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<thead>
<tr>
<th>Title</th>
<th>Ikeda’s conjecture on the period of the Ikeda lift (Automorphic representations, automorphic $L$-functions and arithmetic)</th>
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</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>KATSURADA, Hidenori; KAWAMURA, Hisa-aki</td>
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Ikeda's conjecture on the period of the Ikeda lift

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Abstract

As an affirmative answer to the Duke-Imamoğlu conjecture, Ikeda constructed a certain lifting of classical cusp forms on the special linear group $SL_2$ towards Siegel cusp forms, namely cuspidal automorphic forms on the symplectic group $Sp_{2n}$ of general even genus $2n$. Afterwards he also proposed a certain conjecture concerning the periods (Petersson norms squared) of such forms. In this paper, we would like to explain a brief sketch of a proof of the conjecture. Details will appear elsewhere.

1 Introduction

For each positive integer $n \in \mathbb{Z}$, the symplectic modular group $Sp_{2n}(\mathbb{Z})$ of genus $2n$ is defined to be

$$Sp_{2n}(\mathbb{Z}) = \{ \gamma \in GL_{2n}(\mathbb{Z}) \mid {}^t\gamma J \gamma = J, J = \left( \begin{smallmatrix} 0_n & 1_n \\ -1_n & 0_n \end{smallmatrix} \right) \} .$$

For either an integer or a half-integer $\kappa \in \frac{1}{2}\mathbb{Z}$, we denote the complex vector space consisting of all Siegel cusp forms of weight $\kappa$ with respect to a suitable congruence subgroup $\Gamma$ of $Sp_{2n}(\mathbb{Z})$ by $S_\kappa(\Gamma)$. Then for each $F, G \in S_\kappa(\Gamma)$, we define the Petersson scalar product $\langle F, G \rangle$ by

$$\langle F, G \rangle := [Sp_{2n}(\mathbb{Z}) : \Gamma \cdot \{ \pm 1_{2n} \}]^{-1} \int_{\Gamma \backslash \mathfrak{H}_{2n}} F(Z) \overline{G(Z)} \det(\text{Im}(Z))^{\kappa} dZ^* ,$$

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where $Z = X + \sqrt{-1}Y \in \mathcal{H}_n = \{ Z \in \text{Mat}_{n \times n}(\mathbb{C}) \mid \mathcal{T}Z = Z, \text{Im}(Z) > 0 \}$ and $dZ^* = \det Y^{-(n+1)}dXdY$ is a finite volume scalar product on $\text{Sp}_{2n}(\mathbb{Z}) \backslash \mathcal{H}_n$. As is well-known, this defines a Hermitian scalar product on the space $S_k(\Gamma)$ and hence we can introduce the norm $\| F \|^2 := \langle F, F \rangle$ for each $F \in S_k(\Gamma)$. We note that if $F$ is a Hecke eigenform, that is, a common eigenfunction of all Hecke operators, then the Petersson norm squared $\| F \|^2$ plays an important role within the framework of studying critical values of the standard $L$-function $L(s, F, \text{st})$ attached to $F$ (cf. [1]).

On the other hand, for a couple of positive even integers $n$ and $k$ such that $k > n + 1$, let $f \in S_{2k-n}(\text{Sp}_2(\mathbb{Z})) = S_{2k-n}(\text{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Then we can consider the lift of $f$ towards the space $S_k(\text{Sp}_{2n}(\mathbb{Z}))$ as follows. Namely, Ikeda ([9]) showed that there exists a Hecke eigenform $F_f \in S_k(\text{Sp}_n(\mathbb{Z}))$ such that

$$L(s, F_f, \text{st}) = \zeta(s) \prod_{i=1}^{n} L(s + k - i, f),$$

where $\zeta(s)$ and $L(s, f)$ are the Riemann zeta function and the Hecke $L$-function associated with $f$, respectively. We note that the above lifting coincides with the Saito-Kurokawa lifting in case $n = 2$, and the existence of the lifting was firstly conjectured by Duke and Imamoglu in case $n > 2$ (cf. [2]). More precisely, Ikeda explicitly constructed $F_f$ by Fourier expansions of $f$ and a Hecke eigenform $g \in S_{k-n/2+1/2}(\Gamma_0^{(2)}(4))$ corresponding to $f$ under the Shimura correspondence, where $\Gamma_0^{(2)}(4) = \{ \gamma \in \text{SL}_2(\mathbb{Z}) \mid \gamma \equiv (\begin{smallmatrix} \ast & \ast \\ 0 & \ast \end{smallmatrix}) \pmod{4} \}$. In this paper, we simply call $F_f$ the Ikeda lift of $f$.

