamma_{0,n} \times \Gamma_{0,n} \) affords an \( H \)-admissible tower in \( G \) and the limit multiplicity formula (2.1) is equivalent to

\[
\lim_{n \to \infty} \frac{\mathbb{P}_\tau(\Gamma_n)_{\pi}}{\text{vol}(\Gamma_n \cap H \backslash H)} = \frac{\text{mult}_{K_0}(\tau_0^{\vee}, \pi_0)}{\dim \tau_0} d(\pi_0).
\]

2.1. Limit period formula.

2.1.1. Problem. The group case suggests that the main term of \( \mathbb{P}_\tau(\Gamma)_{\pi} \) as \( \Gamma \to \{e\} \) should be \( \text{vol}(\Gamma_H \backslash H) \). Now, we raise the following question:

Let \( (\pi, \mathcal{H}_\pi) \in \hat{G} \) and \( (\tau, F_\tau) \in \hat{K} \) be such that the condition (2.2) is satisfied. Let \( \{ \Gamma_n \} \) be an \( H \)-admissible tower in \( G \). Does the limit

\[
\lim_{n \to \infty} \frac{\mathbb{P}_\tau(\Gamma_n)_{\pi}}{\text{vol}(\Gamma_n \cap H \backslash H)}
\]

exists? If exists, what is the limit value? □

If the limiting value is non zero, we infer that \( \mathbb{P}_\tau(\Gamma_n)_{\pi} \) is non vanishing for sufficiently large \( n \), which in turn yields a new proof of the existence of a realization of \( \pi \) in the space \( L^2(\Gamma_n \backslash G) \).

We put a remark here. Let \( \Gamma \in \mathcal{L}_G^H, \pi \in \hat{G} \) and \( \tau \in \hat{K} \). The non-vanishing of \( \mathbb{P}_\tau(\Gamma)_{\pi} \) imposes the following restriction on the data \( (\Gamma, \pi, \tau) \).

- \( m_{\Gamma}(\pi) \neq 0 \);
- The (local) compatibility condition of \( \pi \) and \( \tau \):

\[
(2.2) \quad \exists \ell \in (\mathcal{H}_\pi^{\infty})^H, \exists \theta \in (\mathcal{H}_\pi^{\infty}[\tau])^{H \cap K} \text{ s.t. } \ell(\theta) \neq 0,
\]

in particular,

\[
F_{\tau}^{H \cap K} \neq \{0\}, \quad (\mathcal{H}_\pi^{\infty})^H \neq \{0\}
\]

Here, \( \mathcal{H}_\pi^{\infty} \) denotes the space of \( C^\infty \)-vectors of \( \pi \), \( \mathcal{H}_\pi^{\infty}[\tau] \) the \( \tau \)-isotypic part of \( \mathcal{H}_\pi^{\infty} \) and \( \mathcal{H}_\pi^{-\infty} \) the space of distribution vectors of \( \pi \).

2.1.2. Relative discrete series of \( H \backslash G \). Let \( G, H \) be as in 1.2. An irreducible unitary representation \( (\pi, \mathcal{H}_\pi) \) of \( G \) is called to be \( H \)-spherical if \( (\mathcal{H}_\pi^{-\infty})^H \neq 0 \); \( \pi \) is called to be a relative discrete series representation of \( H \backslash G \) if \( \mathcal{L}_\pi \neq 0 \). Here, \( (\mathcal{H}_\pi^{-\infty})^H \) is the space of \( H \)-invariant distribution vectors of \( \pi \), and \( \mathcal{L}_\pi \) is the space of all those \( \ell \in (\mathcal{H}_\pi^{-\infty})^H \) such that

\[
\exists v \in \mathcal{H}_\pi^{\infty} \text{ s.t. } \int_{H \backslash G} |\ell(\pi(g)v)|^2 \, dg < +\infty
\]

We denote by \( \hat{G}^H \) the set of equivalence classes of all \( H \)-spherical irreducible unitary representations of \( G \) and by \( \hat{G}_d^H \) the subset of \( \hat{G}^H \) of those classes containing a relative discrete series.
2.1.3. Formal degree. We define an analogue of formal degree as follows. Let $\pi \in \hat{G}_d^H$ and $\tau \in \hat{K}$ are such that

$\diamondsuit_1 \quad \dim L_{\pi} = 1$ (multiplicite one condition).

