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Paramodular form on $GSp(2,\mathbb{A})$

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Abstract

We give constructions of automorphic forms on $GSp(2,\mathbb{A})$ which are fixed by paramodular groups.

Introduction.

Let $\mathbb{A}$ be the adele of $\mathbb{Q}$. Let $\Pi(G)$ be the set of equivalence classes of admissible representations of G (G may local or global). The θ-lift from $GO(2,2,\mathbb{A})$ to $GSp(2,\mathbb{A})$ provides Siegel modular form whose spinor L-function (of degree 4) is of the following type A) or B),

A). a product $L(s,\sigma_1)L(s,\sigma_2)$ for $\sigma_i \in \Pi(GL(2,\mathbb{A}))$. σ_1, σ_2 have a common central character.

B). $L(s,\sigma)$ for $\sigma \in \Pi(GL(2,L_{\mathbb{A}}))$. L is a real quadratic field, and σ has a central character which factor through the norm map $L^\times \to \mathbb{Q}^\times$.

We can identify $GL(2,\mathbb{Q}) \times GL(2,\mathbb{Q})$ or $GL(2,L)$ with $GO(2,2,\mathbb{Q})$, roughly. By Howe and Piatetski-Shapiro [3], it is known these θ-lift is non-vanishing and generic, in almost all cases (remark there is a non-generic case, see Theorem 5). However, nobody gave how to construct them and what congruence sugbroups fixes the θ-lifts, as far as we know. So, in this article, we shall give the method to construct them. To state our results, we racall some notation and define some groups. For an arbitrary commutative ring A, $GSp(n,A)$ is the group of $g \in GL(2n,A)$ such that for some $\lambda(g) \in A^\times$

$$g \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix} {}^t g = \lambda(g) \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix},$$

where ${}^t g$ denotes the transpose of g. In the case of $n=2$, typical unipotent elements of the maximal parabolic subgroup of $Sp(2)$ are written as

$$u = \begin{bmatrix} 1 & t & * & * \\ 0 & 1 & * & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{bmatrix}. \tag{1}$$

We fix the standard additive character ψ on $\mathbb{Q}\backslash\mathbb{A}$. For $\pi_v \in \Pi(GSp(2,\mathbb{Q}_v))$ and $c_1, c_2 \in \mathbb{Q}$, let $\mathcal{W}(\psi_{c_1,c_2},\pi_v) \subset \pi_v$ be the space of functions satisifying

$$W(ug) = \psi(c_1 t + c_2 s) W(g)$$

for $u \in U(\mathbb{Q}_v)$.

(Gamma zero type congruence group)

$$\Gamma_0^{(n)}(N) = \begin{bmatrix} M(n,\mathbb{Z}) & M(n,\mathbb{Z}) \\ NM(n,\mathbb{Z}) & M(n,\mathbb{Z}) \end{bmatrix} \cap GSp(n,\mathbb{Z}).$$

Local $\Gamma_0^{(n)}(N\mathbb{Z}_v)$ is defined, similarly (and other groups are also).

(Paramodular group (of degree 2)) Paramodular group $K(N)$ of conductor N (i.e., of the polarization

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$(1, N)$) is defined by

$$K(N) = \begin{bmatrix} \mathbf{Z} & \mathbf{Z} & N^{-1}\mathbf{Z} & \mathbf{Z} \\ N\mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ N\mathbf{Z} & N\mathbf{Z} & \mathbf{Z} & N\mathbf{Z} \\ N\mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} \end{bmatrix} \cap GSp(2, \mathbf{Q}).$$

(Semi-paramodular group) Semi-paramodular group $K(N, N')$ of level $(N,, N')$ by

$$K(N, N') = \begin{bmatrix} \mathbf{Z} & \mathbf{Z} & N^{-1}\mathbf{Z} & \mathbf{Z} \\ N\mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ NN'\mathbf{Z} & N\mathbf{Z} & \mathbf{Z} & N\mathbf{Z} \\ N\mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} \end{bmatrix} \cap GSp(2, \mathbf{Q}).$$

For $\eta_v \in \widehat{\mathbf{Q}_v^\times}$, we say a function f on $GSp(2, \mathbf{Q}_v)$ is η_v-semistable on $K(N\mathbf{Z}_v, \mathbf{N}'\mathbf{Z}_v)$, if f satisfies

$$f(zg) = \eta_v(z) f(g), \quad \text{and} \quad f(gu) = \eta_v(d_1) f(g)$$

for $g \in GSp(2, \mathbf{Q}_v)$, $z \in \mathcal{Z}(GSp(2, \mathbf{Q}_v)) \simeq \mathbf{Q}_v^\times$, and

$$u = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & d_1 & * \\ * & * & * & * \end{bmatrix} \in K(N\mathbf{Z}_v, N'\mathbf{Z}_v).$$

Then, we shall describe for the case A).