As will be explained precisely in the subsequent part, Ikeda also conjectured in [10] that the ratio $\| F_f \|^2/\| g \|^2$ should be expressed in terms of special values of certain $L$-functions attached to $f$. The purpose of this paper is to explain a proof of the conjecture. We note that $F_f$ could not necessarily be realized as a theta lift except for the case $n = 2$. Thus we cannot use a general method for evaluating Petersson scalar products of theta lifts due to Rallis (cf. [24]). The method we use is to give explicit formulae for several kinds of Dirichlet series of Rankin-Selberg type attached to Siegel modular forms and then to compare their residues.

We note that we can consider an application of the main result to a problem concerning congruences between Ikeda lifts and some genuine Siegel modular forms. This has been announced in [13, 16], and the details will be discussed in [14].
2 Main results

Throughout this section, we fix a pair of positive even integers \( n, k \in \mathbb{Z} \) such that \( k > n + 1 \).

2.1 Construction of the Ikeda lift

Let \( \text{Sym}^*_n(\mathbb{Z})_+ \) be the set of all positive definite half-integral symmetric matrices of size \( n \). For each \( B \in \text{Sym}^*_n(\mathbb{Z})_+ \) and a rational prime \( p \), we put

\[
 b_p(B; s) := \sum_{R \in \text{Sym}_n(\mathbb{Z}[p^{-1}])/\text{Sym}_n(\mathbb{Z})} e(\text{tr}(BR)) p^{-s\mu_p(R)},
\]

where \( e(x) = \exp(2\pi \sqrt{-1}x) \) for \( x \in \mathbb{C} \), and \( \mu_p(R) = [\mathbb{Z}_p^n R + \mathbb{Z}_p^n : \mathbb{Z}_p^n] \).

As is known by Kitaoka ([18]), we have that there exists a unique polynomial \( F_p(B; X) \in \mathbb{Z}[X] \) such that

\[
 b_p(B; s) = F_p(B; p^{-s}) \times \frac{(1-p^{-s}) \prod_{i=1}^{n/2}(1-p^{2i-2s})}{1-\chi_B(p)p^{n/2-s}},
\]

where \( \chi_B : \mathbb{Z} \to \{ \pm 1, 0 \} \) denotes the Kronecker character corresponding to the quadratic field extension \( \mathbb{Q}(\sqrt{\mathfrak{D}_B})/\mathbb{Q} \) with \( \mathfrak{D}_B := (-1)^{n/2} \det(2B) \). In addition, we can write \( \mathfrak{D}_B = \mathfrak{o}_B \mathfrak{f}_B^2 \) in terms of a fundamental discriminant \( \mathfrak{o}_B \), that is, the discriminant of \( \mathbb{Q}(\sqrt{\mathfrak{D}_B})/\mathbb{Q} \) and \( \mathfrak{f}_B = \sqrt{\mathfrak{D}_B/\mathfrak{o}_B} \in \mathbb{Z} \). Then it is also known that the Laurent polynomial \( \tilde{F}_p(B; X) := X^{-\text{ord}_p(\mathfrak{f}_B)}F_p(B; p^{-(n+1)/2}X) \) is invariant under \( X \mapsto X^{-1} \) (cf. [12]).