$\diamondsuit_2 \quad \text{mult}_K(\tau, \pi) = 1$.

$\diamondsuit_3 \quad (\exists \ell \in L_{\pi})(\exists \theta \in \mathcal{H}_{\pi}^{\infty}[\tau])^{K_H}(\ell(\theta) \neq 0)$ (cf. (2.2)).

Then, there exists $d_{\tau}^{H\backslash G}(\pi)$ such that

$$\int_{H\backslash G} \ell(\pi(g)v) \cdot \overline{\ell(\pi(g)w)} dg = \frac{d_{\tau}^{H\backslash G}(\pi)^{-1}|\ell(\theta)|^2}{\dim \tau \Vert \theta \Vert^2} \cdot (v|w)_{\pi}, \quad \forall v, w \in \mathcal{H}_{\pi}^{\infty}.$$  

Note that the number $d_{\tau}^{H\backslash G}(\pi)$ is independent of the choice of $(\ell, \theta)$.

2.1.4. Limit period formula. Now, from the experience of the group case, we pose the following.

Conjecture: Let $\pi \in \hat{G}_d^H$ and $\tau \in \hat{K}$ be such that the conditions $(\Diamond)_i$ $(i = 1, 2, 3)$ in 1.4.3 are satisfied. Let $\{\Gamma_n\}$ be an $H$-admissible tower in $G$. Then,

$$(2.3) \quad \lim_{n \to \infty} \frac{\mathbb{P}_\tau(\Gamma_n)_{\pi}}{\text{vol}(\Gamma_n \cap H\backslash H)} = d_{\tau}^{H\backslash G}(\pi).$$

For $\pi \in \hat{G} - \hat{G}_d^H$ and $\tau \in \hat{K}$, the same limit should be zero. $\square$

Note that this conjecture is compatible with the group case.

3. Results

We consider the case

$$G = \text{SO}_0(d, 1), \quad (d \geq 2),$$

$$H = \text{SO}_0(d - p, 1) \times \text{SO}(p), \quad (1 \leq p < [d/2]),$$

and report a partial result to the conjecture for some $\pi$ and for an $H$-admissible tower of congruence subgroups of $G$.

3.1. Setting. Let $F$ be an algebraic number field such that $F/\mathbb{Q}$ is totally real and $n_F = [F : \mathbb{Q}]$ is greater than 1. We enumerate all the embeddings of $F$ to $\mathbb{R}$ as $\iota_\nu : F \hookrightarrow \mathbb{R}$ $(1 \leq \nu \leq n_F)$. Let $V$ be an $F$-vector space of dimension $d + 1$ ($\geq 2$) and $Q$ a non-degenerate $F$-quadratic form on $V$. Define $G$ be the restriction of scalars from $F$ to $\mathbb{Q}$ of the orthogonal $O(Q)$ of the quadratic space $(V, Q)$. Thus, for a $\mathbb{Q}$-algebra $A$,

$$G(A) = \{g \in \text{GL}(V \otimes_{\mathbb{Q}} A)| Q \circ g = Q\}.$$  

For each $\nu$, let $V^{(\nu)} = V \otimes_{F, \iota_\nu} \mathbb{R}$ and $Q^{(\nu)}$ the $\mathbb{R}$-quadratic form on $V^{(\nu)}$ induced by $Q$. From now on, we suppose

$$\text{sgn}(Q^{(1)}) = (d+, 1-),$$

$$\text{sgn}(Q^{(\nu)}) = ((d + 1)+, 0-), \quad (2 \leq \nu \leq n_F)$$

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$$\text{sgn}(Q^{(\nu)}) = ((d + 1)+, 0-), \quad (2 \leq \nu \leq n_F)$$
Set $\tilde{G} = O(Q^{(1)})$ and $G = \tilde{G}^\circ$. Then,
\[
G(\mathbb{R}) \cong \tilde{G} \times \prod_{\nu=2}^{n_F} O(Q^{(\nu)}) \overset{pr_1}{\to} \tilde{G}
\]
$\tilde{G} \cong O(d, 1)$ (real rank one)
$O(Q^{(\nu)}) \cong O(d+1)$ (compact) ($\nu \geq 2$)