Theorem 1 — *Let $\sigma_1, \sigma_2 \in \Pi(GL(2, \mathbf{A}))$ be satisfying the follwings.*

- *$\sigma_2 \in \Pi(GL(2, \mathbf{A}))$ is cuspidal.*
- *σ_1, σ_2 have a common central unitary character η.*
- *At every v both of σ_{1v}, σ_{2v} have Whitakker models associated to ψ (we denote by $\mathcal{W}(\sigma_{iv}, \psi_v)$ the space of such Whitakker models).*

Take automorphic forms $f_1 \in \sigma_1^\vee, f_2 \in \sigma_2$ so that $W_1 = \otimes_v W_{1v} \in \mathcal{W}(\sigma_1^\vee, \psi)$ ($\sigma_1^\vee$ is the contragradient of σ_1) and $W_2 = \otimes_v W_{2v} \in \mathcal{W}(\sigma_2, \psi)$

$$W_{1p}(z_1 g) = W_{1p}(g \begin{bmatrix} * & * \\ * & z_1 \end{bmatrix}) = \eta_p^{-1}(z_1) W_{1p}(g), \quad \begin{bmatrix} * & * \\ * & z_1 \end{bmatrix} \in \Gamma_0^{(1)}(N_1 \mathbf{Z}_p)$$

$$W_{2p}(z_2 g) = W_{2p}(g \begin{bmatrix} z_2 & * \\ * & * \end{bmatrix}) = \eta_p(z_2) W_{2p}(g), \quad \begin{bmatrix} z_2 & * \\ * & * \end{bmatrix} \in \Gamma_0^{(1)}(N_2 \mathbf{Z}_p).$$

Then,
1) the θ-lift $F = \theta_2(f_1 \boxtimes f_2, \varphi)$ (defined in (5)) has a global Whittaker function $W_F^{1,-1} = \otimes_v W_{F,v}^{1,-1}$ such as $W_F^{1,-1}(1) \neq 0$. $W_{F,p}$ is η_p^{-1}-semistable on

$$K(\mathfrak{c}(\sigma_{1p})\mathfrak{c}(\sigma_{2p}), \mathfrak{c}(\eta_p))$$

for the conductors $\mathfrak{c}(\sigma_{ip}), \mathfrak{c}(\eta_p)$. At archimedean place, if W_i has weight κ_i, then the highest weight of F is

$$\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{-|\kappa_1 - \kappa_2|}{2}\right).$$

(Remark that $\kappa_1 \pm \kappa_2$ is even, since σ_1, σ_2 have the common central character.)
2) If both of W_{ip} are newforms, then

$$Z_N(s, W_{Fp}^{1,-1}) = L(s, \sigma_{1p}^\vee) L(s, \sigma_{2p}^\vee)$$

where $Z_N(s, W_{F_v}^{1,-1})$ is Novodvorsky's zeta integral.
3) F is cuspidal unless σ_1 is not cuspidal or $\sigma_1 = \sigma_2$. If σ_1 is not cuspidal, then the degenarate Whitakker

function $W_F^{1,0}$ *is not zero. If* $\sigma_1 = \sigma_2$, *then the degenarate Whitakker function* $W_F^{0,1}$ *is not zero.*
4-i) If σ_{1p}, σ_{2p} *are principal representation* $\pi(\mu_1, \eta_p\mu_1^{-1}), \pi(\mu_2, \eta_p\mu_2^{-1})$,

$$W_{Fp}^{1,-1} \in \eta_p\mu_2^{-1}\mu_1^{-1} \times \mu_2\mu_1^{-1} \rtimes \mu_1.$$

Here $\sigma_{2p} \rtimes \eta_p$ *is the Borel parabolically induced representation (see [10] for the definition.)*
4-ii) Let $\nu : \mathbb{Q}_p \ni x \to |x|_p$. *Suppose* σ_{1p} *is a principal series reperesentation* $\chi\pi(\nu^a, \eta'\nu^{-a})$, *where* χ *is a character such as* $\chi^2\eta' = \eta_p$ *,and* $\chi\pi$ *means the* χ*-twist of* π *for* $\pi \in \Pi(GL(2))$. *Then,*

$$W_{Fp}^{1,-1}, W_{Fp}^{1,0} \in \chi^{-1}\nu^{-a}\sigma_{2p} \rtimes \chi\nu^a,$$

the Siegel parabolically induced representation.
4)-iii) Suppose σ_{1p} *is a special reperesentation* $\chi\sigma(\nu^{1/2}, \nu^{-1/2})$. *Then,*

$$W_{Fp}^{1,-1}, W_{Fp}^{1,0} \in G(\nu^{1/2}\chi^{-1}\sigma_{2p}, \nu^{-1/2}\chi)$$

which is a generic constituent of $\nu^{1/2}\chi^{-1}\pi \rtimes \nu^{-1/2}\chi$ *(explained below).*
4-iv) If $\sigma_{1p} = \sigma_{2p}$,

$$W_{Fp}^{1,-1}, W_{Fp}^{0,1} \in 1_{GL(1)} \rtimes \sigma_{2p},$$

the Klingen parabolically induced representation.

Remark 2 *In the case of* $\eta_p = 1$, *if we take newforms* W_{1p}, W_{2p}, *then* $W_{F,p}$ *is the newform, i.e., the paramodular vector for* $K(N_1N_2\mathbb{Z}_p)$ *(c.f. [9]).*

Remark 3 *Except the cases 4-i),ii),iii),* $W_{Fp}^{1,-1}$ *belongs to supercuspidal representations.*

As explained later, we need the Whitakker models for the nonvanishing of global $W_F^{1,-1}$. However, in the case

$$\sigma_1 = \chi(\det) \in \Pi(GL(2, \mathbb{A}))$$

also, we have the nonvanishing of $W_{Fp}^{1,0}$. We recall the local Saito-Kurokawa representation due to Schmidt [11].