On the other hand, let

\[
 f(\tau) = \sum_{m \geq 1} a_f(m)e(m\tau) \in S_{2k-n}(\text{SL}_2(\mathbb{Z})) \quad (\tau \in \mathfrak{H}_1)
\]

be a Hecke eigenform normalized as \( a_f(1) = 1 \). Then we can associate \( f \) with a Hecke eigenform

\[
 g(\tau) = \sum_{m \geq 1, (-1)^{k-n/2}m \equiv 0,1 \pmod{4}} c_g(m)e(m\tau) \quad (\tau \in \mathfrak{H}_1)
\]

in Kohnen's plus space \( S_{k-n/2+1/2}^+(\Gamma_0^2(4)) \) of half-integral weight \( k - n/2 + 1/2 \), that is, a subspace of \( S_{k-n/2+1/2}^+(\Gamma_0^2(4)) \) characterized by the Shimura's Hecke-equivariant isomorphism

\[
 S_{k-(n-1)/2}^+(\Gamma_0^2(4)) \xrightarrow{\cong} S_{2k-n}(\text{SL}_2(\mathbb{Z}))
\]
Then Ikeda’s lifting theorem is stated as follows:

**Theorem I** (cf. [9]). For each $B \in \text{Sym}^n_+(\mathbb{Z})$, we put

$$C_{F_f}(B) := c_\varphi(|d_B|) \left| \frac{k-n/2-1/2}{2} \prod_{p \mid d_B} \tilde{F}_p(B; \alpha_p) \right|,$$

where $\alpha_p + \alpha_p^{-1} = p^{-k+n/2+1/2}a_f(p)$. Then

$$F_f(Z) = \sum_{B \in \text{Sym}^n_+(\mathbb{Z})} C_{F_f}(B) e(\text{tr}(BZ)) \quad (Z \in \mathfrak{H}_n)$$

belongs to the space $S_k(\text{Sp}_{2n}(\mathbb{Z}))$, and forms a Hecke eigenform such that

$$L(s, F_f, \text{st}) = \zeta(s) \prod_{i=1}^{n} L(s+k-i, f).$$

We do not consider Eisenstein series here. However, one can formally look at the Ikeda lift as an analogy to the association between Siegel Eisenstein series $E_k^{(2n)}$ of weight $k$ with respect to $\text{Sp}_{2n}(\mathbb{Z})$ and Eisenstein series $E_{2k-n}^{(2)}$ of weight $2k-n$ with respect to $\text{SL}_2(\mathbb{Z})$. Namely, we have

$$L(s, E_k^{(2n)}, \text{st}) = \zeta(s) \prod_{i=1}^{n} L(s+k-i, E_{2k-n}^{(2)}).$$

### 2.2 Ikeda’s conjecture and the main theorem

In order to state Ikeda’s conjecture precisely, we introduce some notations of $L$-functions as follows. For a given normalized Hecke eigenform $f \in S_{2k-n}(\text{SL}_2(\mathbb{Z}))$ as in the previous section, we put

$$\begin{cases} 
\tilde{\xi}(s) := \Gamma_C(s) \zeta(s), \\
\tilde{\Lambda}(s, f) := \Gamma_C(s)L(s, f), \\
\tilde{\Lambda}(s, f, \text{ad}) := \Gamma_C(s)\Gamma_C(s+2k-n-1)L(s, f, \text{ad}),
\end{cases}$$

where $\Gamma_C(s) := 2(2\pi)^{-s}\Gamma(s)$ and $L(s, f, \text{ad})$ denotes the adjoint $L$-function of $f$ defined by

$$L(s, f, \text{ad}) = \prod_p \{(1-p^{-s})(1-\alpha_p^2p^{-s})(1-\alpha_p^{-2}p^{-s})\}^{-1}.$$
1, f, Ad)/||f||^2 is an algebraic number for each 1 ≤ i < k − n/2. In particular, we have \( \tilde{\Lambda}(1, f, \text{Ad}) = 2^{2k-n}||f||^2 \) (cf. [26]). Then Ikeda proposed the following:

**Conjecture I** (cf. [10]). Under the same situation as in Theorem I, there exists \( \alpha(n, k) \in \mathbb{Z} \) such that

\[
\frac{||F_f||^2}{||g||^2} = 2^{\alpha(n,k)} \Lambda(k, f) \tilde{\xi}(n) \prod_{i=1}^{n/2-1} \tilde{\xi}(2i) \tilde{\Lambda}(2i + 1, f, \text{ad}).
\]

When \( n = 2 \), it has been already known by Kohnen and Skoruppa that the above conjecture holds true (cf. [21], see also [23]). Then the main theorem in this paper is stated as follows.

**Theorem 2.1.** Conjecture I holds true for any positive even \( n \).

In the subsequent sections, we will explain a proof of Theorem 2.1 by using a three step-wise approach.