Let $U \subset V$ be an $F$-subspace such that $Q^{(\nu)}|U^{(\nu)} > 0$ for all $\nu$. We suppose $p := \dim_F(U) \in [1, [d/2] - 1]$. Set

\[H = \text{Res}_{F/Q}(\text{Stab}_{O(Q)}(U))\]
and
\[H = \text{pr}_1 H(\mathbb{R})^\circ \subset G.\]

Thus, $H$ is a connected symmetric subgroup of $G$ such that
\[G \cong SO_0(d, 1), \quad H \cong SO_0(d-p, 1) \times SO(p).\]

Let $\mathcal{L}$ be an $\mathfrak{o}_F$-lattice in $V$ such that $\mathcal{L} = (\mathcal{L} \cap U) \oplus (\mathcal{L} \cap U^\perp)$. Let $\mathfrak{a} \subset \mathfrak{o}_F$ an $\mathfrak{o}_F$-ideal. Set

\[
\tilde{\Gamma}_{\mathcal{L}}(\mathfrak{a}) = \{\gamma \in \tilde{\Gamma}_{\mathcal{L}}(\mathfrak{o}_F) | \gamma \mathfrak{v} - \mathfrak{v} \in \mathfrak{a} \mathcal{L} (\forall \mathfrak{v} \in \mathcal{L})\},
\]
\[\Gamma_{\mathcal{L}}(\mathfrak{a}) = \text{pr}_1(\tilde{\Gamma}_{\mathcal{L}}(\mathfrak{a})) \cap G.\]

Then, $\Gamma_{\mathcal{L}}(\mathfrak{a})$ is a uniform lattice of $G$ belonging to $\mathcal{L}_G^H$. If $\{a_n\}$ is a sequence of $\mathfrak{o}_F$-ideals such that $a_{n+1} \subset a_n$ and such that the distance from $0$ to $a_n - \{0\}$ in $F \otimes_{\mathbb{Q}} \mathbb{R}$ tends $+\infty$ with $n$. Then, $\Gamma_n = \Gamma_{\mathcal{L}}(a_n)$ is an $H$-admissible tower in $G$.

We fix a maximal compact subgroup $K \cong SO(d)$ of $G$ such that $K \cap H$ is maximally compact in $H$. The unitary dual $\hat{K}$ is parametrized by the set of dominant integral weights, which are $\delta$-tuples
\[
[l_1, l_2, \ldots, l_\delta] \in (\mathbb{Z}/2)^\delta, \quad (\delta = [d/2])
\]
such that
\[
l_1 \geq \ldots \geq l_\delta \geq 0 \quad (d : \text{odd})
\]
\[
l_1 \geq \ldots \geq l_{\delta-1} \geq |l_\delta| \quad (d : \text{even}).
\]

We remark that $(\tau_\lambda)^{H \cap K} \neq 0$ if and only if
\[
\lambda = [l_1, \ldots, l_p, 0, \ldots, 0].
\]

Let $( , )$ be the bilinear form on $V^{(1)}$ associated with $Q^{(1)}$:
\[
(v, w) = 2^{-1}\{Q^{(1)}(v + w) - Q^{(1)}(v) - Q^{(1)}(w)\}.
\]
We may suppose that $K$ is the stabilizer in $G$ of a vector $v_0 \in V^{(1)}$ such that $Q^{(1)}(v_0) = -1$, $v_0 \perp U^{(1)}$. Thus, the tangent space of $G/K$ at the origin $o = eK$ is identified with the orthogonal complement of $v_0$ in the natural way: $T_o(G/K) \cong (v_0)^\perp$. Then, the restriction $( , )|v_0^\perp$ is a positive definite bilinear form, which propagates a $G$-invariant metric on $G/K$. The associated Riemannian volume form is denoted by $d\mu_{G/K}$. Fix the Haar measure $dk$. 

\[\]
with the total volume 1. Then, we fix the Haar measure $dg$ of $G$ in such a way that the quotient $dg/dk$ coincides with $d\mu_{G/K}$. We fix a Haar measure $dh$ of $H$ by a similar construction.