Proposition 4 (local Saito-Kurokawa representations.(Schmidt [11])) — *Let* π *be a local, irreducible, admissible, infinite-dimensional representation of* $PGL(2)$. *Let* τ *be a character of* $GL(1)$. *Assume* $\pi \neq \pi(\nu^{3/2}, \nu^{-3/2})$. *Then,*

$$\nu^{1/2}\pi \rtimes \nu^{-1/2}\tau = Q(\nu^{1/2}\pi, \nu^{-1/2}\tau) + G(\nu^{1/2}\pi, \nu^{-1/2}\tau).$$

$G(\nu^{1/2}\pi, \nu^{-1/2}\tau)$ *is the unique irreducible subrepresentation, which is generic.* $Q(\nu^{1/2}\pi, \nu^{-1/2}\tau)$ *is the unique irreducible quotient, which is not generic.*

They call $Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ local Saito-Kurokawa representation. Remark the elements of $G(\nu^{1/2}\pi, \nu^{-1/2}\tau)$ are given by 4-iii) in Theorem 1.

Theorem 5 — *Let* $\sigma_1 = \chi(\det)$ *and* $\sigma_2 \in \Pi(GL(2, \mathbb{A}))$ *be as in Theorem 1, (consequently* $\eta = \chi^2$*.) Then, the Whitakker function* $W_F^{1,-1}$ *of* $F = \theta_2(\chi(\det) \boxtimes f_2, \varphi)$ *is vanishing. But the degenerate* $W_F^{1,0}$ *is not vanishing, and*

$$W_F^{1,0} \in Q(\nu^{1/2}\chi^{-1}\sigma_{2p}, \nu^{-1/2}\chi).$$

Next, we shall describe for the case B), i.e., from Hilbert modular forms over real quadratic field L.

Theorem 6 — *We fix addtiive character on* $L\backslash L_{\mathbb{A}}$ *by*

$$\psi_L := \psi \circ Tr_{L/\mathbb{Q}}.$$

Take irreducible unitary cuspidal representations $\sigma \in \Pi(GL(2, L_{\mathbb{A}}))$ *so that*

- *the central character ω of σ is written as $\eta \circ N_{L/\mathbf{Q}}$ by some character η of $\mathbf{Q}^\times\backslash\mathbb{A}^\times$:*

$$\omega = \eta \circ N_{L/\mathbf{Q}}.$$

- *At every place w of L, σ_w has Whitakker models associated to ψ_{Lw}.*

Take an automorphic form $f \in \sigma^\vee$ whose Whitakker function $W = \otimes_w W_w \in \mathcal{W}(\sigma^\vee, \psi_L)$ satisfies

$$W_w(zg) = W_w(g\begin{bmatrix} z & * \\ * & * \end{bmatrix}) = \omega_p^{-1}(z) W_w(g),$$

for any

$$\begin{bmatrix} z & * \\ * & * \end{bmatrix} \in \begin{bmatrix} \delta_{L_w}^{-1} & \\ & 1 \end{bmatrix} \Gamma_0^{(1)}(\mathfrak{c}(\sigma_w)) \begin{bmatrix} \delta_{L_w} & \\ & 1 \end{bmatrix}$$

where δ_L is the discriminant. Then,
1) the θ-lift $F = \theta_2(\eta f, \varphi)$ (defined in (9)) has a global Whittaker function $W_F^{1,-1} = \otimes_v W_{F,v}^{1,-1}$ such as $W_F^{1,-1}(1) \neq 0$. $W_{F,p}$ is η_p^{-1}-semistable on

$$K\Big(N_{L/\mathbf{Q}}(\mathfrak{c}(\sigma)\delta_L^2), \mathfrak{c}(\eta)\Big).$$

At archimedean place, if W has a (multiple) weight (κ_1, κ_2) (remark that $\kappa_1 \pm \kappa_2$ is even, by the condition for the central character), then the highest weight of F is

$$\Big(\frac{\kappa_1 + \kappa_2}{2}, \frac{-|\kappa_1 - \kappa_2|}{2}\Big).$$

2) If W_w is the newform, then

$$Z_N(s, W_{Fp}^{1,-1}) = \begin{cases} L(s, \sigma_{\mathfrak{P}_1}^\vee) L(s, \sigma_{\mathfrak{P}_2}^\vee) & \text{if } p \text{ is decomposed to } \mathfrak{P}_1\mathfrak{P}_2, \\ L(s, \sigma_p^\vee) & \text{otherwise} \end{cases}$$

3) Let χ_L be the quadratic character of $\mathbf{Q}$ assoiciated to $L/\mathbf{Q}$. F is cuspidal, unless

$$\sigma \text{ is a base change lift of } \sigma_1 \in \Pi(GL(2,\mathbb{A})) \text{ and } \eta = \omega_{\sigma_1}|_{\mathbb{A}^\times}\chi_L.$$

In this case, the degenarate Whitakker function $W_F^{0,1}$ is not zero.
4) When p is decomposed, things are similar to Theorem 1. We mention about the case p is ramified or innert.
4-i) Suppose and $\sigma_p = \pi(\mu, \omega_p\mu^{-1})$. Take μ_1 so that $\mu_1^2 = \mu$. Then,

$$W_{Fp}^{1,-1} \in \eta_p \chi_L \mu \times \chi_L \rtimes \mu_1.$$

4-iii) If σ_p is a local base change of $\sigma_{1p} \in \Pi(GL(2))$, then $W_{Fp}^{0,1}$ is not vanishing and

$$W_{Fp}^{1,-1}, W_{Fp}^{0,1} \in \chi_L \rtimes \sigma_{1p}.$$

Remark 7 *The additive character ψ_{Lw} has conductor δ_{Lw}^{-1}. Hence the Whitakker function W_w is semi-invariant on*

$$\begin{bmatrix} \mathcal{O}_w & \delta_{L_w}^{-1}\mathcal{O}_w \\ \delta_{L_w}\mathfrak{c}(\sigma_w) & \mathcal{O}_w \end{bmatrix} \cap GL(2, L_w).$$