## 3 Rankin-Selberg method for the Fourier-Jacobi expansion of the Ikeda lift

For the moment, let us review the theory of Fourier-Jacobi expansions of Siegel modular forms of genus \( 2n \geq 4 \) and its application towards the evaluation of Petersson norm squared.

For each positive \( k \in \mathbb{Z} \), let \( F \in S_k(\text{Sp}_{2n}(\mathbb{Z})) \) possess the Fourier expansion

\[
F(Z) = \sum_{B \in \text{Sym}_n^+(\mathbb{Z})} C_F(B) e(\text{tr}(BZ)) \quad (Z \in \mathfrak{H}_n).
\]

Then by decomposing each point \( Z \in \mathfrak{H}_n \) into the form

\[
\left( \begin{array}{ll} \tau' & z \\ t_z & \tau \end{array} \right) \quad ((\tau, z) \in \mathfrak{H}_{n-1} \times \mathbb{C}^{n-1}, \tau' \in \mathfrak{H}_1),
\]

we obtain the Fourier-Jacobi expansion

\[
F\left( \left( \begin{array}{ll} \tau' & z \\ t_z & \tau \end{array} \right) \right) = \sum_{m=1}^{\infty} \phi_m(\tau, z)e(m\tau'),
\]
where

\[ \phi_m(\tau, z) := \sum_{(T, r) \in \text{Sym}_{n-1}(\mathbb{Z}) \times \mathbb{Z}^{n-1}} C_F \left( \begin{array}{c} m \frac{r/2}{T} \\ t \frac{r/2}{T} \end{array} \right) \text{e}(\text{tr}(T\tau + t^T z)) \].

We note that for each \( m \), the function \( \phi_m \) belongs to the complex vector space \( J^{cusp}_{k,m}(\text{Sp}_{2n-2}(\mathbb{Z})^J) \) consisting of all holomorphic Jacobi cusp forms of weight \( k \) and index \( m \) with respect to the Jacobi modular group \( \text{Sp}_{2n-2}(\mathbb{Z})^J := \text{Sp}_{2n-2}(\mathbb{Z}) \times (\mathbb{Z}^{2n-2} \times \mathbb{Z}) \) of genus \( 2n - 2 \) (cf. [28]). Then we define the Dirichlet series \( D(s, F) \) attached to \( F \) by

\[ D(s, F) := \zeta(2s-2k+2n) \sum_{m=1}^{\infty} \|\phi_m\|^2 m^{-s}, \]

where \( \|\phi_m\|^2 \) denotes the Petersson norm squared of \( \phi_m \in J^{cusp}_{k,m}(\text{Sp}_{2n-2}(\mathbb{Z})^J) \) introduced to be

\[ \|\phi_m\|^2 := \int_{\text{Sp}_{2n-2}(\mathbb{Z})^J \setminus G \times \mathbb{H}^{n-1}} |\phi_m(\tau, z)|^2 \det(\text{Im}(\tau))^k \times \exp(-4\pi m \text{Im}(z) \text{Im}(\tau)^t \text{Im}(z)) d\tau^* dz. \]

We easily see that the Dirichlet series \( D(s, F) \) converges absolutely for Re\( (s) > k \). Moreover, Yamazaki showed the following:

**Theorem II** (cf. [27], see also [22]). The function

\[ D^*(s, F) := \pi^{k-n-1}(2\pi)^{1-2s} \Gamma(s) D(s, F) \]

has a meromorphic continuation to the whole \( s \)-plane, and has simple poles at \( s = k, k-n \) with the residue \( \|F\|^2 \). Furthermore, it satisfies the functional equation

\[ D^*(2k-n-s, F) = D^*(s, F). \]

Then, as the first main ingredient of the proof of Theorem 2.1, we have the following:

**Theorem 3.1** (cf. [15]). Let \( n, k \) be as in \( \S 2 \). If \( f \in S_{2k-n}(\text{SL}_2(\mathbb{Z})) \) is a normalized Hecke eigenform, then

\[ D(s, F_f) = \|\phi_{f,1}\|^2 \zeta(s-k+1)\zeta(s-k+n)L(s, f), \]

where \( \phi_{f,1} \) denotes the first coefficient of the Fourier-Jacobi expansion of \( F_f \).
Moreover, by comparing residues at $s = k$ on both sides, we also obtain

**Corollary 3.1.** Under the same situation as above, we have

$$\frac{||F_{f}||^{2}}{||\phi_{f,1}||^{2}} = 2^{-k+n-1}\Lambda(k, f)\tilde{\xi}(n).$$

(1)

When $n = 2$, the above two results have been obtained by Kohnen and Skoruppa ([21]).