3.2. The case $p = 1$ (i.e. $H \cong SO_0(d - 1, 1)$). Let $P = MAN$ be a minimal parabolic subgroup of $G = SO_0(d, 1)$. Then,

$$M \cong SO(d - 1), \quad A \cong \mathbb{R}_{>0}.$$ 

For any $s \in \mathbb{C}$, the $K$-spherical principal series $\pi_0(s)$ is defined to be the representation of $G$ (unitarily) induced from the character $1_M \otimes e^s \otimes 1_N$ of $P$:

$$\pi_0(s) = \text{Ind}_P^G(1_M \otimes e^s \otimes 1_N).$$

The following properties of $\pi_0(s)$ is known:

1. $\pi_0(s)|_{K=SO(d)} \cong \bigoplus_{l \in \mathbb{N}} \tau_{\lfloor l,0,\ldots,0 \rfloor}$, 

2. $\pi_0(s)$ is irreducible unitarizable iff 

   $$s \in \sqrt{-1}\mathbb{R} \cup (-\rho, \rho) \quad (\text{where} \rho = \frac{d-1}{2}).$$

3. $\pi_0(s)$ (Re$(s) > 0$) is reducible iff 

   $$s = \rho + k, \quad \exists k \in \mathbb{N} = \{0, 1, \ldots\}.$$ 

4. For $k \in \mathbb{N}$, $\pi_0(\rho + k)$ has a unique irreducible $(\mathfrak{g}, K)$-submodule 

   $$\delta_k \leftarrow \pi_0(\delta + k).$$

5. Set $\delta_{-1} = \pi_0(\rho - 1)$ if $d \geq 4$. Then 

   $$\hat{G}_d^H = \left\{ \begin{array}{ll}
\{ \delta_k | k \in \mathbb{N} \}, & d = 2, 3, \\
\{ \delta_k | k \in \mathbb{N} \} \cup \{ \delta_{-1} \}, & d \geq 4.
\end{array} \right.$$ 

**Theorem 1.** Let $\{a_n\}$ be a sequence of $\sigma_F$-ideals such that $a_{n+1} \subset a_n$ and such that the Euclidean distance from 0 to the lattice points $a_n - \{0\}$ in $F \otimes_{\mathbb{Q}} \mathbb{R}$ tends infinity with $n$. set $\Gamma_n = \Gamma_L(a_n)$.

1. If 

   $$\pi = \delta_k, \quad \tau = \tau_{\lfloor k+1,0,\ldots,0 \rfloor}, \quad (k \in \mathbb{N}),$$

   then

   $$\lim_{n \to \infty} \frac{\mathbb{P}_{\tau}(\Gamma_n)|_{\pi}}{\text{vol}(\Gamma_n \cap H \backslash H)} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\rho + k + 1/2)}{\Gamma(\rho + k)} = d_{\tau}^{H \backslash G}(\pi)$$

2. If $\pi \in \hat{G} - \hat{G}_d^H$, then for any $\tau \in \hat{K}$,

   $$\lim_{n \to \infty} \frac{\mathbb{P}_{\tau}(\Gamma_n)|_{\pi}}{\text{vol}(\Gamma_n \cap H \backslash H)} = 0$$
Remark:
(i) $\delta_k$ is integrable (i.e. $\rightarrow L^1(H\backslash G)$) if and only if $k \geq 1$
(ii) The first identity in (3.1) for $k = 0$ has been proved by a geometric technique ([1]).
(iii) (3.1) is true even for $\pi = \delta_{-1}$, if we assume the existence of “spectral gap” at $\delta_{-1}$
along $\pi_0(s), i.e.,$

$$\exists \epsilon > 0 (\forall n \in \mathbb{N})$$

$$[(m_{\Gamma_n}(\pi_0(s)) \neq 0, |s| < \rho - 1) \implies (|s| \leq \rho - 1 - \epsilon)]$$

This is a consequence of Arthur’s conjecture (cf. [3], [2]).

**Corollary 2.** Let $k \in \mathbb{N}$ and $\tau = \pi_{[l,0,...,0]}$. Let $\{\Gamma_n\}$ be as in Theorem 1.

1. There exists $n \in \mathbb{N}$ and $\phi : G \rightarrow F_{\tau}$ satisfying

$$\phi(\gamma g k) = \tau(k)^{-1} \phi(g), \quad \forall \gamma \in \Gamma_n, \forall k \in K$$

$$C_{\mathfrak{g}} \phi = 2k(k + \rho) \phi$$

$$(C_{\mathfrak{g}} : \text{Casimir operator})$$

$$\int_{\Gamma_n \cap H \backslash H} \phi(h) dh \neq 0.$$ 

2. $m_{\Gamma_n}(\delta_k) \neq 0$ if $n$ is large enough.