Take a deinite quaternion algebra $D(\mathbf{Q})$ defiined over $\mathbf{Q}$ so that

$$D_w = (D(\mathbf{Q}) \otimes L)_w \text{ splits at every finite place of } L.$$

When σ is holomorphic and

$$\kappa_1 \geq 2, \quad \text{and} \quad \kappa_2 \geq 2,$$

there always exists $\sigma^{JL} \in \Pi(D(L_{\mathbb{A}}^\times))$ with $L(s, \sigma^{JL}) = L(s, \sigma)$. The main theorem of Roberts [8] is that, if σ_w is tempered at every place w, the Yoshida lift $\Theta_2(\eta, \sigma^{JL})$ (to be explained later) is not vanishing. However Blasius [1] showed the condition in this case. Hence we have:

Corollary 8 — *Let $\sigma \in \Pi(GL(2, L_{\mathbb{A}}))$ be in the previous theorem. Assume further that σ is holomorphic of multiple weight (κ_1, κ_2) with $\kappa_i \geq 2$, and is not a base change lift. Take a definite quaternion algebra $D(\mathbb{Q})$ definied over $\mathbb{Q}$ so that*

$$D_v \text{ splits at } v \text{ where } L_v/\mathbb{Q}_v \text{ is not ramified.} \tag{2}$$

Then the Yoshida lift $\Theta_2(\eta, \sigma^{JL})$ has a (global) semi-paramodular vector, too.

Remark 9 *The Yoshida lift $\Theta_2(\eta, \sigma^{JL})$ is always holomophic. Hence it doesnot have global Whitakker functions, although it has local ones at every finite place if we take $D(\mathbb{Q})$ such as (2).*

If the central character ω of σ is trivial, $\Theta_2(1, \sigma)$, (and its complex conjugate), $\Theta_2(1, \sigma^{JL})$ (and its complex conjugate) provide paramodular vectors. In particular, if

$$\kappa_1, = 4, \kappa_2 = 2 \text{ and vice versa,}$$

these paramodular vectors are diffential form of $h^{2,1}, h^{1,2}$ and $h^{3,0}, h^{0,3}$ of the Siegel threefold asociated to the paramodular group.

1 θ-lift.

First, we recall the θ-lift from $GO(2m)$ to $GSp(n)$. Let $(X, (,))$ be a $2m$ dimensional quadratic space over $\mathbb{Q}$. Let d_X be the discriminant of X. We denote by $GO(X)$ the group of $\mathbb{Q}$ linear automorphisims h of X such that $(h(x), h(y)) = \lambda(x, y)$ with a certain $\lambda = \lambda(h) \in \mathbb{Q}^\times$ for any $x, y \in X$. Define the sign map $\mathrm{sgn}(h) := \det(h)\lambda(h)^{-2}$ on $GO(X)$. We set

$$O(X) = \{h \in GO(X) \mid \lambda(h) = 1\}, \ GSO(X) = ker(sgn), \ SO(X) = GSO(X) \cap O(X).$$

Take $s_0 \in O(X) \backslash SO(X)$. By the action $s_0 \cdot h := s_0 h s_0$ on $GSO(X)$, we have the isomorphism $GSO(X) \ltimes \{1, s_0\} \simeq GO(X)$ that takes (h, δ) to $h\delta$. Let

$$\mathcal{R} = \{(g, h) \in GSp(n) \times GO(X) \mid \lambda(g) = \lambda(h)\}.$$

The Weil representation r_v of $Sp(n) \times O(X)$ related to ψ_v is the unitary representation on $L^2(X)$ given by

$$\begin{aligned}
r_v\left(\begin{bmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{bmatrix}, 1\right)\varphi(x) &= \chi_X(\det a)|\det a|^{m/2}\varphi(xa), \\
r_v\left(\begin{bmatrix} I_n & b \\ 0 & I_n \end{bmatrix}, 1\right)\varphi(x) &= \psi(\frac{1}{2}\mathrm{tr}(b(x, x)))\varphi(x), \\
r_v\left(\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, 1\right)\varphi(x) &= \gamma\varphi^\vee(xa), \\
r_v(1, h)\varphi(x) &= \varphi(h^{-1}x).
\end{aligned}$$

Here $\chi_X(*)$ is defined by the Hilbert symbol $\{*, (-1)^m d_X\}_v$. γ is the Weil constant depending only on the anisotropic component of X, n and ψ. The fourier transformation $\varphi^\vee$ of φ is defined by

$$\varphi^\vee(x) = \int_{X_v^n} \psi_v(\mathrm{tr}(x, y))\varphi(y)dy$$

where dx is a self dual Haar measure. The Weil representation r_v is extended to $\mathcal{R}(k_v)$ by

$$r_v(g, h)\varphi_v(x) = \lambda(h)^{-n} r_v\left(g\begin{bmatrix} 1 & \\ & \lambda(g)^{-1} \end{bmatrix}, 1\right)\varphi_v(h^{-1}x).$$