4 The Eichler-Zagier-Ibukiyama isomorphism

Based on the result in the previous section, let us review in this section that there exists a natural correspondence between holomorphic Jacobi forms of integral weight and index 1 and Siegel modular forms of half-integral weight, and explain the coincidence of Petersson norms squared up to scalar.

We put $\Gamma_{0}^{(2n-2)}(4) := \{ \gamma \in \text{Sp}_{2n-2}(\mathbb{Z}) \mid \gamma \equiv (0_{n-1}^{*}) \pmod{4} \}$. Then for each $k \in \mathbb{Z}$, we introduce the generalized Kohnen’s plus space by

$$S_{k-1/2}^{+}(\Gamma_{0}^{(2n-2)}(4))
:= \left\{ F(Z) \in S_{k-1/2}(\Gamma_{0}^{(2n-2)}(4)) \mid C_{F}(A) = 0 \text{ unless } A \equiv (-1)^{k+1}\mathbf{rr} \pmod{4\text{Sym}_{n-1}^{*}(\mathbb{Z})} \text{ for some } \mathbf{r} \in \mathbb{Z}^{n-1} \right\}.
$$

As is mentioned before, for each positive even $k \in \mathbb{Z}$, we have

$$S_{k-1/2}(\Gamma_{0}^{(2)}(4)) \overset{\sim}{\rightarrow} S_{2k-2}(\text{SL}_{2}(\mathbb{Z})).$$

Moreover, Eichler and Zagier ([3]) showed that there exists an isomorphism

$$J_{k,1}^{\text{cusp}}(\text{SL}_{2}(\mathbb{Z})^{J}) \overset{\sim}{\rightarrow} S_{k-1/2}^{+}(\Gamma_{0}^{(1)}(4)),$$

which is compatible with actions of all Hecke operators up to $p = 2$. As a generalization of the isomorphism, Ibukiyama showed the following:

**Theorem III** (cf. [4]). If $n \geq 2$, then for each positive even $k \in \mathbb{Z}$, there exists an isomorphism

$$\sigma: J_{k,1}^{\text{cusp}}(\text{Sp}_{2n-2}(\mathbb{Z})^{J}) \overset{\sim}{\rightarrow} S_{k-1/2}(\Gamma_{0}^{(2n-2)}(4)),$$

which is compatible with actions of Hecke operators up to $p = 2$. 
In addition, Eichler and Zagier ([3]) also showed that the isomorphism $\sigma$ is compatible with Petersson norms squared. As its generalization to higher genus, we obtain the following:

**Theorem 4.1.** Under the same assumption as in Theorem III, for each $\phi \in J_{k,1}^{\text{cusp}}(\text{Sp}_{2n-2}(\mathbb{Z})^J)$, we have

$$\|\phi\|^2 = 2^{2(k-1)(n-1)-1}\|\sigma(\phi)\|^2. \quad (2)$$

**Proof.** The proof proceeds in a similar way to that of Theorem 5.4 in [3]. $\square$

Thus by combining Corollary 3.1 and Theorem 4.1, we can show Theorem 2.1 in case $n = 2$. Indeed, for a given normalized Hecke eigenform $f \in S_{2k-2}(\text{SL}_2(\mathbb{Z}))$, we denote by $g \in S_{k-1/2}^+(\Gamma_0^{(2)}(4))$ and $\phi_{f,1} \in J_{k,1}^{\text{cusp}}(\text{SL}_2(\mathbb{Z})^J)$ a Hecke eigenform corresponding to $f$ under Shimura's isomorphism and the first coefficient of the Fourier-Jacobi expansion of the Saito-Kurokawa lift $F_f \in S_k(\text{Sp}_{2n}(\mathbb{Z}))$ of $f$, respectively. Then we have $\sigma(\phi_{f,1}) = g$, and hence by combining the equations (1) and (2), we obtain