**Remark:** This is not new. Indeed, for $k > 0$, this is a special case of [10], and for $k = 0$, this may be deduced from [7].

3.3. The case $p > 1$ (i.e. $H \cong SO(d-p,1) \times SO(p)$). Let $\pi_{p-1}(s) = \text{Ind}_{P}^{G}(\xi_{p-1} \otimes e^{\partial} \otimes 1_{N})$

$(s \in \mathbb{C})$ be the non-unitary principal series with

$$\xi_{p-1} : M = SO(d-1) \rightarrow GL_{\mathbb{R}}(\wedge^{p-1}\mathbb{R}^{d-1}).$$

The following properties are known.

$\bullet_1 \pi_{p-1}(s)$ is irreducible unitarizable iff

$$s \in \sqrt{-1}\mathbb{R} \cup (-\rho_p, \rho_p) \quad \text{(where } \rho_p = \frac{d-1}{2} - p + 1).$$

$\bullet_2 \pi_{p-1}(s) (\text{Re}(s) > 0)$ is reducible iff

$$[s = \rho_p] \quad \text{or} \quad [s = \rho + k, \exists k \in \mathbb{N} = \{0,1,...\}].$$

$\bullet_3 \pi_{p-1}(\rho_p)$ contains a unique irreducible $(\mathfrak{g}, K)$-submodule $\delta^{(p)} \hookrightarrow \pi_{p-1}(\rho_p)$.

For $k \in \mathbb{N}$, $\pi_{p-1}(\rho + k)$ has a unique irreducible $(\mathfrak{g}, K)$-submodule $\delta_{k}^{(p)} \hookrightarrow \pi_{p-1}(\delta + k)$.

$\bullet_4 \{\delta^{(p)}\} \cup \{\delta_{k}^{(p)}|k \in \mathbb{N}\} \subset \hat{G}_{d}^{H}.$

We remark that $\hat{G}_{d}^{H}$ is not exhausted by $\delta^{(p)}$ and $\delta_{k}^{(p)}$.

**Theorem 3.** Let $\{a_n\}$ and $\Gamma_n = \Gamma(a_n)$ be as in Theorem 1. Suppose the existence of “spectral gap” at $\delta^{(p)}$ along $\pi_{p-1}(s), i.e.,$

$$\exists \epsilon > 0 (\forall n \in \mathbb{N})$$

$$[(m_{\Gamma_n}(\pi_{p-1}(s)) \neq 0, |s| < \rho_p) \implies (|s| \leq \rho_p - \epsilon)]$$

Then, for $\pi = \delta^{(p)}$ and $\tau : K = SO(d) \rightarrow GL(\wedge^{p}\mathbb{R}^{4})$, we have the formula:

$$\lim_{n \rightarrow \infty} \frac{P_{\tau}(\Gamma_n)\pi}{\text{vol}(\Gamma_n \cap H \backslash H)} = \frac{1}{\pi^{p/2}} \frac{\Gamma(\rho_p + 1/2)}{\Gamma(\rho_p)} = d_{\tau}^{H \backslash G}(\pi)$$
Remark : (i) Although we do not settle the case for $\delta^{(p)}_k$'s yet, we expect a similar formula.
(ii) $\delta^{(p)}$ is not integrable (on $H\backslash G$).
(iii) Theorem is true under a weaker hypothesis
\[ (\exists \epsilon > 0)(\forall n \in \mathbb{N}) \]
\[ [(\mathbb{P}_\tau(\Gamma_n)_{\pi_{p-1}(s)}) \neq 0, |s| < \rho_p) \implies (|s| \leq \rho_p - \epsilon)]. \]
(iv) The first identity of (3.2) was conjectured by Bergeron in a geometric form (explained below). His method may yield a proof of the formula under a spectral gap hypothesis for Hodge-Laplacian on $p$-forms.