Occasionally, in order to indicate the dependence of r on n, we will write r^n. For $\varphi = \prod_v \varphi_v \in \mathcal{S}(X(\mathbb{A})^n)$ and $(g, h) \in \mathcal{R}(\mathbb{A})$, we set a θ-series

$$\theta(g, h; \varphi) = \sum_{x \in X(k)^n} r(g, h)\varphi(x),$$

which converges absolutely and is left $\mathcal{R}(k)$ invariant. This θ-series gives the following θ-lifts.

$$\theta_n(f,\varphi)(g) = \int_{O(X,k)\backslash O(X,\mathbb{A})} \theta(g,h_1h;\varphi)f(h_1h)dh_1, \tag{3}$$

where f is a cuspform on $GO(X,\mathbb{A})$, and $(h,g) \in \mathcal{R}(\mathbb{A})$. For a cuspidal $\tau \in \Pi(GO(X,\mathbb{A}))$, we denote by $\Theta_n(\tau)$ the subspace of automorphic forms generated by $\theta_n(f,\varphi)$ for $f \in \tau$ and $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$.

Now, suppose $X(\mathbb{Q})$ is a four dimensional space ($m = 2$) and $d \in (\mathbb{Q}^\times)^2$. Then X is isometric to a quaternion algebra $(B(\mathbb{Q}),(,))$ defined over $\mathbb{Q}$ with $(x,y) = \frac{1}{2}tr(xy^\iota)$. Here ι indicates the canonical involution and $tr(y) = y + y^\iota$. The norm of y will be denoted by $n(y)$. Put

$$H(\mathbb{Q}) = B(\mathbb{Q})^\times \times B(\mathbb{Q})^\times,\ \ H^1(\mathbb{Q}) = \{(b_1,b_2) \in H(\mathbb{Q}) \mid n(b_1) = n(b_2)\}. \tag{4}$$

The action ρ of H on B defined by

$$\rho(b_1,b_2)x = b_1^{-1}xb_2$$

induces an isomorphism $GSO(B) \simeq H/\mathbb{Q}^\times$ and $SO(B) \simeq H^1/\mathbb{Q}^\times$, where $\mathbb{Q}^\times$ is embedded into H diagonally. If $\sigma_1,\sigma_2 \in \Pi(B(\mathbb{A})^\times)$ have a common central character η, then $\sigma_1 \boxtimes \sigma_2$ can be regarded as an element in $\Pi(GSO(B,\mathbb{A}))$ with central character η. If the induced representation $\mathrm{Ind}_{GSO(X,\mathbb{A})}^{GO(X,\mathbb{A})}\sigma_1 \boxtimes \sigma_2$ is irreducible, we denote it also by $\sigma_1 \boxtimes \sigma_2 \in \Pi(GO(X,\mathbb{A}))$. Otherwise, the induced representation is divided into two constituents $(\sigma_1 \boxtimes \sigma_2)^+$ and $(\sigma_1 \boxtimes \sigma_2)^-$. This can happen only when $\sigma_1 = \sigma_2$. However, since $\Theta_2((\sigma_1 \boxtimes \sigma_2)^-)$ is always vanishing, we will treat only $(\sigma_1 \boxtimes \sigma_2)^+$ and denote it also by $\sigma_1 \boxtimes \sigma_2$. When $B(\mathbb{Q})$ is a definite quaternion algebra $\Theta_2(\sigma_1 \boxtimes \sigma_2)$ is the *"Yoshida lift of the first type"* (c.f. [14]).

Next, suppose $X(\mathbb{Q})$ is four dimensional and $d_X \notin (\mathbb{Q}^\times)^2$. In this case, $L := \mathbb{Q}(\sqrt{d_X})$ is a quadratic field. $X(\mathbb{Q})$ is isometric to

$$\{b \in B(L) \mid \bar{b} = b^\iota\} \ \text{ or } \ \{b \in B(L) \mid \bar{b} = -b^\iota\}$$

for a quaternion algebra $B(L) = B(\mathbb{Q}) \otimes L$. Here $\bar{b}$'s coefficients in L are the algebraic conjugate over $\mathbb{Q}$ of those of b. Put

$$H'(\mathbb{Q}) = \mathbb{Q}^\times \times B(L)^\times,\ \ H'^1(\mathbb{Q}) = \{(t,b) \in H'(\mathbb{Q}) \mid n(b) = t^2\}.$$

The action ρ' of H' on X defined by

$$\rho'(t,b)x = t^{-1}b^\iota xb$$

induces an isomorphism $GSO(X) \simeq H'/\mathbb{Q}^\times$ and $SO(X) \simeq H'^1/\mathbb{Q}^\times$, where $\mathbb{Q}^\times$ is embedded into H'^1 by $t \mapsto (t,t)$. If $\tau \in \Pi(B(\mathbb{A}_L)^\times)$ has a central character $\eta \circ N_{L/\mathbb{Q}}$, then we can regard (η,τ) as an element in $\Pi(GSO(X,\mathbb{A}))$ with central character η. If its induced representation to $GO(X,\mathbb{A})$ is irreducible, we denote it also by $(\eta,\tau) \in \Pi(GO(X,\mathbb{A}))$. Otherwise, the induced representation is divided into two constituents $(\eta,\tau)^+$ and $(\eta,\tau)^-$. This can happen only when $\tau \in \Pi(B(\mathbb{A}_L)^\times)$ is a base change lift of some $\sigma \in \Pi(B(\mathbb{A})^\times)$. However, since $\Theta_2((\eta,\tau)^-)$ is always vanishing, we only treat $(\eta,\tau)^+$ and denote it also by (η,σ). When $d_X > 0$ and $B(\mathbb{Q})$ is a definite quaternon algebra, $\Theta_2((\eta,\tau))$ is the *"Yoshida lift of the second type"*.