$$\frac{\|F_f\|^2}{\|g\|^2} = \frac{\|F_f\|^2}{\|\phi_{f,1}\|^2} \cdot \frac{\|\phi_{f,1}\|^2}{\|g\|^2} = 2^{k-2}\Lambda(k, f)\tilde{\xi}(2),$$

and this proves the assertion. $\square$

### 5 Rankin-Selberg method for Siegel modular forms of half-integral weight

In this section, we derive an explicit formulae for certain Dirichlet series attached to Siegel modular forms of half-integral weight and apply it to evaluate Petersson norms squared of such forms.

For each positive even $k \in \mathbb{Z}$, we consider

$$F(Z) = \sum_{A \in \text{Sym}_{n-1}^+(\mathbb{Z})} C_F(A)e(\text{tr}(AZ)) \in S_{k-1/2}(\Gamma_0^{(2n-2)}(4)).$$

Then we define the Dirichlet series $R(s, F)$ attached to $F$ by

$$R(s, F) := \sum_{A \in \text{Sym}_{n-1}^+(\mathbb{Z})+/\text{SL}_{n-1}(\mathbb{Z})} \frac{|C_F(A)|^2}{e(A)\det A^s},$$
where \( e(A) = \# \{ X \in \text{SL}_{n-1}(\mathbb{Z}) \mid {}^t X A X = A \} \). This kind of Dirichlet series has been studied by Shimura ([25]) and Kalinin ([11]) in case of integral weight. Then by using a similar method, we easily see the following:

**Proposition 5.1.** We put \( \Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2), \xi(s) = \Gamma_{\mathbb{R}}(s) \zeta(s) \) and

\[
R^*(s, F) := \gamma_{n-1}(s) \xi(2s - 2k + n + 1) \prod_{i=1}^{n/2-1} \xi(4s - 4k - 2i + 2n + 2) R(s, F),
\]

where \( \gamma_{n-1}(s) = 2^{1-2s(n-1)} \prod_{j=1}^{n-1} \Gamma_{\mathbb{R}}(2s-j+1) \). Then the function \( R^*(s, F) \) has a meromorphic continuation to the whole \( s \)-plane and has a simple pole at \( s = k - 1/2 \) with the residue \( \prod_{i=1}^{n/2-1} \xi(2i+1) \| F \|^2 \).

Then we have an explicit formula for the Dirichlet series \( R(s, \sigma(\phi_{f,1})) \) as follows:

**Theorem 5.2** (cf. [17]). Under the same situation as in Theorem 3.1, we put \( \lambda_n = \frac{1}{2} \prod_{i=1}^{n/2-1} \tilde{\xi}(2i) \). Then we have

\[
R(s, \sigma(\phi_{f,1})) = \frac{\lambda_n}{2^{(n-1)(s+1/2)}} \zeta(2s+n-2k+1)^{-1} \prod_{i=1}^{n/2-1} \zeta(4s+2n-4k+2-2i)^{-1}
\]
\[
\times \{ R(s-n/2+1, g) \zeta(2s-2k+3) \prod_{j=1}^{n/2-1} L(2s-2k+2j+2, f, \text{ad}) \zeta(2s-2k+2j+2)
\]
\[
+ (-1)^{n(n-2)/8} R(s, g) \zeta(2s-2k+n+1) \prod_{j=1}^{n/2-1} L(2s-2k+2j+1, f, \text{ad}) \zeta(2s-2k+2j+1) \}.
\]

Moreover, by comparing residues at \( s = k - 1/2 \), we also obtain

**Corollary 5.2.** Under the same situation as above, we have

\[
\frac{\| \sigma(\phi_{f,1}) \|^2}{\| g \|^2} = 2^{\beta(n,k)} \prod_{i=1}^{n/2-1} \tilde{\xi}(2i) \tilde{\Lambda}(2i+1, f, \text{ad}),
\]