3.4. Application to geometry. Let $G = \text{SO}_0(d, 1)$ and $H = \text{SO}_0(d - p, 1) \times \text{SO}(p)$ with $1 \leq p < [d/2]$. Given a torsion free lattice $\Gamma \in \mathcal{L}_G^H$, we have a $(d - p)$-dimensional cycle
\[
C_H^\Gamma = \Gamma_H \backslash H/K_H \hookrightarrow \Gamma \backslash G/K
\]
on $\Gamma \backslash G/K$. Then, the harmonic Poincare dual form $\omega_H^\Gamma$ of $C_H^\Gamma$ is defined by
\[
[C_H^\Gamma] \in H_{d-p}(\Gamma_H \backslash H/K_H; \mathbb{Z}) \overset{\iota^*}{\rightarrow} H_{d-p}(\Gamma \backslash G/K; \mathbb{Z}) \rightarrow H^{d-p}(\Gamma \backslash G/K; \mathbb{R})^\vee
\]
\[ \cong \{ \text{harmonic } p\text{-forms} \} \ni \omega_H^\Gamma, \]
where PD is the Poincaré duality map. The $L^2$-norm of $\omega_H^\Gamma$ is defined as
\[
\|\omega_H^\Gamma\|^2 = \int_{\Gamma \backslash G/K} \omega_H^\Gamma \wedge * \omega_H^\Gamma,
\]
where $*$ is the Hodge $*$-operator of $\Gamma \backslash G/K$.

Proposition 4. Let $\{\Gamma_n = \Gamma_L(a_n)\}$ be as in Theorem 1. Suppose the 'H-spectral gap hypothesis'
\[
(\exists \epsilon > 0)(\forall n \in \mathbb{N}) \]
\[ [(\mathbb{P}_\tau(\Gamma_n)_{\pi_{p-1}(s)}) \neq 0, |s| < \rho_p) \implies (|s| \leq \rho_p - \epsilon)]. \]
is true if $p > 1$. Then,
\[
\lim_{n \to \infty} \frac{\|\omega_H^{\Gamma_n}\|^2}{\text{vol}(C_H^{\Gamma_n})} = \frac{1}{\pi^{p/2}} \frac{\Gamma(\rho_p + 1/2)}{\Gamma(\rho_p)}.
\]

Remark : (1) The form $\omega_H^\Gamma$ is explicitly constructed as a residue of the analytic continuation of some Poincaré series ([7], [8]).
(2) The formula (3.3) for $p = 1$ is proved by a geometric method [1]. The unconditional validity of (3.3) for $p > 1$ is also conjectured by [1].
4. A FEW WORDS ON PROOFS

Following [11] (where the case $G = U(p, q), H = U(p - 1, q) \times U(1)$ is discussed), we prove Theorem 2 by showing the two inequalities:

\[
\lim_{n \to \infty} \sup \frac{\mathbb{P}_\tau(\Gamma_n)_{\pi}}{\text{vol}(\Gamma_n \cap H\backslash H)} \leq \frac{1}{\pi^{p/2}} \frac{\Gamma(\rho_p + 1/2)}{\Gamma(\rho_p)}.
\]

To prove this, we follow the argument used by [9] in the proof of the limit multiplicity formula.

\[
\lim_{n \to \infty} \inf \frac{\mathbb{P}_\tau(\Gamma_n)_{\pi}}{\text{vol}(\Gamma_n \cap H\backslash H)} \geq \frac{1}{\pi^{p/2}} \frac{\Gamma(\rho_p + 1/2)}{\Gamma(\rho_p)}.
\]

This part is accomplished by a form of relative trace formula.

5. REMARKS

- Similarly, we can treat the cases:
  - $G = U(p, q), H = U(p - 1, q) \times U(1)$
  - $G = SO_0(p, q), H = SO_0(p - 1, q)$
  - $G = U(n, 1), H = U(n - p, 1) \times U(p) \ (1 \leq p < n)$

- We expect the same method works at least when the split rank of $H\backslash G$ is 1.

- The following (naive) question seems natural. For $S \subset \hat{G}$, set

\[
\mu^H_\tau(\Gamma; S) = \sum_{\pi \in S} \mathbb{P}_\tau(\Gamma)_{\pi}.
\]

Does the measure

\[
S \mapsto \frac{\mu^H_\tau(\Gamma_n; S)}{\text{vol}(\Gamma_n \cap H\backslash H)},
\]

approximate the spectral measure (Plancherel measure) of the decomposition of $L^2(H\backslash G; \tau)$? By extending the argument in [11], we already have a regorous result on this observation for the case $(G, H) = (U(p, q), U(p - 1, q))$ ([12]).

REFERENCES


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