2 Schwartz function.

First, let us treat the case A). Take $W_1 = \otimes_v W_{1v} \in \mathcal{W}(\sigma_1^\vee,\psi)$ and $W_2 = \otimes_v W_{2v} \in \mathcal{W}(\sigma_2,\psi)$ as in Theorem 1. Corresponding them, we define Schwartz function $\varphi_v \in \mathcal{S}(M_2(\mathbb{Q}_v)^2)$ as follows.
(*At archimedean place* ∞) We choose two polynomials of $M_2(\mathbb{R})$

$$P_1(x) = i(d_x - a_x) - (b_x + c_x),\ P_2(x) = i(a_x + d_x) + (c_x - d_x),$$

where we write

$$x = \begin{bmatrix} a_x & b_x \\ c_x & d_x \end{bmatrix}.$$

These polynomials have the properties

$$P_1(u_{\theta_1}^{-1}xu_{\theta_2}) = e^{-i(\theta_1+\theta_2)}P_1(x), P_2(u_{\theta_1}^{-1}xu_{\theta_2}) = e^{i(\theta_1-\theta_2)}P_2(x).$$

for $u_\theta = [\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}]$. Let s_1, s_2 be indeterminants. Define $\varphi_{\infty_j} \in \mathcal{S}(M_2(\mathbb{R})^2) \otimes \mathbb{C}[s_1, s_2]$ by

$$\begin{aligned} \varphi_\infty(x_1, x_2) &= \exp(-\pi \operatorname{tr}(R[x_1, x_2]))P_1(s_1x_1 + s_2x_2)^\alpha \\ &\quad \times \begin{cases} P_2(s_2x_1 - s_1x_2)^\beta & \text{if } \kappa_1 \le \kappa_2, \\ \overline{P}_2(s_2x_1 - s_1x_2)^\beta & \text{otherwise,} \end{cases} \end{aligned}$$

where $\alpha = \frac{\kappa_{1j}+\kappa_{2j}}{2}$ and $\beta = \frac{|\kappa_{1j}-\kappa_{2j}|}{2}$. Here symmetic matrix $R \in M_4(\mathbb{Q})$ is chosen so that $R[x_i] = a_{x_i}^2 + b_{x_i}^2 + c_{x_i}^2 + d_{x_i}^2$ and x_i is regarded as a line vector (so $R[x_1, x_2] = {}^t(R[x_1, x_2]) \in M(2, \mathbb{R})$). R is an Hermite's minimal majorant of the symmetric matirix Q corresponding to the quadratic form in $M(2)$, i.e., $RQ^{-1}R = Q$.
(*At finite place p*) Let $\mathfrak{c}_i = \mathfrak{c}(\sigma_{ip})$ be the conductor of σ_{ip} and $\mathfrak{c} = \mathfrak{c}_1\mathfrak{c}_2$. Let $\mathfrak{f}$ be the conductor of the common central character η_p. In the case of $\mathfrak{f} = 0$, define

$$\varphi_p(x_1, x_2) = \text{the characteristic function of } \left(\begin{bmatrix} \mathfrak{c}_2 & \mathbf{Z}_p \\ \mathfrak{c} & \mathfrak{c}_1 \end{bmatrix} \oplus M(2, \mathbf{Z}_p) \right).$$

In the case of $\mathfrak{f} > 0$, we define

$$\varphi_p(x_1, x_2) = \begin{cases} \eta_p(b_{x_1}), & \text{if } (x_1, x_2) \in \begin{bmatrix} \mathfrak{c}_2 & \mathbf{Z}_p^\times \\ \mathfrak{c} & \mathfrak{c}_1 \end{bmatrix} \oplus M_2(\mathbf{Z}_p), \\ 0, & \text{otherwise.} \end{cases}$$

Then (3) is written as

$$\theta_2(f_1 \boxtimes f_2, \varphi)(g) = \int_{\mathbf{A}^\times H^1(\mathbb{Q})\backslash H^1(\mathbf{A})} \sum_{x_i \in M_2(\mathbb{Q})} (r(g,h)\varphi)(x_1, x_2) f_1 \boxtimes f_2(hh')dh. \tag{5}$$

Whittaker function of $\theta(f_1 \boxtimes f_2, \varphi)$: We select a pair of elements

$$e_{-1} = \begin{bmatrix} & -1 \\ & \end{bmatrix}, \; \alpha_{-1} = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}.$$

Put

$$\begin{aligned} Z_0(\mathbb{Q}) &= \{h = (h_1, h_2) \in H^1(\mathbb{Q}) \mid \rho(h)e_{-1} = e_{-1}, \rho(h)\alpha_{-1} = \alpha_{-1}\} && (6) \\ &= \left\{ \left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & -x \\ & 1 \end{bmatrix} \right) \mid x \in \mathbb{Q} \right\}. && (7) \end{aligned}$$

The global Whittaker function

$$W_F^{1,-1}(g) := \int_{U(k)\backslash U(\mathbf{A})} \overline{\psi_{1,-1}}(u) F(ug) du$$

of $F = \theta(g; f_1 \boxtimes f_2)$ is calculated as

$$\int_{Z_0(\mathbf{A})\backslash H^1(\mathbf{A})} r(g,h)\varphi(e_{-1}, \alpha_{-1}) W_1(h_1) W_2(h_2) dh. \tag{8}$$