(3)

where \( \beta(n, k) = -3k(n-2) + n(n-3)/2 + 1. \)
Therefore, by combining the three equations (1), (2) and (3), we can show Theorem 2.1. Indeed, we have

\[
\frac{\|F_f\|^2}{\|g\|^2} = \frac{\|F_f\|^2}{\|\phi_{f,1}\|^2} \cdot \frac{\|\phi_{1}\|^2}{\|\sigma(\phi_{f,1})\|^2} \cdot \frac{\|\sigma(\phi_{f,1})\|^2}{\|\sigma(\phi_{f,1})\|^2}
\]

\[
= 2^{-k+n-1} \Lambda(k, f) \xi(n) \cdot 2^{2(k-1)(n-1)-1} \cdot 2^{\beta(n,k)} \prod_{i=1}^{n/2-1} \xi(2i) \Lambda(2i+1, f, \text{ad})
\]

\[
= 2^{-(n-3)(k-n/2)-n+1} \Lambda(k, f) \xi(n) \prod_{i=1}^{n/2-1} \xi(2i) \Lambda(2i+1, f, \text{ad}),
\]

and this proves the assertion. \(\square\)

6 Proof of Theorem 5.2

The rest of the paper is devoted to a sketch of a proof of Theorem 5.2. Details will appear in [17]. For each positive \(m \in \mathbb{Z}\), we simply write \(S_{m,p} = \text{Sym}_m^*(\mathbb{Z}_p)\) and \(S_{m,p}^x = S_{m,p} \cap \text{GL}_m(\mathbb{Q}_p)\). In particular, if \(m\) is odd, then we put

\[S_{m,p}^{(1)} := \{ A \in S_{m,p} | A + {}^trr \in 4S_{m,p} \text{ for some } r \in \mathbb{Z}_p^m \}\].

For each \(A \in S_{m,p}^{(1)}\), we put

\[\tilde{F}_p^{(1)}(A; X) := \tilde{F}_p( \left( \begin{array}{cc} 1 & r/2 \\ t_r/2 & (A + {}^trr)/4 \end{array} \right); X),\]

where \(r = r_A \in \mathbb{Z}_p^{n-1}\) such that \(A + {}^trr \in 4S_{n-1,p}\). For each \(A \in S_{m,p}^x\) and \(e \geq 0\), we put

\[A_e(A, A) = \{ X \in \text{Mat}_{n-1 \times n-1}(\mathbb{Z}_p)/p^e \text{Mat}_{n-1 \times n-1}(\mathbb{Z}_p) | {}^tXAX - A \in p^eS_{m,p} \}\]

and

\[\alpha_p(A, A) := \frac{1}{2} \lim_{e \to \infty} p^{e\{-m^2+m(m+1)/2\}} \#A_e(A, A).\]

For each \(\varnothing \in \mathbb{Z}_p\) and a \(\text{GL}_{n-1}(\mathbb{Z}_p)\)-invariant function \(\omega_p\) on \(S_{n-1,p}^x\), we put

\[H_p^{(n-1)}(\varnothing, \omega_p; X, t) := \sum_{i=0}^{\infty} \sum_{A \in A_p(\varnothing, l)} \omega_p(A) \frac{|\tilde{F}_p^{(1)}(A; X)|^2}{\alpha_p(A, A)} e^{\text{ord}_p(\det A)},\]
where $\mathcal{A}_p(0, l) = \{ A \in S_{n-1, p}^{(1)} | \det A = 0 p^{2l+(n-2)\delta_{2, p}} \}/GL_{n-1}(\mathbb{Z}_p)$. As for $\omega_p : S_{n-1, p}^{\times}/GL_{n-1}(\mathbb{Z}_p) \rightarrow \{ \pm 1, 0 \}$, we consider either the constant function $\iota_p$ on $S_{n-1, p}^{\times}$ taking the value 1 or the function $\epsilon_p$ assigning the Hasse invariant of $A$ for $A \in S_{n-1, p}^{\times}$ (cf. [19]). Then by using the same method to Ibukiyama and Saito ([8]), similarly to [5, 6], we have

**Theorem 6.1.** We have

$$
R(s, \sigma(\phi_{f1})) = \kappa_{n-1} \sum_{\mathfrak{d}} |c_g(0)|^2 |\mathfrak{d}|^{-k+n/2+1/2} \times \left\{ \prod_p H_p^{(n-1)}(\mathfrak{d}, \iota_p; \alpha_p, p^{-s+k-1/2}) + \prod_p H_p^{(n-1)}(\mathfrak{d}, \epsilon_p; \alpha_p, p^{-s+k-1/2}) \right\},
$$

where the summation is taken over all fundamental discriminant $\mathfrak{d} \in \mathbb{Z}$ such that $(-1)^{n/2} \mathfrak{d} > 0$ and we put $\kappa_{n-1} = 2^{(n-2)(n-1)/2-\delta_{n, 2}} \pi^{-n(n-1)/4} \prod_{i=1}^{n-1} \Gamma(i/2)$.

Moreover, we obtain the following explicit formulae for the power series $H_p^{(n-1)}(\mathfrak{d}, \omega_p; X, t)$:

**Theorem 6.2.** Let $\mathfrak{d} \in \mathbb{Z}$ be a fundamental discriminant and $\xi = (\frac{0}{p})$, where $(\frac{\_}{p})$ denotes the Kronecker symbol associated with $\mathfrak{d}$.

1. For $\omega_p = \iota_p$, we have

$$
H_p^{(n-1)}(\mathfrak{d}, \iota_p; X, t) = \frac{(2^{-(n-1)(n-2)/2} t^{n-2})^{\delta_{2,p}}}{\prod_{i=1}^{n/2-1} (1-p^{-2i})} (p^{-1}t)^{ord_p(\mathfrak{d})} (1-p^{-n}t^2) \prod_{i=1}^{n/2-1} (1-p^{-2n+2i}t^4) \times \frac{(1+p^{-2}t^2)(1+\xi^2 p^{-3}t^2) - 2\xi p^{-5/2} (X+X^{-1})t^2}{(1-p^{-2}X^2t^2)(1-p^{-2}X^{-2}t^2)(1-p^{-2}t^2)^2} \times \frac{1}{\prod_{i=1}^{n/2-1} (1-p^{-2i}X^2t^2)(1-p^{-2i}X^{-2}t^2)(1-p^{-2i}t^2)^2}.
$$

2. For $\omega_p = \epsilon_p$, we have

$$
H_p^{(n-1)}(\mathfrak{d}, \epsilon_p; X, t) = \frac{((-1)^{n(n-2)/8} 2^{-(n-1)(n-2)/2} t^{n-2})^{\delta_{2,p}}}{\prod_{i=1}^{n/2-1} (1-p^{-2i})} (p^{-n/2}t)^{ord_p(\mathfrak{d})} (1-p^{-n}t^2) \prod_{i=1}^{n/2-1} (1-p^{-2n+2i}t^4) \times \frac{(1+p^{-n}t^2)(1+\xi^2 p^{-n-1}t^2) - 2\xi p^{-1/2-n} (X+X^{-1})t^2}{(1-p^{-n}X^2t^2)(1-p^{-n}X^{-2}t^2)(1-p^{-n}t^2)^2} \times \frac{1}{\prod_{i=1}^{n/2-1} (1-p^{-2i}X^2t^2)(1-p^{-2i}X^{-2}t^2)(1-p^{-2i}t^2)^2}.
$$

78
where \((*, *)_p\) denotes the Hilbert symbol over \(\mathbb{Q}_p\).

On the other hand, by using the same argument as in Theorem 6.1, we obtain the following:

**Proposition 6.3.** Let \(f\) and \(g\) be a couple of Hecke eigenforms as in §2. Then we have

\[
R(s, g) = L(2s - 2k + n + 1, f, \text{ad}) \sum_{\mathcal{D}} |c_g(\mathcal{D})|^2 |\mathcal{D}|^{-s} \\
\times \prod_{p} \{(1 + p^{-2s + 2k - n - 1}) (1 + \left(\frac{\mathcal{D}}{p}\right)^2 p^{-2s + 2k - n - 2} - 2 \left(\frac{\mathcal{D}}{p}\right) a_f(p) p^{-2s + 2k - n - 3/2})\},
\]

where the summation is taken over all fundamental discriminant \(\mathcal{D} \in \mathbb{Z}\) such that \((-1)^{n/2} \mathcal{D} > 0\).

By combining Theorems 6.1, 6.2 and Proposition 6.3, we can prove Theorem 5.2. \qed

**References**


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