We can calculate each local factors of (8)

$$I_v(g) = \int_{Z_0(\mathbb{Q}_v)\backslash H^1(\mathbb{Q}_v)} r_v(1,h)\varphi_v(e_{-1}, \alpha_{-1}) W_{1v}(h_1) W_{2v}(h_2) dh.$$

at arbitary g. In particular, $I_v(1) \ne 0$. Hence $W_F^{1,-1} \ne 0$. By calculating $W_{F_v}^{1,-1}(g) = I_v(g)$, 2) is obtained. The noncuspidality 3) of $\Theta_2(\sigma_1 \boxtimes \sigma_2)$ is obtained by cheking the degenerated Whittaker function $W_F^{1,0}$ or $W_F^{0,1}$ is not zero. 4) is also obtained by calculating $W_{F_v}^{1,-1}(g), W_{F_v}^{1,0}(g), W_{F_v}^{0,1}(g)$.

Next, we treat the case B). Write $L = \mathbf{Q}(\epsilon)$ by $\epsilon \in \mathcal{O} = \mathcal{O}_L$, where $\epsilon^2 \in \mathbf{Z}$ is squarefree. For the character η and Hilbert modular form f in Theorem 6, we put

$$f_\eta((t,h)) := \eta^{-1}(t)f(h).$$

Let c be the generator of $\mathrm{Gal}(L/\mathbf{Q})$. Let

$$\begin{aligned} X(\mathbf{Q}) &= \{x \in M_2(L) \mid c(x^\iota) = -x\} \\ &= \left\{ \begin{bmatrix} a_x & b_x \\ c_x & -a_x^c \end{bmatrix} \mid b_x, c_x \in \mathbf{Q} \right\}. \end{aligned}$$

and set the quadratic form $(x,y) = \mathrm{tr}(xy^\iota)$ in $X(\mathbf{Q})$. Then, we define Schwartz functions, corresponding to the above η and W in Theorem 6, as follows.
(At ∞): With the same polynoimial P_1, P_2 as in the case A), define

$$\begin{aligned} \varphi_\infty(x_1,x_2) &= \exp\Big(-\pi\sum_{i=1}^{2}(a_{x_i}^2 + (a_{x_i}^c)^2 + b_{x_i}^2 + c_{x_i}^2)\Big)P_1(s_1x_1+s_2x_2)^\alpha \\ &\times \begin{cases} P_2(s_2x_1 - s_1x_2)^\beta & \text{if } \kappa_1 \le \kappa_2, \\ \overline{P}_2(s_2x_1 - s_1x_2)^\beta & \text{otherwise,} \end{cases} \end{aligned}$$

where $\alpha = \frac{\kappa_1+\kappa_2}{2}$ and $\beta = \frac{|\kappa_1-\kappa_2|}{2}$. Here $(a_{x_i}^2 + (a_{x_i}^c)^2 + b_{x_i}^2 + c_{x_i}^2)$ is corresponding to a minimal majorant of the quadratic form (,).
(*At* $\mathfrak{p} = \mathfrak{P}^2$, $c(\mu_\mathfrak{p}) = 0$ *case*): Se set, for $y \in X_p$

$$\varphi_p^0(y) = \begin{cases} \chi_{L,p}(pb_y), & \text{if } y \in \begin{bmatrix} \mathfrak{O}_\mathfrak{P} & \varpi^{-1}\mathbf{Z}_p^\times \\ p\mathbf{Z}_p & \mathfrak{O}_\mathfrak{P} \end{bmatrix}, \\ 0, & \text{otherwise.} \end{cases}$$

Take the summation

$$\varphi_p^1(y) = \varphi_p^0(y) + \sum_{i\in\mathbf{Z}/p\mathbf{Z}} r_p^1\Big(\begin{bmatrix} i & -1 \\ 1 & \end{bmatrix}, 1\Big)\varphi_p^0(y),$$

which is not identically zero. Let $\mathfrak{e} = \mathrm{ord}_\mathfrak{P}(\mathfrak{c}(\sigma_\mathfrak{P}))$. When, η_p is trivial, define

$$\varphi_p(x_1,x_2) = \varphi_p^1\Big(p^{-1}\mathfrak{c}^{-1}\begin{bmatrix} \epsilon^\mathfrak{e} & \\ & 1 \end{bmatrix} x_1 \begin{bmatrix} (\epsilon^c)^{-\mathfrak{e}} & \\ & 1 \end{bmatrix}\Big)\varphi_p^1(x_2).$$

When $\eta_\mathfrak{P}$ is not trivial, define

$$\varphi_p^\eta(y) = \eta_\mathfrak{P}(p^1c_y)\varphi_p^0(y)$$

for $y \in X_p$ and

$$\varphi_p(x_1,x_2) = \varphi_p^\eta\Big(p^{-1}\mathfrak{c}^{-1}\begin{bmatrix} \epsilon^\mathfrak{e} & \\ & 1 \end{bmatrix} x_1 \begin{bmatrix} (\epsilon^c)^{-\mathfrak{e}} & \\ & 1 \end{bmatrix}\Big)\varphi_p^1(x_2).$$

(*At* $\mathfrak{p} = \mathfrak{P}$ *inert in* L, $\mathfrak{c}(\mu_p) = 0$ *case*): Set

$$\varphi_p^0(x) = \mathrm{ch}(X_p \cap M_2(\mathcal{O}_p); x) \in \mathcal{S}(X_p),$$

ch denotes the characteristic function. Define

$$\varphi_p(x_1,x_2) = \varphi_p^0(\mathfrak{c}^{-1}\begin{bmatrix} p^\mathfrak{e} & \\ & 1 \end{bmatrix} x_1 \begin{bmatrix} p^{-\mathfrak{e}} & \\ & 1 \end{bmatrix}\varphi_p^0(x_2)$$

with $\mathfrak{e} = \mathfrak{c}(\sigma_p)$.
(*At* $p = \mathfrak{P}$, $\mathfrak{c}(\mu_p) > 0$ *case*): Let $\mu^0(y) = \mu(y)\cdot\mathrm{ch}(\mathfrak{o}_p^\times; y)$. We define

$$\varphi_p^\mu(x) = \mu^0(c_x)\varphi_p^0(x)$$

and

$$\varphi_p(x_1, x_2) = \varphi_p^{\mu}(p^{-e}\begin{bmatrix} p^e & \\ & 1 \end{bmatrix} x_1 \begin{bmatrix} p^{-e} & \\ & 1 \end{bmatrix})\varphi_p^1(x_2).$$

Put

$$GSp_2(\mathbb{Q})^N = \{g \in GSp_2(\mathbb{Q}) \mid \nu(g) \in N_{L/\mathbb{Q}}(L^\times)\}.$$

Define

$$\theta(\eta f, \varphi)(g) = \int_{\mathbb{A}^\times (H')^1(k)\backslash (H')^1(\mathbb{A})} \sum_{x_i \in M_2(k)} (r(g,h)\varphi)(x_1, x_2) f_\eta(hh')dh. \tag{9}$$

Here $h' = (1, h_0') \in H(\mathbb{A})$ is chosen so that $\nu(g) = N_{L/k}\det(h_0')^{-1}$, and we embed $\mathbb{A}^\times \ni t \mapsto (t^2, t) \in (H')^1(\mathbb{A})$. Since $\theta(\eta f, \varphi)$ is left $GSp_2(\mathbb{Q})^N$-invariant, we can extend $\theta(\eta f, \varphi)$ to a function on $GSp_2(\mathbb{Q})\backslash GSp_2(\mathbb{A})$ by insisting that it is left $GSp_2(\mathbb{Q})$-invariant and zero outside of $GSp_2(\mathbb{Q})GSp_2(\mathbb{A})^N$.

Remarks. When both of $\sigma_1, \sigma_2 \in \Pi(GL(2,\mathbb{A}))$ are holomorphic of weight 2 in the case A and when $\sigma \in \Pi(GL(2, L_{\mathbb{A}}))$ is holomorphic of weight $(2,2)$ in the case B, the θ-lifts can be the generic Siegel modular forms corresponding some abelian surface, similar to the Yoshida lift. That is,

(A) Let $f_1, f_2 \in S_2(\Gamma_0(N_i))$ be elliptic cuspforms of weight 2 with level N_1, N_2. Then there exists

$$F \in M_{2,0}(K(N_1 N_2))$$

corresponding to the $GSp(2)$-valued Galois representation $\rho_{f_1} \oplus \rho_{f_2}$.

(B) Let $f \in S_{(2,2)}(\Gamma_0(\mathfrak{n}))$ be a Hilbert cuspform of multiple weight $(2,2)$ of level $\mathfrak{n}$. Then there exists

$$F \in M_{2,0}(K((N_{L/\mathbb{Q}}(\mathfrak{n}\delta_L^2)))$$

corresponding to the $GSp(2)$-valued Galois representation $\mathrm{Ind}_{\mathrm{Gal}(\overline{L}/L)}^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}\rho_f$.

(A) means that all jacobian varieties of elliptic modular curves of genus 2 are Siegel modular in the generic sence, e.g., product of elliptic curves. (B) means all motives of Hilbert modular forms over a real quadratic field of weight $(2,2)$ are also Siegel modular, e.g, jacobian of Shimura curves obtained by indefinite quaternion algebras, and abelian surface with complex multiplication of quartic CM-field. However, according to Przebinda [7], the archimedan component of F belongs a P_1-principal series reperesentation (c.f p.904 of [6]), not a (limit of) discrete series representation.

If L is an imaginary quadratic field, we can also consider θ-lift to $GSp(2)$ from certain classes in $\Pi(GL(2,L))$. In this case, we identify $GL(2,L)$ with $GO(3,1,\mathbb{Q})$. But, different from the real quadratic case, the space $\Theta_2(\sigma)$ of the imeages of θ-lift is decomposed as follows.

$$\begin{aligned} \Theta_2(\sigma) &= \Theta_2(\sigma)^{gen} \text{ of highest weight } (N,1) \\ &+ \Theta_2(\sigma)^{gen} \text{ of highest weight } (N,0) \\ &+ \Theta_2(\sigma)^{hol} \text{ of highest weight } (N,2). \end{aligned}$$

(More strictly, we have three ways to extend σ_∞ to $\Pi(GO(3,1,\mathbb{R}))$ for nontrivial θ-lift. see §3 and table in p. 394 of [4]). Similar to Theorem 6, we have non-vanishing of $\Theta^{gen}(\sigma)$ for such classes $\sigma \in \Pi(GL(2, \mathcal{L}_{\mathbb{A}})$, i.e., we can always generic Siegel modular forms which are semistable on semi-paramodular groups.

But, different from the real quadratic case, this θ-lift may provides holomorphic Siegel modular forms. We cannot say $\Theta_2^{hol}(\sigma) \neq 0$. See [4] for some nonvanishing conditions.

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